Revista de la
Unión Matemática Argentina
Volumen 25, 1970.

SINGULAR INTEGRALS AND MULTIPLIERS ON HOMOGENEOUS SPACES

Ronald R. Coifman and Miguel de Guzmán*

Dedicado al profesor Alberto González Domínguez

The purpose of this note is to show that, with very little modification, it is possible to extend the estimates of Calderón and Zygmund [1] to homogeneous spaces.**

As an application we obtain a criterium for the boundedness in L^p for operators commuting with a group action. This condition in the Euclidean case yields the Marcinkiewicz multiplier theorem. (As proved by L. Hörmander [2]). In the case of the sphere we obtain a multiplier theorem for expansions in spherical harmonics.

Since the natural setting for our theorem is locally compact Abelian groups or compact Riemannian Symmetric spaces, we chose to state our results in such a general context.

§1.

Let X be a topological space and $\rho(x,y): X \to R^+$ a "distance" function on X satisfying:

1° $\rho(x,y) = \rho(y,x) > 0$ for all $x \neq y$;

2° There is a constant $C > 0$ such that
for all x,y,z $\rho(x,y) \leqslant C(\rho(x,z) + \rho(z,y))$;

3° The sets

$$S_r(x) = \{y \in X : \rho(y,x) < r\}$$

form a base of open neighborhoods of x ;

* The research of the first named author was supported in part by the U.S.Army Research Office (Durham), Contract No. DA-31--124-ARO(D)-58.

** Similar extensions have been obtained independently by N. Rivière and A.Korányi and S.Vági by somewhat different methods.

4° There exists a number N such that for every $x \in X$ and $r > 0$ there are no more than N points $x_i \in S_r(x)$ such that $\rho(x_i, x_j) \geq \frac{r}{2}$.

We should observe that condition 4° is a "homogeneity" condition on the space X; it is satisfied whenever there exists a Borel measure μ on X such that for some $C' > 0$ and for all $x, x' \in X$ and $r > 0$ we have:

$$(1.1) \qquad 0 < \mu(S_r(x)) \leq C'\mu(S_{r/2}(x')) < \infty$$

The following lemma is an analogue of a lemma of Whitney.

LEMMA 1. *Every open set* $O \subset X$, $O \neq X$ *can be represented as*

$$O = \bigcup_{i=1}^{\infty} S_{r_i}(x_i)$$

where each point in O *belongs to no more than* M *"balls"* $S_{r_i}(x_i)$, $(M \leq 2N\ell g_2(8C^2))$. *Moreover, for each* i, $S_{kr_i}(x_i) \not\subseteq O$ *for* $k > 8C^2$.

The next lemma is well known in a different setting and a generalization of Wiener's covering lemma.

LEMMA 2. *Let* $E \subset X$ *be a bounded set (i.e.* E *is contained in some "ball") and* $S_{r(x)}(x)$ *a cover of* E. *Then there exist* $x_i \in E$ *such that* $S_{r(x_i)}(x_i)$ *are disjoint and* $E \subseteq \bigcup_{i=1}^{\infty} S_{kr(x_i)}(x_i)$, *where* k *is a constant depending only on* C *and* N.

We now let μ be a measure satisfying condition (1.1). By a standard argument, one obtains from lemma 2 a Hardy-Littlewood maximal estimate, which, together with lemma 1, yields the following analogue to the estimate of Calderón and Zygmund. (See also E.M.Stein [3]).

LEMMA 3. *Let* $f(x) \geq 0$, $f \in L'(X, d\mu)$. *Then, for every* $\lambda > 0$, *there exist* $S_{r_i}(x_i)$ *such that:*

a) $f(x) \leq \lambda$ *for a.e.* $x \notin \cup S_{r_i}(x_i)$;

b) $$\frac{1}{\mu(S_{r_i}(x_i))} \int_{S_{r_i}(x_i)} f(y)\,d\mu(y) \leqslant k\lambda \quad ;$$

c) $$\sum_i \mu(S_{r_i}(x_i)) \leqslant K \frac{\int_X f(y)\,d\mu(y)}{\lambda} \quad ;$$

d) *No point in* X *belongs to more than* M *"balls"* $S_{r_i}(x_i)$,
where the constants K, M *depend only on* C' *and* C.(*)

From lemma 3 we obtain the following generalization of the theorem of Calderón and Zygmund.

THEOREM 1. *Let* M *be a bounded operator on* $L^2(X,d\mu)$ *of the form*

$$M(f)(x) = \int_X k(x,y)f(y)\,d\mu(y) \quad .$$

If there exist k *and* C *such that, for all* $x, x_o, r > 0$, *we have*

$$(1.2) \qquad \int_{X-S_{kr}(x_o)} |k(x,y)-k(x,x_o)|\,d\mu(x) \leqslant C \text{ for } y \in S_r(x_o) \quad ;$$

then M *is a bounded operator on* $L^p(X,d\mu)$ *for* $1 < p \leqslant 2$ *and*

$$(1.3) \qquad \mu(\{x \in X : M(f)(x) > \lambda\}) \leqslant C' \frac{\|f\|_1}{\lambda} \quad .$$

§2.

Let G be a locally compact σ-compact group, dx a left invariant Haar measure on G. We assume the existence of a base U_i, $i \in Z$, of open neighborhoods of the identity e of G, satisfying the following conditions:

(*) Lemma 3 is of independent interest since it implies also that the estimates of John and Nirenberg concerning functions of bounded mean oscillation are valid in this general setting.

1° $U_i = U_i^{-1}$;

2° $\bigcup_{i=-\infty}^{\infty} U_i = G$;

3° $U_i \cdot U_i \subseteq U_{i+1}$;

4° $|U_{i+1}| \leqslant C|U_i| < \infty$, where $|U_i|$ is the left Haar measure of U_i. Examples of groups for which such a base exists can be found in Edwards and Hewitt [4] .

We now define

$$\rho(x) = \inf\{|U_i| : x \in U_i\}$$

and

$$\rho(x,y) = \rho(x\, y^{-1}) \quad .$$

It is immediate that $\rho(x,y)$ satisfies the conditions 1°,2°,3°,4° of §1.

We should observe that for convolution operators the condition (1.2) of theorem 1 reads as follows:

$$(2.1) \quad \int_{X-U_{i+1}} |K(x\, y^{-1}) - K(x)|dx \leqslant C \quad \text{for all} \quad y \in U_i \ .$$

Our purpose now is to give conditions under which a bounded linear operator on $L^2(G)$, commuting with left or right translations, can be represented as a "convolution" with a kernel satisfying condition (2.1).

Let $\phi_r \geqslant 0$ be a family of functions in $L'(G) \cap L^2(G)$, $0 < r < \infty$ such that

a) $\int_G \phi_r(x)dx = 1$, $\int_G \phi_r^2(x)dx \leqslant \frac{c}{r}$;

b) $\phi_r * \phi_s = \phi_s * \phi_r$;

c) $\int_{\rho(x)>t} \phi_r(x)dx \leqslant C(\frac{r}{t})^{\varepsilon}$ and

$$\int |\phi_r(xy^{-1}) - \phi_r(x)|\,dx \leqslant C' \left(\frac{\rho(y)}{r}\right)^{\varepsilon'}$$

for some $C, C', \varepsilon, \varepsilon' > 0$.

If we now define $\psi_r = \phi_r - \phi_{r/2}$, for $r > 0$, we have the following theorem:

THEOREM 2. *Let* M *be a bounded linear operator on* $L^2(G)$ *commuting with left (or right) translations of* G. *If, for some* $\varepsilon, C > 0$ *we have*

(2.2) $$\|M(\psi_r)\rho^{\frac{1}{2}+\varepsilon}\|_2 \leqslant Cr^{\varepsilon} ,$$

then M *is a bounded operator on* $L^p(G)$ *for* $1 < p < \infty$ *and is of weak type* (1.1).

A basic example of an approximate identity ϕ_r satisfying the conditions a), b), c), is obtained in the following way.

Let

(2.3) $$S_r = \{x \in G : \rho(x) < r\}$$

and

(2.4) $$\phi_r(x) = \begin{cases} \frac{1}{|S_r|} & \text{if} \quad x \in S_r \\ 0 & \text{if} \quad x \notin S_r \end{cases}$$

Condition b) is satisfied if G is Abelian or G is compact, and the sets S_r are invariant under conjugation. As for condition c) we must add the assumption

(2.5) $$\frac{1}{|S_r|}\,|S_r - yS_r| \leqslant C\left(\frac{\rho(y)}{r}\right)^{\varepsilon} \quad \text{for some} \quad C, \varepsilon > 0 ,$$

which is satisfied for the examples cited below. The other conditions are then obvious.

Most approximate identities useful in Analysis, like the Poisson kernel or Gauss-Weirstrass kernels, satisfy the conditions a),b), c). We should observe also that, instead of defining ψ_r as above, we can, with suitable modifications, let

$$\psi_r = r \frac{\partial}{\partial r} \phi_r$$

and theorem 2° remains valid.

Let now G/K be a compact Riemannian symmetric space. Fix a G-invariant Riemannian metric on G/K and define $\rho(x)$ to be the volume of the smallest closed ball centered at 0 containing x. (0 is the base point of G/K). Define S_r, ϕ_r by (2.3) and (2.4); condition (2.5) is then satisfied for $\varepsilon = \frac{1}{n}$, n being the dimension of G/K. Theorem 2 remains true for operators on $L^2(G/K)$ which commute with the action of G. As a consequence of theorem 2 we obtain in the case of the sphere $\Sigma_n = SO(n+1)/SO(n)$, a multiplier theorem for expansions in spherical harmonics:

Let $Y_i^k(x)$, $i = 1,2,\dots,d_k$, $k = 0,1,2,\dots$, be an orthonormal base of spherical harmonics. An operator M on $L^2(\Sigma_n)$ which commutes with the action of SO(n+1) is of the form:

$$(2.6) \qquad M\Big(\sum_{k=0}^{\infty}{}' \sum_{i=1}^{d_k}{}' a_i^{(k)} Y_i^{(k)}(x)\Big) = \sum_{k=0}^{\infty}{}' m_k \sum_{i=1}^{d_k} a_i^{(k)} Y_i^{(k)}(x)$$

where m_k is a bounded sequence.

THEOREM 3. *An operator* M *defined by* (2.6) *is bounded on* $L^p(\Sigma_n)$, $1 < p < \infty$, *and is of weak type* (1-1) *if the sequence* $\{m_k\}$ *is bounded and satisfies*

$$(2.7) \qquad \sum_{K=2^N}^{2^{N+1}} |\Delta^j m_k|^2 \leqslant C\, 2^{-(2j-1)N} \quad \textit{for} \quad j = [\tfrac{n}{2}] + 1 ,$$

where $\Delta^1 m_k = m_k - m_{k+1}$ *and* $\Delta^j m_k = \Delta^1(\Delta^{j-1} m_k)$.

Remark: condition (2.7) is satisfied if $|\Delta^j m_k| < \frac{c}{K^j}$ for $j = [\frac{n}{2}] + 1$.

In conclusion, we observe that, in the case of R^n, T^n, and the p-adic number fields (see Taibleson [5]), condition (2.2) is easily seen to be equivalent to a condition involving estimates on derivatives or differences of the Fourier transform of M. This equivalence is still valid for compact symmetric spaces as will be shown explicitely in a forthcoming paper.

REFERENCES

[1] CALDERON A.P. and ZYGMUND A., *"On the Existence of Certain Singular Integrals"*, Acta Math., Vol. 88 (1952), pp. 85-139.

[2] HÖRMANDER L., *"Estimates for Translation Invariant Operators in L^p Spaces"*, Acta Math., Vol. 104 (1960), pp. 93-140.

[3] STEIN F.M., *"A Course in Harmonic Analysis on Euclidean Spaces"*, Presented to the University of Paris at Orsay in 1967-8. Notes in French.

[4] EDWARDS R.E. and HEWITT E., *"Pointwise Limits for Sequences of Convolution Operators"*, Acta Math., Vol. 113 (1965), pp. 181-218.

[5] TAIBLESON M.H., *"Harmonic Analysis on n-dimensional Vector Spaces over Local Fields III. Multipliers"*, Math. Ann., Vol. 187 (1970), pp. 259-71.

Washington University.
St. Louis, Missouri.

Recibido en julio de 1970.