Revista de la
Unión Matemática Argentina
Volumen 25, 1970.

MULTIPLIER TRANSFORMATIONS OF FUNCTIONS ON SU(2) and $\sum_2$

by Ronald R. Coifman and Guido Weiss*

Dedicado al profesor Alberto González Domínguez

1. INTRODUCTION. Suppose f is a function belonging to $L^p(-\pi,\pi)$, $1 \leqslant p \leqslant \infty$. We write

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ik\theta} \tag{1.1}$$

to indicate that the sum on the right is the Fourier series of f. An important class of linear operators mapping a space $L^p(-\pi,\pi)$ into a space $L^q(-\pi,\pi)$, $1 \leqslant q \leqslant \infty$, is the class of *multiplier transformations*. Each such transformation M is characterized by a sequence $\{\hat{M}(k)\}$ having the property that if f has the Fourier series (1.1) then Mf has the Fourier series

$$\sum_{k=-\infty}^{\infty} \hat{M}(k)\hat{f}(k)e^{ik\theta} . \tag{1.2}$$

Considerable work has been done on the problem of finding conditions on the sequence $\{\hat{M}(k)\}$ which guarantee that M is a bounded linear transformation from $L^p(-\pi,\pi)$ to $L^q(-\pi,\pi)$. Perhaps the best known result of this type is a theorem of Marcinkiewicz [3] which can be stated in the following way:

THEOREM. *If* $\{\hat{M}(k)\}$ *and* $\{\sum_{j=2^{k-1}}^{2^k-1} |\hat{M}(j+1)-\hat{M}(j)|\}$ *are bounded then* M *is a bounded linear transformation from* $L^p(-\pi,\pi)$ *into itself for* $1 < p < \infty$. *More precisely, if* $f \in L^p(-\pi,\pi)$ *has the Fourier series* (1.1) *then, under these conditions,* (1.2) *is the Fourier series of a function* $Mf \in L^p(-\pi,\pi)$ *and there exists a constant* A_p, *independent of* f, *such that*

* Research supported in part by the U.S.Army research office (Durham) under contract No. DA-31-124-ARO(D)-58.

$$\left(\int_{-\pi}^{\pi} |(Mf)(\theta)|^p d\theta \right)^{\frac{1}{p}} = \|Mf\|_p \leqslant A_p \|f\|_p = A_p \left(\int_{-\pi}^{\pi} |f(\theta)|^p d\theta \right)^{\frac{1}{p}}.$$

The purpose of this paper is to establish an analogous result for functions on $G = SU(2)$, the special unitary group of 2×2 complex matrices. In order to do this we will have to set up some notation and announce certain classical results. We refer the reader to Vilenkin [5] and to Coifman and Weiss [1] for their proofs.

An element $u \in SU(2)$ is a 2×2 matrix having the form

$$u = \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix}$$

where z_1 and z_2 are complex numbers satisfying $|z_1|^2 + |z_2|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 = 1$.

In general, if $a = [\![a_{ij}]\!]$ is an $n \times n$ matrix with complex entries, its *Hilbert-Schmidt* norm $|\!|\!| a |\!|\!|$ is defined by

$$|\!|\!| a |\!|\!|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 \quad .$$

The *operator norm* of a will be denoted by $\|a\|$. That is, $\|a\|$ is the least constant A such that

$$\sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}x_j|^2 \leqslant A^2 \sum_{j=1}^{n} |x_j|^2$$

for all n-tuples $x = (x_1, x_2, \ldots, x_n)$ of complex numbers. We shall have occasion to use both of these norms. For technical reasons, however, we use the Hilbert-Schmidt norm to introduce a metric d on G by letting $d(u,v) = |\!|\!| u - v |\!|\!|$. In particular, the function whose values are $\rho(u) = |\!|\!| u - e |\!|\!|$, where e is the identity element of G, will play an important role in our development.

A function f on G is called *central* if its values depend only on the classes of conjugate elements of G. That is, $f(v^{-1}uv) = f(u)$ for all $u,v \in G$. It is easy to see that the function ρ we have just introduced is central since multiplying a column (row) on the left (right) by a unitary matrix does not change the Euclidean norm of the column (row). Thus, the Hilbert-Schmidt norms of $v^{-1}uv - e = v^{-1}(u-e)v$ and $u-e$ are the same. The fact that ρ is central enables us to obtain a particularly simple expression of $\rho(u)$ in terms of the proper values of u. Since u is unitary and its determinant is 1, the proper values of u must have the form $e^{-i\lambda/2}$ and $e^{i\lambda/2}$ for $0 \leqslant \lambda \leqslant 2\pi$. Moreover, we can find a unitary matrix v such that $v^{-1}uv$ is the diagonal matrix

$$\delta_\lambda = \begin{bmatrix} e^{-i\frac{\lambda}{2}} & 0 \\ 0 & e^{i\frac{\lambda}{2}} \end{bmatrix}$$

Consequently, since $\rho(u) = \rho(v^{-1}uv) = \rho(\delta_\lambda) = \sqrt{|1-e^{-i\frac{\lambda}{2}}|^2 + |1-e^{i\frac{\lambda}{2}}|^2}$, we must have

$$\rho(u) = \sqrt{8}\,\sin\frac{\lambda}{4} \quad . \tag{1.3}$$

We shall also use the fact that integration of a central function with respect to Haar measure has a particulary simple expression in terms of the parameter λ. Suppose $f \in L^1(G)$ is such that $f(u) = F(\lambda)$, where $e^{\pm i\frac{\lambda}{2}}$ are the proper values of u, then

$$\int_G f(u)du = \frac{1}{\pi}\int_0^{2\pi} F(\lambda)\sin^2\frac{\lambda}{2}\,d\lambda \quad , \tag{1.4}$$

where du is the element of Haar measure on $G = SU(2)$.

In order to discuss the analog of the Fourier series expansions (1.1) for functions defined on G we have to introduce some of the basic facts concerning the irreducible unitary representations of SU(2). First, we recall that a *unitary representation* of G is a continuous map, $u \to T(u)$, of G into the class of unitary operators on a Hilbert space H that satisfies the relation $T(uv) = T(u)T(v)$ for all $u,v \in G$. A subspace $M \subset H$ is said to be *invariant* under the action of T if T(u) maps M into itself for all $u \in G$. If $\{0\}$ and H are the only invariant subspaces, then the representation T is said to be *irreducible*. A basic result in the theory of representations of compact groups is

(1.5) *If the representation* T, *acting on the Hilbert space* H, *is irreducible then* H *is finite dimensional.*

If T is an irreducible representation acting on H, we can choose an orthonormal basis of H, which must be finite by (1.5), and express T as a unitary matrix $[\![t_{ij}]\!]$ with respect to this basis. We let the symbol T represent the matrix $[\![t_{ij}]\!]$ as well. In fact, for the remainder of this paper we will assume that our irreducible representations are unitary matrix valued maps $u \to T(u) = [\![t_{ij}(u)]\!]$ and the fact that multiplication is preserved under such mappings is expressed by the formula

$$t_{ij}(uv) = \sum_{k=1}^{d} t_{ik}(u)t_{kj}(v)$$

for all $u,v \in G$ and $1 \leq i,j \leq d$, where d is the dimension of the space H on which T acts.

Two representations S and T acting on the Hilbert spaces H and K are said to be *equivalent* when there exists an invertible linear transformation $L: H \to K$ such that $T(u)L = LS(u)$ for all $u \in G$. A system $\{T^{\ell}\}$, ℓ belonging to some indexing set $\mathcal{L}$, of irreducible representations of G is said to be *complete* if, given any irreducible representation T, there exists a unique index ℓ such that T and T^{ℓ} are equivalent.

Proposition (1.5), together with the following one, constitute a formulation of the Peter-Weyl theorem in the special case when $G = SU(2)$:

(1.6) *Let* $\mathcal{L} = \{\ell : \ell = \frac{n}{2}, n = 0,1,2,\dots\}$ *be the collection of half-integers. A complete system of irreducible matrix valued representations of* SU(2) *can be indexed by the set* $\mathcal{L}$ *in such a way that, if* $\{T^{\ell}\}$, $\ell \in \mathcal{L}$, *is the indexed system then:*

(i) $T^{\ell} = [\![t^{\ell}_{m,n}]\!]$ *is a* $(2\ell+1) \times (2\ell+1)$ *matrix, where* $-\ell \leq m, n \leq \ell$;

(ii) *the collection of functions* $\sqrt{2\ell+1}\, t^{\ell}_{m,n}$ *is an orthonormal basis of* $L^2(SU(2))$.

Thus, if $f \in L^2(G)$, $G = SU(2)$, we can define its (matrix valued) *Fourier transform* $\hat{f}$ by letting

$$\hat{f}(\ell) = \int_G f(u)T^{\ell}(u^{-1})du$$

for $\ell \in \mathcal{L}$. It follows from (1.6) (ii) that

$$f \sim \sum_{2\ell=0}^{\infty} (2\ell+1)\operatorname{tr}\{\hat{f}(\ell)T^{\ell}\} , \tag{1.7}$$

where the series on the right converges to f in the L^2-norm. This expansion is the desired analog of the Fourier series expansion (1.1).

If $f,g \in L^1(G)$ we define their convolution $f * g$ by letting

$$(f * g)(v) = \int_G f(u)g(vu^{-1})du$$

for all $v \in G$. It follows readily from this definition that if $(f * g)^\wedge$ is the Fourier transform of $f * g$ then

$$(f * g)^\wedge(\ell) = \hat{f}(\ell)\ \hat{g}(\ell) \tag{1.8}$$

for all $\ell \in \mathcal{L}$. It is well known that, in the classical case, multiplier transforms arise from convolution operators (generally , from convolution with a distribution). Motivated by the definitions we have made(1), therefore, the multiplier transformations M that we consider are those that transform a function with development (1.7) into one, Mf, whose development is

$$\sum_{2\ell=0}^{\infty} (2\ell+1)\operatorname{tr}(\hat{M}(\ell)\hat{f}(\ell)T^{\ell}) \quad . \tag{1.9}$$

As in the classical case, those multiplier transformations that map $L^2(G)$ boundedly into $L^2(G)$ are the easiest to characterize:

(1) Since SU(2) is not commutative, the operation of convolution we introduced is not commutative. The reader should observe, therefore, that the multiplier theorem we are developing is, really, a statement about "left" multipliers. We leave it to the reader to formulate the related results that arise because of this lack of commutativity.

THEOREM (1.10). *The multiplier transform* M *maps* $L^2(G)$ *boundedly into itself if and only if the operator norms* $\hat{M}(\ell)$ *are bounded independently of* $\ell \in \mathcal{L}$.

We shall sketch a proof of this theorem. We first observe that if

$$\sum_{2\ell=0}^{\infty} (2\ell+1) \sum_{m,n=-\ell}^{\ell} \alpha_{m,n}^{\ell} \, t_{m,n}^{\ell}$$

is the *development* of a function $f \in L^2(G)$ (thus , $\alpha_{m,n}^{\ell} = \int_G f(u) t_{n,m}^{\ell}(u) du$, since $T^{\ell}(u^{-1})$ is the conjugate transpose matrix to T^{ℓ}), then, by (1.6) part (ii)

$$\int_G |f(u)|^2 du = \|f\|_2^2 = \sum_{2\ell=0}^{\infty} (2\ell+1) \sum_{m,n=-\ell}^{\ell} |\alpha_{m,n}^{\ell}|^2 \ . \tag{1.11}$$

If $\mu_{m,n}^{\ell}$ are the coefficients of the multiplier matrix $\hat{M}(\ell)$ then the L^2 norm of a function having expansion (1.9) is

$$\sum_{2\ell=0}^{\infty} (2\ell+1) \sum_{m,n=-\ell} \Big|\sum_{j=-\ell}^{\ell} \mu_{m,j}^{\ell} \, \alpha_{j,n}^{\ell}\Big|^2 \ .$$

If the operator norms of the matrices $\hat{M}(\ell)$ are bounded, say $\|\hat{M}(\ell)\| \leqslant A$ for all $\ell \in \mathcal{L}$, then

$$\sum_{m=-\ell}^{\ell} \Big|\sum_{j=-\ell}^{\ell} \mu_{m,j}^{\ell} \, \alpha_{j,n}^{\ell}\Big|^2 \leqslant A^2 \sum_{j=-\ell}^{\ell} |\alpha_{j,n}^{\ell}|^2 \ .$$

Summing over n and ℓ, we then obtain $\|Mf\|_2^2 \leqslant A^2 \|f\|_2^2$. Conversely, if the last inequality is satisfied by all $f \in L^2(G)$ it follows that $\|\hat{M}(\ell)\| \leqslant A$ for all $\ell \in \mathcal{L}$. We see this by applying M to functions whose development is, say,

$$(2\ell+1) \sum_{j=-\ell}^{\ell} \alpha_{j,1}^{\ell} \, t_{j,1}^{\ell} \ .$$

Having set down this background material we can now turn to the development of the multiplier theorem mentioned above.

We would also like to thank our friend and colleague I. I. Hirschman Jr. for having read the manuscript carefully and having con -

tributed several valuable suggestions.

2. MULTIPLIER TRANSFORMATIONS THAT PRESERVE $L^p(G)$, $1 < p < \infty$.

Our treatment of the multiplier theorem has certain features that are similar to that of Hörmander's version of the multiplier theorem associated with n-dimensional Euclidean spaces. The similarity lies in the fact that the multiplier theorem is reduced to results concerning certain Calderón-Zygmund singular integrals (see [2]). These results have the following analog for SU(2): Suppose M is defined on $C^\infty(G)$ by

$$(Mf)(u) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\rho(v) > \varepsilon} m(v) f(uv^{-1}) dv \quad ,$$

where m is locally integrable on SU(2) - {e} and satisfies

$$\text{(2.1)} \qquad \int_{\rho(u) > 2\rho(v)} |m(uv^{-1}) - m(u)| du \leqslant C < \infty \quad ,$$

where C is independent of v. Then, if M is bounded as an operator on L^2 (i.e. there exists $A < \infty$ such that $\|Mf\|_2 \leqslant A\|f\|_2$ for all $f \in C^\infty(G)$), M is also bounded as an operator on $L^p(G)$, $1 < p < \infty$; that is, there exists a constant A_p such that $\|Mf\|_p \leqslant A_p\|f\|_p$ for all $f \in C^\infty(G)$.

We state this result without proof since it is not substantially different from that found in Hörmander [2] (1). Our development will make use of a result, obtained by de Guzmán and the first author of this paper, which permits us to replace condition (2.1)

(1) If $z_j = x_j + iy_j$, j=1,2, the correspondence $u = \left\|\begin{matrix} z_1 & -z_2 \\ z_2 & \bar{z}_1 \end{matrix}\right\| \longleftrightarrow$ $\longleftrightarrow (x_1, y_1, x_2, y_2)$ can be used to obtain a natural identification of SU(2) with the surface of the unit sphere in Euclidean 4-dimensional space. The operators described above, in terms of this identification, are Calderón-Zygmund type singular integrals associated with the surface of the unit sphere in R^4. Such operators have been studied by many authors. In particular, the results we have just announced can be obtained by applying the method of Hörmander to the singular integrals developed by Morley [4].

by a condition on the behaviour of M when applied to a specific approximation to the identity. This result, in its full generality , is contained in the preceeding article of this volume; at present, we limit ourselves to stating the special case associated with SU(2):

For $0 < r \leqslant 2\pi$ let $S_r = \{u \in G : \rho(u) < \sqrt{8}\sin\frac{r}{4}\}$. By virtue of (1.3), if the proper values of u are written in the form $e^{\pm i\frac{\lambda}{2}}$ for $\lambda = \lambda_u \in [0,2\pi]$, this is equivalent to $S_r = \{u \in G : \lambda_u < r\}$. Let χ_{S_r} be the characteristic function of S_r and $\phi_r = \chi_{S_r}/|S_r|$, where, in general, $|E|$ denotes the Haar measure of $E \subset SU(2)$. If $\psi_r = \phi_r - \phi_{\frac{r}{2}}$ and M is a linear operator which is defined on $L^\infty(G)$, bounded on $L^2(G)$, commutes with left translations and satisfies

$$\int_{SU(2)} |(M\psi_r)(u)|^2 [\rho(u)]^4 du \leqslant Cr \quad , \tag{2.2}$$

where $C < \infty$ is independent of r, then M is an operator of the Calderón-Zygmund type described above. In particular, M is bounded on L^p , $1 < p < \infty$ (1).

Suppose, then, that we do have a multiplier operator of the type described in (1.9). We shall suppose that the operator norms of the matrices $\hat{M}(\ell)$ are bounded; thus, by (1.10) M maps $L^2(G)$ boundedly into itself. It is readily verified that M does commute with left translations. Thus, if we can find suitable conditions which imply (2.2) we would have a theorem concerning multiplier transforms. Our task, therefore, is to study the effect, on the L^2 norm, obtained by applying M to the function ψ_r and then multiplying the resulting function by ρ^2.

(1) The reader should observe that the analog of condition (2.2) for R^3 (the number 3 being the dimension of SU(2)) is equivalent to estimates of the L^2-norm of the 1st and 2nd order derivatives of the Fourier transform of M (applied to an appropriate analog of ψ_r). A similar situation arises in Hörmander [2] .

We first suppose that $M\psi_r$ has the development

$$M\psi_r \sim \sum_{2\ell=0}^{\infty} (2\ell+1)\left(\sum_{m,n=-\ell}^{\ell} \alpha^{\ell}_{m,n} t^{\ell}_{m,n}\right) .$$

Moreover, let us agree that, for the remainder of this paper, we have chosen the particular complete system of irreducible representations of SU(2) that is introduced in the third chapter of Vilenkin's book [5]. This will enable us to refer directly to this source for many formulae and, in particular, for the calculations of certain Clebsch-Gordan coefficients. These coefficients arise when the tensor product of two irreducible representations are decomposed into irreducible parts. In particular, products of the form $t^{\ell'}_{m',n'} t^{\ell}_{m,n}$ can be expressed as linear combinations of the functions $t^{k}_{p,q}$ with $|\ell-\ell'| \leqslant k \leqslant \ell+\ell'$, $-k \leqslant p,q \leqslant k$. We shall be interested in the precise values of the coefficients involved in these linear combinations when $\ell' = \frac{1}{2}$ and (m',n') is either $(\frac{1}{2},\frac{1}{2})$ or $(-\frac{1}{2},-\frac{1}{2})$. The reason for this is the following:

Because of (1.3), $[\rho(u)]^2 = 8\sin^2\frac{\lambda}{4} = 4(1-\cos\frac{\lambda}{2})$, where $e^{\pm i\frac{\lambda}{2}}$ are the proper values of u. But $2\cos\frac{\lambda}{2} = e^{-i\frac{\lambda}{2}} + e^{i\frac{\lambda}{2}} = \mathrm{trace}(u) = t^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} + t^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}$. Thus, the effect of multiplication by $[\rho(u)]^2$ is easily derived from the formulae expressing $t^{\frac{1}{2}}_{\pm\frac{1}{2},\pm\frac{1}{2}} t^{\ell}_{m,n}$ as linear combinations of $t^{\ell\pm\frac{1}{2}}_{m\pm\frac{1}{2},n\pm\frac{1}{2}}$. These can be found in §8 of chapter III of Vilenkin and, from there, we obtain

$$\begin{aligned}(2.3)\qquad 4\sin^2\tfrac{\lambda}{4}\, t^{\ell}_{m,n} = 2t^{\ell}_{m,n} - \frac{1}{2\ell+1}\Big\{ & \sqrt{(\ell-m)(\ell-n)}\; t^{\ell-\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} + \\ & + \sqrt{(\ell+m)(\ell+n)}\; t^{\ell-\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}} + \sqrt{(\ell+m+1)(\ell+n+1)}\; t^{\ell+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} + \\ & + \sqrt{(\ell-m+1)(\ell-n+1)}\; t^{\ell+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}} \Big\}, \quad \ell=0,\tfrac{1}{2},1,\tfrac{3}{2},\dots, -\ell \leqslant m,n \leqslant \ell\end{aligned}$$

Let us now define $A^{\ell}_{m}(\varepsilon,\delta) = \sqrt{\frac{1}{2} + \frac{\varepsilon(1+4\delta m)}{2\ell+1}}$ for $-\ell \leqslant m \leqslant \ell$ and $\varepsilon,\delta = -\frac{1}{2}, \frac{1}{2}$ (1). Then, from (2.3) we have (formally)

(2.4)
$$4\sin^2\frac{\lambda}{4}\sum_{2\ell=0}^{\infty}(2\ell+1)\sum_{m,n=-\ell}^{\ell}\alpha^{\ell}_{m,n}t^{\ell}_{m,n} =$$
$$\sum_{2\ell=0}^{\infty}(2\ell+1)\sum_{m,n=-\ell}^{\ell}\left\{2\alpha^{\ell}_{m,n} - \sum_{\varepsilon,\delta=-\frac{1}{2},\frac{1}{2}}A^{\ell}_{m}(\varepsilon,\delta)A^{\ell}_{n}(\varepsilon,\delta)\alpha^{\ell+\varepsilon}_{m+\delta,n+\delta}\right\}t^{\ell}_{m,n}.$$

We see therefore, that the effect of multiplication by ρ^2 is to produce certain differences of the coefficients involved in the series developement of $M\psi_r$. In view of (1.9), these coefficients $\alpha_{m,n}$ involve both those of the multiplier transform and of the function ψ_r. We shall now examine this relationship carefully and, by obtaining estimates on the differences of the coefficients of ψ_r, we will then be able to determine conditions on the coefficients of the $\hat{M}(\ell)$'s, and their differences, assuring us that condition (2.2) is satisfied. This will then give us a multiplier theorem whose basic features are not unlike those the the classical result of Marcinkiewicz.

By the nature of the definition of ϕ_r (and ψ_r) in terms of the central function ρ, it follows that ϕ_r (and ψ_r) is also central. It is well known (see §4 of chapter I of Vilenkin) that the characters

$$\chi^{\ell}(u) = \sum_{m=-\ell}^{\ell} t^{\ell}_{m,m}$$

of the irreducible representations of G form a complete orthonor-

(1) The reader should observe that these coefficients will multiply $\alpha^{\ell+\varepsilon}_{m+\delta,n+\delta}$. These last expressions may have no meaning when $\varepsilon = -\frac{1}{2}$ and m (or n) equals ℓ or $-\ell$; when this is the case, $A^{\ell}_{m}(\varepsilon,\delta) = 0$ and, thus, these terms will not appear in the summation occuring in (2.4).

mal system in the space of central functions. In particular,

$$\hat{\Phi}_r(\ell) = \int_G \phi_r(u) T^\ell (u^{-1}) du$$

is a scalar $(2\ell+1) \times (2\ell+1)$ matrix. That is, if I_ℓ denotes the $(2\ell+1) \times (2\ell+1)$ identity matrix and

$$\hat{\phi}_r^\ell = \frac{1}{2\ell+1} \int_G \phi_r(u) \overline{\chi^\ell(u)} du$$

we have $\hat{\phi}_r^\ell I_\ell = \hat{\Phi}(\ell)$. The development (1.7) becomes in this case

$$\phi_r \sim \sum_{2\ell=0}^{\infty} (2\ell+1) \mathrm{tr}\{\hat{\Phi}_r(\ell) T^\ell\} = \sum_{2\ell=0}^{\infty} (2\ell+1) \hat{\phi}_r^\ell \chi^\ell .$$

Similarly, if we let

$$\hat{\Psi}_r(\ell) = \int_G \psi_r(u) T^\ell (u^{-1}) du$$

and

$$\hat{\Psi}_r^\ell = \frac{1}{2\ell+1} \int_G \psi_r(u) \overline{\chi^\ell(u)} du$$

we have

$$\psi_r \sim \sum_{2\ell=0}^{\infty} (2\ell+1) \mathrm{tr}\{\hat{\Psi}(\ell)\ T^\ell\} = \sum_{2\ell=0}^{\infty} (2\ell+1) \hat{\Psi}_r^\ell \chi^\ell .$$

By making use of formula (1.4) and the fact that χ^ℓ has the values

$$\chi^\ell(u) = \frac{\sin(\ell + \frac{1}{2})\lambda}{\sin \frac{\lambda}{2}}$$

(see §7 of chapter III of [5]), we can calculate the coefficients $\hat{\phi}_r^\ell$ and $\hat{\psi}_r^\ell$ explicitly. We first observe that

$$|S_r| = \int_o^r \sin^2 \frac{\lambda}{2}\, d\lambda = \frac{r-\sin r}{2} .$$

Thus,

$$\hat{\phi}^{\ell}_{r} = \frac{1}{(2\ell+1)|S_r|}\int_{S_r}\chi^{\ell}(u)du = \frac{1}{(2\ell+1)}\,\frac{2}{r-\sin r}\int_0^r \sin\frac{\lambda}{2}\,\sin(2\ell+1)\frac{\lambda}{2}\,d\lambda$$

$$= \frac{1}{(2\ell+1)(r-\sin r)}\int_0^r [\cos \ell\lambda - \cos(\ell+1)\lambda]\,d\lambda .$$

Hence,

$$(2.5) \qquad \hat{\phi}^{\ell}_{r} = \frac{1}{2\ell+1}\,\frac{\frac{\sin \ell r}{\ell} - \frac{\sin(\ell+1)r}{\ell+1}}{r-\sin r}, \quad \ell = 0,\tfrac{1}{2},1,\tfrac{3}{2},\ldots .$$

Since $\hat{\phi}^{\ell}_{r} - \hat{\phi}^{\ell}_{r/2} = \hat{\psi}^{\ell}_{r}$ (2.5) will also give us an explicit formula for the coefficients $\hat{\psi}^{\ell}_{r}$.

For any sequence $\{\alpha^{\ell}\}$, $\ell = 0,\frac{1}{2},1,\frac{3}{2},\ldots$ we let $D\alpha^{\ell} = \alpha^{\ell-\frac{1}{2}} - \alpha^{\ell}$ and $D^2\alpha^{\ell} = 2\alpha^{\ell} - \alpha^{\ell+\frac{1}{2}} - \alpha^{\ell-\frac{1}{2}}$. We then have:

LEMMA (2.6). *Each of the series*

$$\sum_{2\ell=0}^{\infty} |\hat{\psi}^{\ell}_{r}|^2(2\ell+1)^{-2}, \quad \sum_{2\ell=0}^{\infty} |D\hat{\psi}^{\ell}_{r}|^2, \quad \sum_{2\ell=0}^{\infty} |D^2\hat{\psi}^{\ell}_{r}|^2(2\ell+1)^2$$

is bounded by a constant times r*; that is, each series is* $O(r)$.

We shall indicate, briefly, how these estimates can be obtained. In each case it is convenient to split the sum into one involving those terms for which $\ell r \leq 1$ plus one involving those terms for which $\ell r > 1$. In order to estimate $\hat{\psi}^{\ell}_{r} = \hat{\phi}^{\ell}_{r} - \hat{\phi}^{\ell}_{r/2}$ we can make use of (2.5) and the power series expansion for the sine function to obtain $\hat{\psi}^{\ell}_{r} = O(\ell^2 r^2)$. Thus,

$$\sum_{\ell\leq 1/r}(2\ell+1)^{-2}|\hat{\psi}^{\ell}_{r}|^2 = O\{\sum_{\ell\leq 1/r}\ell^2 r^4\} = r^4 O\{r^{-3}\} = O(r) .$$

For the rest of the estimates it suffices to examine the coefficients $\hat{\phi}_r^\ell$. For example, applying the mean value theorem to $\sin rx/x$ we obtain $\hat{\phi}_r^\ell = O(1)$ for $\ell > 1/r$. Thus, $\hat{\psi}_r^\ell = O(1)$ and

$$\sum_{\ell > 1/r} (2\ell+1)^{-2} |\hat{\psi}_r^\ell|^2 = O\{\sum_{\ell > 1/r} \ell^{-2}\} = O(r).$$

We can conclude, therefore, that

$$\sum_{2\ell = 1} (2\ell+1)^{-2} |\hat{\psi}_r^\ell|^2 = O(r)$$

and this is the first estimate of lemma(2.6). Similarly, it can be shown that, for

$\ell r \leqslant 1$, $D\hat{\phi}_r^\ell = O(r^2\ell)$ and, thus, $\sum_{\ell \leq 1/r} |D\hat{\psi}_r^\ell|^2 = r^4 O\{\sum_{\ell \leq 1/r} \ell^2\} = O(r)$.

Moreover, $D^2\hat{\phi}_r^\ell = O(r^2)$ which implies $\sum_{\ell \leq 1/r} (2\ell+1)^2 |D^2\hat{\psi}_r^\ell|^2 =$

$= r^4 O\{\sum_{\ell \leq 1/r} \ell^2\} = O(r)$. The mean value theorem can then be used to obtain estimates for $D\hat{\phi}_r^\ell$ and $D^2\hat{\phi}_r^\ell$, when $\ell r > 1$, that allow us to complete the proof of the lemma. We leave the details to the reader.

Suppose we denote the coefficients of the matrix $\hat{M}(\ell)$ by $\mu_{m,n}^\ell$, $-\ell \leqslant m$, $n \leqslant \ell$; that is, $\hat{M}(\ell) = [\![(\mu_{m,n}^\ell)]\!]$. Then the coefficients involved in the series (1.9), when $f = \psi_r$, are those of the matrix

$$\hat{M}(\ell)\ \hat{\Psi}_r(\ell) = \hat{\psi}_r^\ell\ \hat{M}(\ell) = \hat{\psi}_r^\ell [\![(\mu_{m,n}^\ell)]\!] .$$

Thus, the coefficients $\alpha_{m,n}^\ell$ involved in the expansion of $M\psi_r$ are

(2.7) $$\alpha_{m,n}^\ell = \hat{\psi}_r^\ell\ \mu_{m,n}^\ell .$$

By straight-forward computations we obtain

LEMMA (2.8). $$2\hat{\psi}_r^{\ell}\mu_{m,n}^{\ell} - \sum_{\varepsilon,\delta=\frac{1}{2},-\frac{1}{2}} A_m^{\ell}(\varepsilon,\delta)A_n^{\ell}(\varepsilon,\delta)\hat{\psi}_r^{\ell+\varepsilon}\mu_{m+\delta,n+\delta}^{\ell+\varepsilon} =$$

$$= \hat{\psi}_r^{\ell}\{2\mu_{m,n}^{\ell} - \sum_{\varepsilon,\delta=\frac{1}{2},-\frac{1}{2}} A_m^{\ell}(\varepsilon,\delta)A_n^{\ell}(\varepsilon,\delta)\mu_{m+\delta,n+\delta}^{\ell+\varepsilon}\} +$$

$$+ (D\hat{\psi}_r^{\ell}) \sum_{\varepsilon,\delta=\frac{1}{2},-\frac{1}{2}} 2\varepsilon A_m^{\ell}(\varepsilon,\delta)A_n^{\ell}(\varepsilon,\delta)\mu_{m+\delta,n+\delta}^{\ell+\varepsilon} +$$

$$+ (D^2\hat{\psi}_r^{\ell})(\sum_{\delta=-\frac{1}{2},\frac{1}{2}} A_m^{\ell}(\tfrac{1}{2},\delta)A_n^{\ell}(\tfrac{1}{2},\delta)\mu_{m+\delta,n+\delta}^{\ell+\frac{1}{2}} .$$

It follows from (2.4) and the remarks following (1.11) that

$$\frac{1}{16}\int_G |(M\psi_r)(u)|^2 [\rho(u)]^4 du = \sum_{2\ell=0}^{\infty}(2\ell+1)\sum_{m,n=-\ell}^{\ell}|\beta_{m,n}^{\ell}|^2,$$

where $\beta_{m,n}^{\ell}$ is the expression within the curly brackets in (2.4). But is clear from this expression, (2.7) and (2.8) that $\int_G |M\psi_r|^2 \rho^4 du$ is majorized by a constant times

$$\sum_{2\ell=0}^{\infty}(2\ell+1)\{|\hat{\psi}_r^{\ell}|^2 |||\Delta^2\hat{M}(\ell)|||^2 + |D\hat{\psi}_r^{\ell}|^2 |||\Delta\hat{M}(\ell)|||^2 + |D^2\hat{\psi}_r^{\ell}|^2 |||\hat{M}(\ell)|||^2\},$$

where we recall that, for any $(2\ell+1) \times (2\ell+1)$ matrix $P = [\![p_{m,n}]\!]$, $-\ell \leqslant m,n \leqslant \ell$, $|||P||| = (\sum_{m,n=-\ell}^{\ell}|p_{m,n}|^2)^{1/2}$ denotes its Hilbert-Schmidt norm and Δ, Δ^2 are the "difference" operators which, when applied to the family of matrices $\hat{M}(\ell)$, have the coefficients

$$(\Delta\hat{M}(\ell))_{m,n} = 2\sum_{\varepsilon,\delta=\frac{1}{2},-\frac{1}{2}} \varepsilon A_m^{\ell}(\varepsilon,\delta)A_n^{\ell}(\varepsilon,\delta)\mu_{m+\delta,n+\delta}^{\ell+\varepsilon}$$

and

$$(\Delta^2\hat{M}(\ell))_{m,n} = 2\mu^{\ell}_{m,n} - \sum_{\varepsilon,\delta=\frac{1}{2},-\frac{1}{2}} A^{\ell}_{m}(\varepsilon,\delta)A^{\ell}_{n}(\varepsilon,\delta)\mu^{\ell+\varepsilon}_{m+\delta,n+\delta} .$$

It now follows immediately from theorem (1.10) , (2.2) and lemma (2.6) that we obtain the following multiplier theorem:

THEOREM I. *Suppose* M *is a multiplier transform with multiplier matrices* $\hat{M}(\ell)$ *whose operator norms are bounded and, moreover , satisfy*

(i) $|||\hat{M}(\ell)||| = O(\ell^{\frac{1}{2}})$, (ii) $|||\Delta\hat{M}(\ell)||| = O(\ell^{-\frac{1}{2}})$, (iii) $|||\Delta^2\hat{M}(\ell)||| = O(\ell^{-\frac{3}{2}})$

then M *is a bounded operator on* $L^p(G)$, $1 < p < \infty$. (1)

Let us now investigate the operators Δ and Δ^2 in more detail. We shall do this in order to obtain a better understanding of the meaning of conditions (ii) and (iii). A simple calculation shows that for $-\ell+1 \leq m,n \leq \ell-1$

(1) At the beginning of this paper we stated the classical multiplier theorem of Marcinkiewicz. There we imposed conditions on certain sums of absolute values of differences of coefficients. The reader can easily check that by making appropriate changes in the statement of lemma (2.6) we could obtain a result which, instead of conditions (i), (ii) and (iii) (iii) , we would have

$$\Big(\sum_{2^{k-1}\leq\ell<2^k} |||\hat{M}(\ell)|||^2\Big)^{1/2} = O(2^k), \Big(\sum_{2^{k-1}\leq\ell<2^k} |||\Delta\hat{M}(\ell)|||^2\Big)^{1/2} = O(1)$$

and $$\Big(\sum_{2^{k-1}\leq\ell<2^k} |||\Delta^2\hat{M}(\ell)|||^2\Big)^{1/2} = O(2^{-k}) .$$

$$
(2.9)\begin{cases}
(\Delta\hat{M}(\ell))_{m,n} = \left[(1+\frac{m+n}{2\ell+1})\,\Delta_+^1 + (1-\frac{m+n}{2\ell+1})\,\Delta_-^1\right]\mu_{m,n}^{\ell} + \\
\quad + \frac{1}{2}\sum_{\varepsilon,\delta=-\frac{1}{2},\frac{1}{2}} \left[\frac{1}{2\ell+1} - 2\varepsilon R_{m,n}^{\ell}(\varepsilon,\delta)\right]\mu_{m+\delta,n+\delta}^{\ell+\varepsilon} \\
(\Delta^2\hat{M}(\ell))_{m,n} = \left[(1+\frac{m+n}{2\ell+1})\,\Delta_+^2 + (1-\frac{m+n}{2\ell+1})\,\Delta_-^2 - \frac{\Delta_+^1+\Delta_-^1}{2\ell+1}\right]\mu_{m,n}^{\ell} - \\
\quad - \sum_{\varepsilon,\delta=-\frac{1}{2},\frac{1}{2}} \frac{1}{2} R_{m,n}^{\ell}(\varepsilon,\delta)\mu_{m+\delta,n+\delta}^{\ell+\varepsilon} \quad ,
\end{cases}
$$

where $\Delta_+^1\mu_{m,n}^{\ell} = \dfrac{\mu_{m+\frac{1}{2},n+\frac{1}{2}}^{\ell+\frac{1}{2}} - \mu_{m-\frac{1}{2},n-\frac{1}{2}}^{\ell-\frac{1}{2}}}{2}$, $\quad \Delta_-^1\mu_{m,n}^{\ell} = \dfrac{\mu_{m-\frac{1}{2},n-\frac{1}{2}}^{\ell+\frac{1}{2}} - \mu_{m+\frac{1}{2},n+\frac{1}{2}}^{\ell-\frac{1}{2}}}{2}$,

$$
\Delta_+^2\mu_{m,n}^{\ell} = \mu_{m,n}^{\ell} - \frac{\mu_{m+\frac{1}{2},n+\frac{1}{2}}^{\ell+\frac{1}{2}} + \mu_{m-\frac{1}{2},n-\frac{1}{2}}^{\ell-\frac{1}{2}}}{2}, \quad \Delta_-^2\mu_{m,n}^{\ell} = \mu_{m,n}^{\ell} - \frac{\mu_{m-\frac{1}{2},n-\frac{1}{2}}^{\ell+\frac{1}{2}} + \mu_{m+\frac{1}{2},n+\frac{1}{2}}^{\ell-\frac{1}{2}}}{2},
$$

and, finally, $R_{m,n}^{\ell}(\varepsilon,\delta) = (A_m^{\ell}(\varepsilon,\delta) - A_n^{\ell}(\varepsilon,\delta))^2$.

When $m = -\ell$ or $m = \ell$ we have already observed that undefined terms (like $\mu_{\ell+\frac{1}{2},n+\frac{1}{2}}^{\ell-\frac{1}{2}}$) appear with coefficients that are 0. We can, therefore, interpret the above formulae in this case as well by making the convention that such undefined terms are absent. It is of interest, however, to note that we can also rewrite these differences in these extreme cases in the following way when $m = \ell$:

$$
(2.9')\quad
\begin{cases}
(\Delta\hat{M}(\ell))_{m,n} = (\mu^{\ell+\frac{1}{2}}_{\ell+\frac{1}{2},n+\frac{1}{2}} - \mu^{\ell-\frac{1}{2}}_{\ell-\frac{1}{2},n-\frac{1}{2}}) + \\
\quad + \frac{1}{2\ell+1}(\mu^{\ell-\frac{1}{2}}_{\ell-\frac{1}{2}\,n-\frac{1}{2}} + \mu^{\ell+\frac{1}{2}}_{\ell-\frac{1}{2},n-\frac{1}{2}}) + \\
\quad + \sum_{\varepsilon,\delta}{}' (\delta\frac{n-\ell}{2\ell+1} - \varepsilon\, R^{\ell}_{\ell,n}(\varepsilon,\delta))\, \mu^{\ell+\varepsilon}_{\ell+\delta,n+\delta}, \\
(\Delta^2\hat{M}(\ell))_{m,n} = (2\mu^{\ell}_{m,n} - \mu^{\ell+\frac{1}{2}}_{\ell+\frac{1}{2},n+\frac{1}{2}} - \mu^{\ell-\frac{1}{2}}_{\ell-\frac{1}{2},n-\frac{1}{2}}) + \\
\quad + \frac{1}{2\ell+1}(\mu^{\ell-\frac{1}{2}}_{\ell-\frac{1}{2},n-\frac{1}{2}} - \mu^{\ell+\frac{1}{2}}_{\ell-\frac{1}{2},n-\frac{1}{2}}) + \\
\quad + \sum_{\varepsilon,\delta}{}' (\frac{2\delta\varepsilon(\ell-n)}{2\ell+1} - \frac{1}{2}R^{\ell}_{\ell,n}(\varepsilon,\delta))\, \mu^{\ell+\varepsilon}_{\ell+\delta,n+\delta},
\end{cases}
$$

where $\sum'_{\varepsilon,\delta}$ denotes the summation over all $\varepsilon,\delta = \frac{1}{2},-\frac{1}{2}$ except the case $\varepsilon = -\frac{1}{2}$ and $\delta = \frac{1}{2}$. Similar formulae are valid for $m = -\ell$ (and $n = \ell,-\ell$).

Part (i) of the following lemma follows immediately from the inequality $(a^2+b^2)(a-b)^2 \leqslant (a^2-b^2)^2$ applied to the non-negative numbers $a = A^{\ell}_{m}(\varepsilon,\delta)$ and $b = A^{\ell}_{n}(\varepsilon,\delta)$. Part (ii) is an obvious identity.

LEMMA (2.10). (i) $R^{\ell}_{m,n}(\varepsilon,\delta) \leqslant \dfrac{(m-n)^2}{2(2\ell+1)(\ell-\lfloor\frac{m+n}{2}\rfloor)}$,

$$\text{(ii)}\quad \frac{\ell - n}{2\ell + 1} = \frac{(\ell - n)^2}{(2\ell + 1)\,2\,(\ell - \frac{\ell + n}{2})}\ , \qquad -\ell \leqslant m,n \leqslant \ell\ .$$

We can now obtain the following more detailed version of theorem I

THEOREM II. *Suppose* $\hat{M}(\ell) = [\![(\mu^{\ell}_{m,n})]\!]$, $-\ell \leqslant m,n \leqslant \ell$, $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ *is a sequence of matrices having bounded operator norms and satisfying the conditions*

$$\text{(a)}\quad \sum_{m,n=-\ell}^{\ell} \frac{|m-n|^4}{(2\ell - |m+n|)^2}\, |\mu^{\ell}_{m,n}|^2 = O(\ell^{-1})\ ,$$

$$\text{(b)}\quad \sum_{|m-\delta|,\, |n-\delta| \leq \ell - \frac{1}{2}} |2\mu^{\ell}_{m,n} - \mu^{\ell+\frac{1}{2}}_{m+\delta,n+\delta} - \mu^{\ell-\frac{1}{2}}_{m-\delta,n-\delta}|^2 = O(\ell^{-3}), \delta = -\tfrac{1}{2}, \tfrac{1}{2}\ ,$$

then the multiplier operator defined by the sequence $\hat{M}(\ell)$ *is a bounded operator on* $L^p(G)$, $1 < p < \infty$.

Since $\sum\limits_{\substack{m,n=-\ell \\ m\neq n}}^{\ell} (2\ell)^{-2} |\mu^{\ell}_{m,n}|^2 \leqslant \sum\limits_{m,n=-\ell}^{\ell} \frac{|m-n|^4}{(2\ell - |m+n|)^2} |\mu^{\ell}_{m,n}|^2$, condition (a) assures us that the contribution to the Hilbert-Schmidt norm of $\hat{M}(\ell)$ by the terms off the main diagonal is less than or equal to a constant (independent of ℓ) times $\sqrt{\ell}$. Since the operator norms of the sequence $\{\hat{M}(\ell)\}$ are bounded, each of the $2\ell+1$ diagonal elements of $\hat{M}(\ell)$ has absolute value not exceeding a bound for these norms. Thus, the diagonal elements also contribute, at most, a constant, independent of ℓ, times $\sqrt{\ell}$ to the Hilbert - Schimdt norm of $\hat{M}(\ell)$. Therefore, it suffices to show that conditions (ii) and (iii) of theorem I are satisfied. In order to this we first show that (b) implies

$$\text{(2.11)}\qquad \sum_{m,n=-\ell}^{\ell} |\mu^{\ell}_{m,n} - \mu^{\ell+\frac{1}{2}}_{m+\delta,n+\delta}|^2 = O(\ell^{-1})\ .$$

For $\delta = \frac{1}{2}$ we first establish the identity

$$(2.12) \qquad \mu^{\ell+\frac{1}{2}}_{m+\frac{1}{2},n+\frac{1}{2}} - \mu^{\ell}_{m,n} = 2 \sum_{2j=1}^{\infty} \Delta_{+}^{2} \mu^{\ell+j}_{m+j,n+j}$$

This equality is a consequence of the fact that the series on the right is absolutely convergent (condition (b) assures us that this series is termwise majorized by a constant times $\sum_{2j\geq 1} j^{-3/2}$) and the fact that $a_j = \mu^{\ell+j+1/2}_{m+j+1/2,n+j+1/2} - \mu^{\ell+j}_{m+j,n+j}$ tends to 0 as $j\to\infty$ (from (2.12) applied to a_j and condition (c) we see that $\{a_j\}$ is a Cauchy sequence; thus, $\lim_{j\to\infty} a_j$ exists. The fact that this limit is 0 is a consequence of the fact that the a_j's are differences of the bounded sequence $\mu^{\ell+j}_{m+j,n+j}$). When $\delta = -\frac{1}{2}$ we have an analogous identity for $\mu^{\ell+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}} - \mu^{\ell}_{m,n}$ in which there is an obvious change in some of the signs preceeding the indices j and 1/2. We now can obtain (2.11) by taking the Hilbert-Schmidt norms of the matrices involved in (2.12).

By examining the expression for $\Delta^2\hat{M}(\ell)$ in (2.9) we see, therefore, that each of the first two differences, involving Δ_{+}^{2} and Δ_{-}^{2}, give us matrices which, by virtue of (b), have Hilbert-Schmidt norms that are $O(\ell^{-3/2})$. Because of (2.11) the same is true for the terms involving Δ_{+}^{1} and Δ_{-}^{1}. Lemma (2.10) and condition (a) show that this also holds for the last term involving the coefficients $R^{\ell}_{m,n}$. Hence, (iii) of theorem I is satisfied (care should be taken to account for the extreme cases arising when m or n are $\pm\ell$ by making use of (2.9') and (2.10), part (ii)).

For the same reasons, the first two summands occuring in the expression for $\Delta\hat{M}(\ell)$ in (2.9) have Hilbert-Schmidt norms that are $O(\ell^{-1/2})$. The fact that $|||\hat{M}(\ell)||| = O(\ell^{1/2})$, condition (a) and lemma (2.10) assure us that the same is true for the remaining terms. Thus, (ii) of theorem I is satisfied; this proves theorem II.

An immediate corollary of this theorem is the following result for "diagonal" multipliers:

COROLLARY I. *Suppose the matrices* $\hat{M}(\ell)$ *are diagonal; that is,*

$$\hat{M}(\ell) = \begin{bmatrix} \mu^{\ell}_{-\ell} & 0 & \cdots & 0 \\ 0 & \mu^{\ell}_{-\ell+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu^{\ell}_{\ell} \end{bmatrix}$$

Then the multiplier operator they define is bounded on $L^p(G)$, $1 < p < \infty$, *provided there exists a constant* $C > 0$ *such that* $|\mu^{\ell}_m| < C$ *and* $|2\mu^{\ell}_m - \mu^{\ell+\frac{1}{2}}_{m+\delta} - \mu^{-\frac{1}{2}}_{m-\delta}| \leqslant C\ell^{-2}$ *for all* m,ℓ *(with* $|m-\delta| < \ell-\frac{1}{2}$ *and* $\delta = \frac{1}{2},-\frac{1}{2}$.

If we restrict ourselves still further to those "diagonal" multiplier operators of the $\mu^{\ell} I_{\ell}$ (that is, $\mu^{\ell}_m = \mu^{\ell}$ for $-\ell \leqslant m \leqslant \ell$) we obtain the special case (for SU(2)) of theorem 3 in the preceeding paper of Coifman and de Guzmán:

If $\mu^{\ell} = O(1)$ *and* $2\mu^{\ell} - \mu^{\ell+\frac{1}{2}} - \mu^{\ell-\frac{1}{2}} = O(\ell^{-2})$ *then the multiplier operator induced by the matrices* $\hat{M}(\ell) = \mu^{\ell} I_{\ell}$ *is bounded on* $L^p(G)$, $1 < p < \infty$.

In view of the fact that the surface $\sum_2$ of the unit sphere in three dimensional Euclidean space can be realized as the homogeneous space SU(2)/SO(2), we can use theorem II to obtain a multiplier theorem for functions defined on $\sum_2$. We recall that a basis for the spherical harmonics of degree ℓ (ℓ an integer) can be identified with the functions $t^{\ell}_{m,0}$, $-\ell \leqslant m \leqslant \ell$ (see Vilenkin [5], pages 167-8). In fact, using Vilenkin's notation, a function in $L^2(\sum_2)$ can be represented by an expansion of the type

$$f(\phi,0) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} a_k^\ell \, Y_{k,\ell}(\phi,\theta) \quad .$$

Let $\hat{M}(\ell) = (\mu_{m,n}^\ell)$ be a sequence of $(2\ell+1) \times (2\ell+1)$ matrices, $\ell = 0,1,2,\ldots$.

In terms of these matrices we define (formally) the multiplier operator

$$(2.13) \qquad (Mf)(\phi,\theta) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(\sum_{k=-\ell}^{\ell} \mu_{m,k}^\ell \, a_k^\ell \Big) Y_{m,\ell}(\phi,\theta)$$

We then have the following result as a consequence of theorem II:

COROLLARY II. *The operator* M *is a bounded transformation of* $L^p(\sum_2)$ *into itself,* $1 < p < \infty$ *, provided:* (a) *the operator norms of* $\hat{M}(\ell)$ *are* $O(1)$,

$$(b) \qquad \sum_{m,n=-\ell}^{\ell} \frac{|m-n|^4}{(2\ell - |m+n|)^2} \, |\mu_{m,n}^\ell|^2 = O(\ell^{-3}) \quad \textit{and}$$

$$(c) \qquad \sum_{|m-\delta|,\,|n-\delta| \le \ell-1} \left| 2\mu_{m,n}^\ell - \mu_{m+\delta,n+\delta}^{\ell-1} - \mu_{m-\delta,n-\delta}^{\ell-1} \right|^2 = O(\ell^{-3}) \quad \textit{for } \delta=-1,1.$$

If the matrices $\hat{M}(\ell)$ are diagonal we obtain the simpler operator

$$(2.14) \qquad (Mf)(\phi,\theta) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mu_k^\ell \, a_k^\ell \, Y_{k,\ell}(\phi,\theta) \quad .$$

In this case the last result becomes:

COROLLARY III. M *maps* $L^p(\sum_2)$ *into itself for* $1 < p < \infty$ *provided* *there exists* $C > 0$ *such that* $|\mu_k^\ell| \leq C$ *and* $|2\mu_k^\ell - \mu_{k+\delta}^{\ell+1} - \mu_{k-\delta}^{\ell-1}| \leq C\ell^{-2}$ *for* $|k-\delta| \leq \ell - 1$, $\ell = 0,1,2,\ldots$ *and* $\delta = -1,1$.

By restricting our attention to other special classes of functions on SU(2) we obtain other corollaries that gives us multiplier theorems for expansions in terms of Jacobi polynomials with integral indices. These results, together with "weak-type" theorems will appear elsewhere. At this point we simply assert that there exists a weak type (1.1) result associated with each of the theorems and corollaries we obtained.

BIBLIOGRAPHY

[1] COIFMAN R.R. and WEISS G., *"Representations of Compact Groups and Spherical Harmonics"*, L'Ens. Math., t. XIV, fasc.2(1968), pp. 121-173.

[2] HÖRMANDER L.,*"Estimates for Translation Invariant Operators in L^p Spaces"*, Acta Math., Vol. 104 (1960), pp. 93-140.

[3] MARCINKIEWICZ J., *"Sur les Multiplicateurs des Séries de Fourier"*, Studia Math., Vol. 8 (1939), 78-91.

[4] MORLEY V., *"On Singular Integrals on the Sphere"*, Ph. D. Thesis, Univ, of Chicago (1958).

[5] VILENKIN N., *"Special Functions and the Theory of Group Representations"*, Moscow (1965).

Washington University,
St Louis, Missouri.

Recibido en agosto de 1970.