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THE (α,β) - CHARACTERISTIC FUNCTION

Ezio Marchi

Dedicado al Profesor Alberto González Domínguez

I. When introducing the characteristic function, one considers all the players of the coalition as exhibiting completely cooperative behavior. This fact is assumed about the anticoalition too.

Here we introduce an extension of the characteristic function by considering a division of both the coalition and the anticoalition into two sets, the support set and the difference set. The support set which is the corresponding set in the subcoalition showing cooperative behavior while the difference set is the subcoalition exhibiting non-cooperative behavior. Intuitively speaking, the new characteristic function assigns to each coalition a semi-cooperative power.

The second part of this paper examines some simple technical results which are used in Part III in introducing the new characteristic function. By using the important result, due to von Neumann and Morgenstern, concerning the simple characteristic function, a game is characterized by a superadditive function. But to each superadditive function can be assigned a semi-cooperative power. This is the motivation behind this paper. From this result arises an intricate question related to some intuitive questions which we do not treat. At the end of Part II a result concerning the superadditive character of the new characteristic function is examined.

The last section extends the concepts of solution and core by considering domination among imputations with respect to the support of the coalition only and not with respect to the whole coalition. The new solutions and new core depend upon the support function. This fact can be given an intuitive interpretation. Finally, as an example, we calculate all these new solutions for the essential three-person game. It is noted that in each case Shapley's conjecture is satisfied.

II. In this section we deal with some aspects of finite zero-sum two-person games whose strategy sets are Cartesian products of finite sets.

Let $$\Gamma = \{\Sigma_1\times\ldots\times\Sigma_m \ , \ \Sigma_{m+1}\times\ldots\times\Sigma_n \ ; \ A\}$$

be a finite zero-sum two-person game. The pure-strategies set of the first player is the Cartesian product of the finite sets $\Sigma_1,\ldots,\Sigma_m$ and the pure-strategies set of the opponent is the Cartesian product of the finite sets $\Sigma_{m+1},\ldots,\Sigma_n$. Then the mixed extension $\tilde{\Gamma}$ is defined by

$$\tilde{\Gamma} = \{\widetilde{\Sigma_1\times\ldots\times\Sigma_m} \ , \ \widetilde{\Sigma_{m+1}\times\ldots\times\Sigma_n} \ ; \ E\} \quad .$$

For any two partitions,

$$\mathcal{P} = \{P_1,\ldots,P_r\} \quad \text{and} \quad \mathcal{Q} = \{Q_1,\ldots,Q_s\} \quad ,$$

of the respective sets

$$M = \{1,\ldots,m\} \quad \text{and} \quad N = \{m+1,\ldots,n\}$$

consider the following sets

$$\Sigma_1(\mathcal{P}) = \Sigma_{P_1}\times\ldots\times\Sigma_{P_r} \ , \ \Sigma_2(\mathcal{Q}) = \Sigma_{Q_1}\times\ldots\times\Sigma_{Q_s} \quad .$$

The symbol Σ_R indicates the Cartesian product of Σ_i with $i \in R$.

We will also use the sets

$$\bar{\Sigma}_1(\mathcal{P}) = \tilde{\Sigma}_{P_1}\times\ldots\times\tilde{\Sigma}_{P_r} \ , \ \bar{\Sigma}_2(\mathcal{Q}) = \tilde{\Sigma}_{Q_1}\times\ldots\times\tilde{\Sigma}_{Q_s}$$

where $\tilde{\Sigma}_R$ indicates the set of probability distributions on Σ_R.

Given the game Γ, consider the associated game

$$\bar{\Gamma}(\mathcal{P},\mathcal{Q}) = \{\bar{\Sigma}_1(\mathcal{P}) \ , \ \bar{\Sigma}_2(\mathcal{Q}) \ ; \ E\}$$

for the partition $\mathcal{P}$ and $\mathcal{Q}$ which we call the semi-mixed extension of Γ by $\mathcal{P}$ and $\mathcal{Q}$. Such a game is equivalent to the mixed extension

$\tilde{\Gamma}$ of the game Γ, when the partitions P and Q each contain only one element.

A general result for such games is:

THEOREM 2.1. *Let* $\Gamma = \{\Sigma_1 \times \ldots \times \Sigma_m , \Sigma_{m+1} \times \ldots \times \Sigma_n ; A\}$ *be a finite zero-sum, two-person game and* P *and* Q *be partitions of the respective sets* M *and* N. *If the expectation function* E *is a bilinear function in the variable*

$$(X,Y) \in \bar{\Sigma}_1(P) \times \bar{\Sigma}_2(Q) \quad ,$$

then there exists a saddle point of the semi-mixed extension of $\bar{\Gamma}$ *by* P *and* Q *:*

$$\bar{\Gamma}(P,Q) = \{\bar{\Sigma}_1(P), \bar{\Sigma}_2(Q); E\} = \{\tilde{\Sigma}_{P_1} \times \ldots \times \tilde{\Sigma}_{P_r}, \tilde{\Sigma}_{Q_1} \times \ldots \times \tilde{\Sigma}_{Q_s}; E\} \quad .$$

Therefore

$$\max_{X \in \bar{\Sigma}_1(P)} \min_{Y \in \bar{\Sigma}_2(Q)} E(X,Y) = \min_{Y \in \bar{\Sigma}_2(Q)} \max_{X \in \bar{\Sigma}_1(P)} E(X,Y)$$

Proof. For each point (X,Y) of the set $\bar{\Sigma}_1(P) \times \bar{\Sigma}_2(Q)$ which is non-empty, compact and convex in a Euclidean space, consider the set

$$\psi(X,Y) = \phi_1(Y) \times \phi_2(X)$$

where

$$\phi_1(Y) = \{U \in \bar{\Sigma}_1(P) : \max_{Z \in \bar{\Sigma}_1(P)} E(Z,Y) = E(U,Y)\}$$

and

$$\phi_2(X) = \{V \in \bar{\Sigma}_2(Q) : \min_{Z \in \bar{\Sigma}_2(Q)} E(X,Z) = E(X,V)\}$$

By the bilinearity of the function E the set $\psi(X,Y)$ is non-empty and convex. Hence, the fixed point theorem of Kakutani applied to the multivalued function

$$\psi\colon \bar{\Sigma}_1(P)\times\bar{\Sigma}_2(Q) \rightarrow \bar{\Sigma}_1(P)\times\bar{\Sigma}_2(Q)$$

defined by $\psi(X,Y)$, which is obviously upper semi-continuous, guarantees the existence of a fixed point $(\bar{X},\bar{Y}) \in \psi(\bar{X},\bar{Y})$. Such a point is a saddle point of the game $\bar{\Gamma}(P,Q)$. Q.E.D.

For this class of finite two-person games, this result is a simple extension of the minimax theorem. The minimax theorem is obtained when the partitions P and Q contain only one member each.

We note that the condition of bilinearity in the above theorem is essential. Without this condition the above result is not always true as is shown by the following example. Consider the game

$$\Gamma = \{\Sigma_1 \times \Sigma_2 \; ; \; \Sigma_3 \; ; \; A\}$$

where

$$\Sigma_1 = \Sigma_2 = \Sigma_3 = \{1,2\}$$

and the payoff function defined by

$$A(\sigma_1,\sigma_2,\sigma_3) = \begin{cases} 1 & \text{if} \quad \sigma_1 = \sigma_2 = \sigma_3 \\ 0 & \text{otherwise} \end{cases}$$

Then for the semi-mixed extension

$$\bar{\Gamma}((\{1\},\{2\}),\{3\}) = \{\tilde{\Sigma}_1\times\tilde{\Sigma}_2 \; , \; \tilde{\Sigma}_3 \; ; \; E\} \quad ,$$

the following equalities, which can be easily obtained by using simple arguments of symmetry are satisfied:

$$\max_{X\in\tilde{\Sigma}_1\times\tilde{\Sigma}_2} \; \min_{Y\in\tilde{\Sigma}_3} \; E(X,Y) = \frac{1}{4}$$

and

$$\min_{Y\in\tilde{\Sigma}_3} \; \max_{X\in\tilde{\Sigma}_1\times\tilde{\Sigma}_3} \; E(X,Y) = \frac{1}{2}$$

These last two equations show that the minimax theorem is not valid for this semi-mixed extension.

Given two partitions P_1 and P_2 of M, suppose that for each $P_1 \in P_1$ there is a $P_2 \in P_2$ such that $P_1 \subset P_2$. Equivalently, each member $P_2 \in P_2$ is a union of members $P_1 \in P_1$. We then say that the partition P_1 is a refinement of the partition P_2.

Consider any two partitions

$$P_1 = \{P_{1,1},\dots,P_{1,r_1}\} \quad \text{and} \quad P_2 = \{P_{2,1},\dots,P_{2,r_2}\}$$

such that the partition P_1 is a refinement of the partition P_2. For each $i = 1,\dots,r_2$ let

$$\rho_i = \{j : P_{1,j} \subseteq P_{2,i}\}$$

be the set of indices of members of the partition P_1 that are members of $P_{2,i}$. Then, with any member $X \in \bar{\Sigma}_1(P_1)$, which is given by

$$X(\sigma) = (X_{P_{1,1}}(\sigma_{P_{1,1}}),\dots,X_{P_{1,1}}(\sigma_{P_{1,r_1}}))$$

for each $\sigma \in \Sigma_1(P_1)$, we can associate a member $X^* \in \Sigma_1(P_2)$ defined by the relation

$$X^*(\sigma) = (\textstyle\prod_{j\in\rho_1} X_{P_{1,j}}(\sigma_{P_{1,j}}),\dots,\prod_{j\in\rho_{r_2}} X_{P_{1,j}}(\sigma_{P_{k,j}}))$$

for each $\sigma \in \Sigma_1(P_2)$. We denote by

$$\bar{\Sigma}_1(P_1)/P_2 \subseteq \bar{\Sigma}_1(P_2)$$

the set in $\bar{\Sigma}_1(P_2)$ of all associate members of the elements in $\bar{\Sigma}_1(P_1)$. We observe that each partition P is a refinement of the partition $\bar{P}$ formed by only one element. In an analogous way, the corresponding sets for the second player are defined.

Given any partitions P_1, P_2 and Q_1, Q_2 such that P_1 is a refinement of P_2 and Q_1 is a refinement of Q_2 consider any points

$$X \in \bar{\Sigma}_1(P_1) \quad \text{and} \quad Y \in \bar{\Sigma}_1(Q_1) \quad ,$$

then for the expectation function the following equality is always true:

$$E(X,Y) = E(X^*,Y^*)$$

where

$$X^* \in \bar{\Sigma}_1(P_1)/P_2 \subseteq \bar{\Sigma}_1(P_2) \quad \text{and} \quad Y^* \in \bar{\Sigma}_2(Q_1)/Q_2 \subseteq \bar{\Sigma}_2(Q_2)$$

are their corresponding associated elements.

A basic result for the subsequent discussion is given by the following theorem.

THEOREM 2.2. *Let* $\Gamma = \{\Sigma_1 \times \ldots \times \Sigma_m \ , \ \Sigma_{m+1} \times \ldots \times \Sigma_n \ ; \ A\}$ *be a finite zero-sum, two person game,* P_1 *and* Q_1 *partitions of the respective sets* M *and* N.

Assume there is a saddle point

$$(\bar{X},\bar{Y}) \in \bar{\Sigma}_1(P_1) \times \bar{\Sigma}_2(Q_1)$$

of the semi-mixed extension

$$\bar{\Gamma}(P_1,Q_1)=\{\bar{\Sigma}_1(P_1),\bar{\Sigma}_2(Q_1);E\}=\{\tilde{\Sigma}_{P_{1,1}} \times \ldots \times \tilde{\Sigma}_{P_{1,r_1}}, \tilde{\Sigma}_{Q_{1,1}} \times \ldots \times \tilde{\Sigma}_{Q_{1,s_1}};E\}$$

Then for any respective partitions P_2 *and* Q_2 *such that* P_1 *is a refinement of* P_2 *and* Q_2 *is a refinement of* Q_1*, the associated element*

$$(\bar{X}^*,\bar{Y}^*) \in \bar{\Sigma}_1(P_2) \times \bar{\Sigma}_2(Q_2)$$

is a saddle point of the semi-mixed extension

$$\bar{\Gamma}(P_2,Q_2)=\{\bar{\Sigma}_1(P_2),\bar{\Sigma}_2(Q_2);E\}=\{\tilde{\Sigma}_{P_{2,1}} \times \ldots \times \tilde{\Sigma}_{P_{2,r_2}}, \tilde{\Sigma}_{Q_{2,1}} \times \ldots \times \tilde{\Sigma}_{Q_{2,s_2}};E\}$$

Proof. First of all, we note that the relation

$$\max_{X\in\bar{\Sigma}_1(P_1)} E(X,\bar{Y}) = E(\bar{X},\bar{Y}) = \min_{Y\in\bar{\Sigma}_2(Q_1)} E(\bar{X},Y) \quad ,$$

is true. It is also always true that

$$\max_{X\in\bar{\Sigma}_1(P_1)} E(X,\bar{Y}) = \max_{X\in\bar{\Sigma}_1(P_1)/P_2} E(X,\bar{Y}) = \max_{X\in\bar{\Sigma}_1(P_2)} E(X,\bar{Y})$$

and

$$\min_{Y\in\bar{\Sigma}_1(Q_1)} E(\bar{X},Y) = \min_{Y\in\bar{\Sigma}_2(Q_1)/Q_2} E(\bar{X},Y) = \min_{Y\in\bar{\Sigma}_2(Q_2)} E(\bar{X},Y) \quad .$$

Using the relations

$$E(X,\bar{Y}) = E(X,\bar{Y}^*) \quad , \quad E(\bar{X},Y) = E(\bar{X}^*,Y) \quad \text{and} \quad E(\bar{X},\bar{Y}) = E(\bar{X}^*,\bar{Y}^*)$$

for each X and Y, where

$$(\bar{X}^*,\bar{Y}^*) \in \bar{\Sigma}_1(P_1)/P_2 \times \bar{\Sigma}_2(Q_1)/Q_2$$

are the corresponding associated elements of $\bar{X}$ and $\bar{Y}$, together with the above equalities we obtain

$$\max_{X\in\bar{\Sigma}_1(P_2)} E(X,\bar{Y}^*) = E(\bar{X}^*,\bar{Y}^*) = \min_{Y\in\bar{\Sigma}_2(Q_2)} E(\bar{X}^*,Y^*) \quad . \qquad \text{Q.E.D.}$$

A particular case of this result arises when both partitions P_2 and Q_2 are constituted by only one member.

COROLLARY 2.3. *Let* $\Gamma = \{\Sigma_1\times\ldots\times\Sigma_m \ , \ \Sigma_{m+1}\times\ldots\times\Sigma_n \ ; \ A\}$ *be a finite zero-sum, two-person game, and* P *and* Q *partitions of the respective sets* M *and* N.

If there is a saddle point

$$(\bar{X},\bar{Y}) \in \bar{\Sigma}_1(P) \times \bar{\Sigma}_2(Q)$$

of the semi-mixed extension

$$\bar{\Gamma}(P,Q) = \{\bar{\Sigma}_1(P) \ , \ \bar{\Sigma}_2(Q) \ ; \ E\} = \{\tilde{\Sigma}_{P_1}\times\ldots\times\tilde{\Sigma}_{P_r} \ , \ \tilde{\Sigma}_{Q_1}\times\ldots\times\tilde{\Sigma}_{Q_s} \ ; \ E\}$$

then for the partitions $\bar{P} = \{M\}$ *and* $\bar{Q} = \{N\}$ *the associated element*

$$(\bar{X}^*, \bar{Y}^*) \in \overbrace{\Sigma_1 \times \ldots \times \Sigma_m} \times \overbrace{\Sigma_{m+1} \times \ldots \times \Sigma_n}$$

is a saddle point of the mixed extension

$$\tilde{\Gamma} = \{\overbrace{\Sigma_1 \times \ldots \times \Sigma_m} \; ; \; \overbrace{\Sigma_{m+1} \ldots, \Sigma_n} \; ; \; E\}$$

A very special case of this result is obtained when the partitions are

$$P = \{\{1\}, \ldots, \{m\}\} \quad \text{and} \quad Q = \{\{m+1\}, \ldots, \{n\}\} \quad .$$

III. We now introduce the (α, β) - characteristic function and examine some basic properties. Finally, we will consider some special points of interest.

Let $\Gamma = \{\Sigma_1, \ldots, \Sigma_n \; ; \; A_1, \ldots, A_n\}$ be a finite n-person game; let $N = \{1, \ldots, n\}$ be the set of players and let $R \subseteq N$ be an arbitrary coalition. The players of the coalition R are divided into two disjoint sets

$$\alpha(R) \subseteq R \quad \text{and} \quad \bar{\alpha}(R) = R - \alpha(R)$$

and the players of the anticoalition N-R are separated into the disjoint sets

$$\beta(R) \subseteq N\text{-}R \quad \text{and} \quad \bar{\beta}(R) = N - (R \cup \beta(R)) \quad .$$

The set $\alpha(R)$ is called the support of the coalition R and the set $\beta(R)$ is said to be the support of the anticoalition N-R.

A function (α, β) defined on the set of all subsets $R \subseteq N$ which assigns to each coalition R the corresponding support $\alpha(R)$ and to each anticoalition N-R its corresponding support $\beta(R)$ is called a support function of the game Γ.

Intuitively speaking, the supports of the coalition R and the anticoalition N-R are the corresponding cooperative subcoalitions.

These subcoalitions have a structure that is very different from that of the subcoalitions $\bar{\alpha}(R)$ of the coalition R and $\bar{\beta}(R)$ of the anticoalition, the behavior of these latter two being non-cooperative.

For a coalition $R \subseteq N$, consider the finite zero-sum two-person game

$$\Gamma_R = \{ \underset{j\in R}{X} \Sigma_j \ , \ \underset{j\in N-R}{X} \Sigma_j \ ; \ \sum_{j\in R} A_j \} \quad .$$

The semi-mixed extension of this game by the partitions

$$\{\alpha(R) \ , \{j_1\},\ldots,\{j_s\}\} \quad \text{and} \quad \{\beta(R) \ , \{k_1\},\ldots,\{k_t\}\} \quad ,$$

where

$$\{j_1,\ldots,j_s\} = \bar{\alpha}(R) \quad \text{and} \quad \{k_1,\ldots,k_t\} = \bar{\beta}(R) \quad ,$$

is denoted by

$$\bar{\Gamma}_R = \{\tilde{\Sigma}_{\alpha(R)} \times Z_{\bar{\alpha}(R)} \ , \ \tilde{\Sigma}_{\beta(R)} \times Z_{\bar{\beta}(R)} \ , \ E_R\} \quad .$$

In this extension, the strategy sets are determined by

$$\Sigma_s = \underset{j\in S}{X} \Sigma_j \ , \quad Z_S = \underset{j\in S}{X} \tilde{\Sigma}_j$$

and E_R is the expectation function of the payoff function $A_R = \underset{j\in R}{E} A_j$.

We define the (α,β)-value of the coalition R in the finite, n-person game Γ as the maximin value of the corresponding semi-mixed extension $\bar{\Gamma}_R$ of the finite zero-sum two-person game Γ_R. Hence the (α,β) value is defined as:

$$v_o(\alpha(R),\beta(R);R) =$$

$$= \max_{(X_{\alpha(R)},U_{\bar{\alpha}(R)})\in\tilde{\Sigma}_{\alpha(R)}\times Z_{\bar{\alpha}(R)}} \ \min_{(Y_{\beta(R)},V_{\bar{\beta}(R)})\in\tilde{\Sigma}_{\beta(R)}\times Z_{\bar{\beta}(R)}} E_R((X_{\alpha(R)},U_{\bar{\alpha}(R)}),(Y_{\beta(R)},V_{\beta(R)})).$$

As has been shown after theorem 2.1, we observe that generally the maximin value $v_o(\alpha(R), \beta(R); R)$ and the minimax value $v^o(\alpha(R), \beta(R); R)$ do not coincide.

Given a finite n-person game Γ, then the (α,β)-characteristic function $v(\alpha(R), \beta(R); R)$ of Γ is a real function on the set of all subsets $R \subseteq N$ giving the (α,β)-value.

The (α,β)-characteristic function is a generalization of the characteristc function introduced by von Neumann-Morgenstern. This simple characteristic function corresponds to the case in which the support function is given by

$$\alpha(R) = R \quad \text{and} \quad \beta(R) = N-R$$

for each coalition $R \subseteq N$.

The (α,β)-characteristic function and simple characteristic function are connected by another more interesting relation which is essentially determined by the well known fact due to von Neumann-Morgenstern, which guarantees the existence of a game with characteristic function v for each superadditive real function v defined on the set of all subsets $R \subseteq N = \{1,\ldots,n\}$, with the zero value for the empty set: $v(\phi) = 0$. This result together with theorem 2.2 is used to obtain the following theorem.

THEOREM 3.1 *Let* v *be a superadditive real function defined on the set of all subsets* $R \subseteq N = \{1,\ldots,n\}$ *with* $v(\phi) = 0$, *then there exists an n-person game* Γ *such that for all support functions* (α,β) *the* (α,β)*-characteristic function of* Γ *is the function* v.

Moreover, if the function v *satisfies*

$$v(R) + v(N-R) = v(N)$$

for each $R \subseteq N$, *then there exists a constant-sum n-person game* Γ *with total sum* $v(N)$, *which is such that for all support functions* (α,β) *the* (α,β)*-characteristic function of* Γ *is the function* v.

Proof. Given such a superadditive function v, consider the well known n - person game of von Neumann - Morgenstern $\Gamma = \{\Sigma_1,\ldots,\Sigma_n ; A_1,\ldots,A_n\}$ defined by the strategy sets:

$$\Sigma_i = \{S \subseteq N : i \in S\} \quad \text{for all} \quad i \in N$$

and the payoff functions of player $i \in N$ by

$$A_i(R_1,\ldots,R_n) = \begin{cases} \dfrac{v(R_i)}{|R_i|} & \text{if} \quad R_i = R_j \quad \text{for all} \quad j \in R_i\,, \\ v(\{i\}) & \text{otherwise} \end{cases}$$

where $|R_i|$ denotes the number of elements of the set R_i.

Then, it is well known that for each coalition $R \subseteq N$, the value $v(R)$ is the value of the game

$$\Gamma_R = \{\Sigma_R, \Sigma_{N-R} ; \sum_{j\in R} A_j\}$$

where

$$\Sigma_R = \mathop{X}_{j\in R} \Sigma_j \quad , \quad \Sigma_{N-R} = \mathop{X}_{j\in N-R} \Sigma_j \quad ,$$

which is reached by using the maximin and minimax pure strategies

$$\bar{\sigma}_R = (R,\ldots,R) \in \Sigma_R \quad \text{and} \quad \bar{\sigma}_{N-R} = (N-R,\ldots,N-R) \in \Sigma_{N-R}$$

for the first player and the second player respectively.

It is equivalent to consider the saddle point

$$(\bar{X}_R, \bar{Y}_{N-R}) \in \tilde{\Sigma}_R \times \tilde{\Sigma}_{N-R}$$

defined by

$$\bar{X}_R = \begin{cases} 1 & \text{if} \quad \sigma_R = \bar{\sigma}_R \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{X}_{N-R} = \begin{cases} 1 & \text{if} \quad \sigma_{N-R} = \bar{\sigma}_{N-R} \\ 0 & \text{otherwise} \end{cases}$$

in the semi-mixed extension

$$\Gamma_R(\bar{P},\bar{Q}) = \{\bar{\Sigma}_1(P) \;,\; \bar{\Sigma}_2(Q) \;;\; E_R\}$$

corresponding to the partitions

$$\bar{P} = \{\{j_1\},\ldots,\{j_r\}\} \quad \text{and} \quad \bar{Q} = \{\{j_{r+1}\},\ldots,\{j_n\}\}$$

where

$$\{j_1,\ldots,j_r\} = R \quad \text{and} \quad \{j_{r+1},\ldots,j_n\} = N\text{-}R$$

with the value

$$E_R(\bar{X}_R,\bar{Y}_R) = v(R) \quad .$$

Now, any support function (α,β) determines partitions

$$P_\alpha = \{\alpha(R),\{j_1\},\ldots,\{j_s\}\} \quad \text{and} \quad Q_\beta = \{\beta(R),\{k_1\},\ldots,\{k_t\}\}$$

where

$$\{j_1,\ldots,j_s\} = \bar{\alpha}(R) \quad \text{and} \quad \{k_1,\ldots,k_t\} = \bar{\beta}(R)$$

of the coalition R and the anticoalition N-R, such that the partitions $\bar{P}$ and $\bar{Q}$ are respectively refinements of P_α and Q_β. Therefore theorem 2.2 assures that the associated strategy

$$(\bar{X}^*_R,\bar{Y}^*_{N-R}) \in \bar{\Sigma}_1(P_\alpha) \times \bar{\Sigma}_2(P_\beta)$$

is a saddle point of the semi-mixed extension

$$\bar{\Gamma}_R(P_\alpha,Q_\beta) = \{\bar{\Sigma}_1(P_\alpha) \;,\; \bar{\Sigma}_2(P_\beta) \;;\; E_R\} \quad ;$$

which implies the equality

$$v(\alpha(R) \;,\; \beta(R) \;;\; R) = v(R) \; .$$

If the function v also has the additional property

$$v(R) + v(N\text{-}R) = v(N) \quad \text{for all} \quad R \subseteq N \quad ,$$

it is well known that there is a constant sum game $\Gamma = \{\Sigma_1,\dots,\Sigma_n ; B_1,\dots,B_n\}$ defined by the strategies sets as before and by the payoff function as:

$$B_i(R_1,\dots,R_n) = A_i(R_1,\dots,R_n) + \frac{1}{n}((v(N)) - \sum_{j\in N} A_j(R_1,\dots,R_n))$$

for $i \in N$, whose total sum is $v(N)$. Moreover, for each coalition $R \subseteq N$ the value $v(R)$ is the value of the game

$$\Gamma_R = \{\Sigma_R , \Sigma_{N-R} ; \sum_{j\in N} B_j\}$$

with the saddle point

$$(\bar{\sigma}_R,\bar{\sigma}_{N-R}) \in \Sigma_R \times \Sigma_{N-R} .$$

Consequently, the associated strategy

$$(\bar{X}^*,\bar{Y}^*_{N-R}) \in \bar{\Sigma}_1(P_\alpha) \times \bar{\Sigma}_2(P_\beta)$$

of $(\bar{X}_R,\bar{Y}_{N-R})$ is a saddle point of the semi-mixed extension

$$\bar{\Gamma}_R(P_\alpha,Q_\beta) = \{\bar{\Sigma}_1(P_\alpha) , \bar{\Sigma}_2(Q_\beta) ; F_R\}$$

where F_R denotes the expectation function of the function $\sum_{j\in N} B_j$. Therefore we have

$$v(\alpha(R) , \beta(R) ; R) = v(R) . \qquad \text{Q.E.D.}$$

Now, based on this result, one might argue that for each game expressed by a superadditive function, which assigns to each coalition the measure of cooperative power, such a measure can also be interpreted as any semicooperative power in the sense described by the (α,β)-characteristic function.

Consider that the following result might be true; for each n and each superadditive function v defined on the set of all subsets $R \subseteq N = \{1,\dots,n\}$, with $v(\phi) = 0$, there exists a game Γ such that the simple characteristic function is v and for all support functions (α,β) different (non-trivially) from that which generates the simple characteristic function, the (α,β)-characteristic and the characteristic function are different.

In such a case, for each superadditive function there is a game, for which the measure of the power of a coalition assigned by the superadditive function, can be only interpreted as the complète cooperative power, and not in any semicooperative manner.

We do not deal here with this very intricate question. However, we are going to examine in general the superadditive property of the (α,β)-characteristic function, which by the above theorem is only guaranteed for some particular cases.

Such a characterization is given in the following result.

THEOREM 3.2. *Let* Γ *be an n-person game such that for each pair of disjoint coalitions* R *and* S*:*

$$\alpha(S) \cup \beta(R \cup S) \subseteq \beta(R)$$

and

$$\alpha(R) \cup \alpha(S) \subseteq \alpha(R \cup S) \quad .$$

Then the (α,β)*-characteristic function is superadditive, i.e.:*

$$v(\alpha(R\cup S),\beta(R\cup S);R\cup S) \geqslant v(\alpha(R),\beta(R);R) + v(\alpha(S),\beta(S);S)$$

for each pair of disjoint coalitions R *and* S.

Proof. We have a maximin strategy

$$(\bar{X}_{\alpha(R)},\bar{U}_{\alpha(R)}) \in \tilde{\Sigma}_{\alpha(R)} \times Z_{\bar{\alpha}(R)}$$

of R in the semi-mixed extension game $\bar{\Gamma}_R$ which satisfies

$$E_R((\bar{X}_{\alpha(R)},\bar{U}_{\bar{\alpha}(R)}),(Y_{\beta(R)},V_{\bar{\beta}(R)})) \geqslant v(\alpha(R),\beta(R);R)$$

for each strategy

$$(Y_{\beta(R)},V_{\bar{\beta}(R)}) \in \tilde{\Sigma}_{\beta(R)} \times Z_{\bar{\beta}(R)}$$

Analogously, we have a maximin strategy of the coalition S in the semi-mixed extension $\bar{\Gamma}_S$:

$$(\bar{X}_{\alpha(S)}, \bar{U}_{\alpha(S)}) \in \tilde{\Sigma}_{\alpha(S)} \times Z_{\bar{\alpha}(S)} \quad ,$$

which fulfills

$$E_S((\bar{X}_{\alpha(S)}, \bar{U}_{\bar{\alpha}(S)}), (Y_{\beta(S)}, V_{\bar{\beta}(S)})) \geqslant v(\alpha(S), \beta(S); S)$$

for each strategy

$$(Y_{\beta(R)}, V_{\bar{\beta}(S)}) \in \tilde{\Sigma}_{\beta(S)} \times Z_{\bar{\beta}(S)} \quad .$$

Now, since the following relations hold:

$$\alpha(R \cup S) = [\alpha(R) \cup \gamma(R)] \cup [\alpha(S) \cup \gamma(S)]$$

and

$$\bar{\alpha}(R \cup S) = [\bar{\alpha}(R) - \gamma(R)] \cup [\bar{\alpha}(S) - \gamma(S)] \quad ,$$

where

$$\gamma(R) \subseteq \bar{\alpha}(R) \quad \text{and} \quad \gamma(S) \subseteq \bar{\alpha}(S) \quad ,$$

one obviously obtains a strategy

$$(\bar{X}_{\alpha(R \cup S)}, \bar{U}_{\bar{\alpha}(R \cup S)}) \in \tilde{\Sigma}_{\alpha(R \cup S)} \times Z_{\bar{\alpha}(R \cup S)}$$

in the semi-mixed extension $\Gamma_{R \cup S}$ by choosing $\bar{X}_{\bar{\alpha}(R \cup S)}$ as the associate of the strategy

$$(\bar{X}_{\alpha(R)}, \bar{U}_{\gamma(R)}, \bar{X}_{\alpha(S)}, \bar{U}_{\gamma(S)}) \in \tilde{\Sigma}_{\alpha(R)} \times Z_{\gamma(R)} \times \tilde{\Sigma}_{\alpha(S)} \times Z_{\gamma(S)}$$

in the set $\tilde{\Sigma}_{\alpha(R \cup S)}$ where $\bar{U}_{\gamma(R)}$ and $\bar{U}_{\gamma(S)}$ are the respective restrictions of the strategies $\bar{U}_{\bar{\alpha}(R)}$ and $\bar{U}_{\bar{\alpha}(S)}$ to the set $\tilde{\Sigma}_{\alpha(R \cup S)}$, by taking each strategy $\sigma_{\alpha(R \cup S)} \in \Sigma_{\alpha(R \cup S)}$ with the probability

$$\bar{X}_{\alpha(R \cup S)}(\sigma_{\alpha(R \cup S)}) = \bar{X}_{\alpha(R)}(\sigma_{\alpha(R)}) \bar{U}_{\gamma(R)}(\sigma_{\gamma(R)}) \bar{X}_{\alpha(S)}(\sigma_{\alpha(S)}) \bar{U}_{\alpha(S)}(\sigma_{\gamma(S)})$$

and $\bar{U}_{\bar{\alpha}(R \cup S)}$ defined for each $\sigma_{\bar{\alpha}(R \cup S)} \in \Sigma_{\bar{\alpha}(R \cup S)}$ by

$$\bar{U}_{\bar{\alpha}(R\cup S)}(\sigma_{\bar{\alpha}(R\cup S)})=\bar{U}_{\bar{\alpha}(R)-\gamma(R)}(\sigma_{\bar{\alpha}(R)-\gamma(R)})\bar{U}_{\bar{\alpha}(S)-\gamma(S)}(\sigma_{\bar{\alpha}(S)-\gamma(S)})$$

where $\bar{U}_{\alpha(R)-\gamma(R)}$ and $\bar{U}_{\bar{\alpha}(S)-\gamma(S)}$ are the respective restrictions of the strategies $\bar{U}_{\alpha(R)}$ and $\bar{U}_{\alpha(S)}$.

By using the strategy $(\bar{X}_{\alpha(R\cup S)},\bar{U}_{\alpha(R\cup S)})$, in the semi-mixed extension $\bar{\Gamma}_{R\cup S}$ against an arbitrary strategy $(Y_{\beta(R\cup S)},V_{\bar{\beta}(R\cup S)})$, the expectation of the payoff to coalition $R\cup S$ becomes

$$E_{R\cup S}\ ((\bar{X}_{\alpha(R\cup S)},\bar{U}_{\bar{\alpha}(R\cup S)}),(Y_{\beta(R\cup S)},V_{\bar{\beta}(R\cup S)})) =$$

$$= \quad E_R((\bar{X}_{\alpha(R)},\bar{U}_{\bar{\alpha}(R)}),((Y_{\beta(R\cup S)},\bar{X}_{\alpha(S)},\bar{U}_{\delta(S)}),(V_{\bar{\beta}(R\cup S)},\bar{U}_{\bar{\alpha}(S)-\delta(S)}))$$

$$+\ E_S((\bar{X}_{\alpha(S)},\bar{U}_{\bar{\alpha}(S)}),((Y_{\beta(R\cup S)},\bar{X}_{\alpha(R)},\bar{U}_{\delta(R)}),(V_{\bar{\beta}(R\cup S)},\bar{U}_{\alpha(R)-\delta(R)})))$$

since

$$\beta(R) = \beta(R\cup S)\cup\alpha(S)\cup\delta(S)\ ,\ \beta(R) = \beta(R\cup S)\cup\alpha(R)\cup\delta(R)$$

and

$$\bar{\beta}(R) = \bar{\beta}(R\cup S)\cup[\bar{\alpha}(S)-\delta(S)]\ ,\ \bar{\beta}(S) = \bar{\beta}(R\cup S)\cup[\bar{\alpha}(R)-\delta(S)]$$

where

$$\delta(R)\subseteq\bar{\alpha}(R) \quad \text{and} \quad \delta(S)\subseteq\bar{\alpha}(S)\ .$$

Now, by choosing $Y_{\beta(R)}$ as the associate of the strategy

$$(Y_{\beta(R\cup S)},\bar{X}_{\alpha(S)},\bar{U}_{\delta(S)})\in\tilde{\Sigma}_{\beta(R\cup S)}\times\tilde{\Sigma}_{\alpha(S)}\times Z_{\delta(S)}$$

in the set $\tilde{\Sigma}_{\beta(R)}$, and $Y_{\beta(S)}$ as the associate of the strategy

$$(Y_{\beta(R\cup S)},\bar{X}_{\alpha(R)},\bar{U}_{\delta(R)})\in\tilde{\Sigma}_{\beta(R\cup S)}\times\tilde{\Sigma}_{\alpha(R)}\times Z_{\delta(R)}$$

in the set $\tilde{\Sigma}_{\beta(S)}$, and finally, the strategies

$$V_{\bar{\alpha}(R)} = (V_{\bar{\beta}(R\cup S)},\bar{U}_{\bar{\alpha}(S)-\delta(S)}) \quad \text{and} \quad V_{\bar{\alpha}(S)} = (V_{\bar{\beta}(R\cup S)},\bar{U}_{\bar{\alpha}(R)-\delta(R)})$$

we obtain

$$E_{R\cup S}((\bar{X}_{\alpha(R\cup S)},\bar{U}_{\bar{\alpha}(R\cup S)}),(Y_{\beta(R\cup S)},V_{\bar{\beta}(R\cup S)})) =$$

$$= E_R((\bar{X}_{\alpha(R)},\bar{U}_{\bar{\alpha}(R)}),(Y_{\beta(R)},V_{\bar{\beta}(R)})) +$$

$$+ E_S((\bar{X}_{\alpha(S)},\bar{U}_{\bar{\alpha}(S)}),(Y_{\beta(S)},V_{\bar{\beta}(S)}))$$

$$\geq v(\alpha(R),\beta(R);R) + v(\alpha(S),\beta(S);S) \ .$$

Therefore

$$v(\alpha(R\cup S),\beta(R\cup S);R\cup S) \geq v(\alpha(R),\beta(R);R) + v(\alpha(S),\beta(S);S)$$

Q.E.D.

The condition on the supports of the coalitions and anticoalitions in the above theorem can be easily transformed to an equivalent condition which has a more symmetric expression, namely: for each pair of disjoint coalitions R and S

$$\alpha(R) \cup \alpha(S) \subseteq \alpha(R \cup S) \quad ,$$

$$\beta(R \cup S) \subseteq \beta(R) \cap \beta(S)$$

and

$$\alpha(N-R) \subseteq \beta(R) \quad .$$

Some particular kinds of support functions, for which the (α,β)-characteristic function satisfies the condition of superadditivity, immediately arise from the last condition, namely:

a) If for each pair of disjoint coalitions R and S:

$$\alpha(R) = \beta(N-R)$$

$$\alpha(R \cup S) = \alpha(R) \cup \alpha(S) \quad .$$

In this case, the characteristic function can be obtained by taking the support of the coalition as the coalition:

$$\alpha(R) = R$$

Another simple case is presented if the support is empty:

$$\alpha(R) = \phi$$

b) If for each pair of disjoint coalitions R and S

$$\alpha(R) = \phi$$

$$\beta(R \cup S) \subseteq \beta(R) \cap \beta(S)$$

and finally,

c) if for each pair of disjoint coalitions R and S

$$\alpha(R \cup S) \supseteq \alpha(R) \cup \alpha(S)$$

$$\beta(R) = N - R$$

IV. Theorem 3.2 assures the fulfillment of the superadditive property by the (α,β)-characteristic functions whose support functions satisfy the mentioned relation, and therefore for these characteristic functions we are in an analogous situation to the description of a game by a superadditive function. Thus, it is natural to try to extend the basic concepts by using the concept of a support function.

In this section we deal with a possible extension of the concept of solution which is based upon a generalization of the concept of domination among imputations. We find all these new solutions of the constant-sum three-person essential game and in the end we consider some remarks; our principal motive is only to illustrate a semi-cooperative extension of the cooperative interpretation of the characteristic function.

Consider an n-person game described by a superadditive function as its (α,β)-characteristic function whose support function (α,β)

satisfies the requirement in theorem 3.2.

Given two imputations

$$a = (a_1,\dots,a_n) \quad \text{and} \quad b = (b_1,\dots,b_n)$$

for the (α,β)-characteristic function v, i.e., vectors whose components satisfy $a_i \geqslant v(\{i\})$ for all $i \in N$ and $\sum_{i\in N}\alpha_i = v(N)$, we say that the imputation b α-dominates the imputation a with respect to $R \subseteq N$, if

$$\alpha(R) \neq \phi \quad , \quad \sum_{i\in R} b_i \leqslant v(R)$$

and

$$b_i > a_i \quad \text{for all} \quad i \in \alpha(R).$$

We note that in the preceding definition the function β does not play any roles. Therefore without loss of generality and for simplicity, we can consider the function β defined by $\beta(R) = N-R$ for all $R \subseteq N$.

This is a natural extension of the classic concept of domination, which was obtained by weakening the last condition.

The imputation b α-dominates the imputation a, if there exists a subset $R \subseteq N$ so that the imputation b α-dominates the imputation a with respect to R.

Now, the extension of the concept of solution of a game in the sense of von Neumann-Morgenstern is immediate. This is obtained by replacing domination by α-domination.

A set of imputations L_α is said to be an α-solution of the game Γ with (α,β)-characteristic function v and support function (α,β), if it satisfies the following conditions:

(i) for each imputation $a \notin L_\alpha$ there exists an imputation $b \in L_\alpha$ which α-dominates a.

(ii) no imputation of L_α α-dominates any other imputation of L_α.

Now we examine all the α-solutions for the essential constant-sum three person game.

For this game the possible support functions, except permutations and trivial differences, which satisfy the requirement of the previous theorem are the following:

α_1 defined by $\alpha_1(R) = \phi$ for all $R \subseteq N$.

α_2 defined by $\alpha_2(\{1,2\}) = \{1,2\}$, $\alpha_2(\{1,2,3\}) = \{1,2,3\}$ and $\alpha_2(R) = \phi$ otherwise .

α_3 defined by $\alpha_3(\{1,2\}) = \{1,2\}$, $\alpha_3(\{1,3\}) = \{1,3\}$, $\alpha_3(\{1,2,3\}) = \{1,2,3\}$ and $\alpha_3(R) = \phi$ otherwise.

Finally, α_4 defined by $\alpha_4(R) = R$ for R with more than one element.

THEOREM 4.1. *Let* v *be the superadditive function defined by*

$$v(\phi) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$$

Then the respective solutions are:

L_{α_1} = *the set of all the imputations*

L_{α_2} = *the set of imputations* $(a_1,a_2,0)$ *with* $a_1+a_2=1$

$L_{\alpha_3}^{f_2,f_3}$ = *the set of imputations* $(p,f_2(p),f_3(p))$ *with* f_2 *and* f_3 *monotonically, non-increasing , non-negative, continuous functions on the variable* $p \in [0,1]$ *and so that* $f_2(p)+f_3(p)=(1-p)$ *for each* $p \in [0,1]$.

L_{α_4} = *the von Neumann-Morgenstern solutions.*

Proof. Since $\alpha_1(R) = \phi$ for each coalition R, no imputation α_1-dominates any other imputation, so the first result is obvious.

Consider the support function α_2. Obviously, an imputation b α_2-dominates another imputation a if and only if

$$b_1 > a_1 \quad \text{and} \quad b_2 > a_2 \quad .$$

Therefore, the set of all imputations $(a_1,a_2,0)$ with $a_1+a_2 = 1$ satisfies the requirements of the external and internal stability.

Moreover, L_{α_2} is evidently the unique solution for the support function α_2.

Now, consider the support function α_3. In this case an imputation b α_3-dominates another imputation a if and only if

$$b_1 > a_1 \quad \text{and} \quad b_2 > a_2 \quad \text{or} \quad b_3 > a_3 \quad .$$

From this condition we see that each solution contains the imputation (1,0,0). We now demonstrate that for each solution L_{α_3} there exists an imputation

$$(p,a_2,a_3) \in L_{\alpha_3}$$

for each $p \in [0,1)$.

Given any $p \in [0,1)$, consider the imputation

$$a_1 = (p \; , \; \frac{1-p}{2} \; , \; \frac{1-p}{2}) \quad .$$

If $a_1 \in L_{\alpha_3}$, the assertion is valid, and if $a_1 \notin L_{\alpha_3}$, by the definition of the solution L_{α_3} there exists an imputation

$$a^2 = (p_2,a_{2,2},a_{3,2}) \in L_{\alpha_3}$$

which α_3-dominates the imputation a_1, i.e.,

$$p_2 > p \quad \text{and} \quad a_{2,2} > \frac{1-p}{2} \quad \text{or} \quad a_{3,2} > \frac{1-p}{2}$$

Obviously, we have $p_2 < \frac{1-p}{2}$.

This latter imputation determines the following set of imputations

$$L(a^2) = \{(p,b_2,b_3) : b_2 \geqslant a_{2,2} , b_3 \geqslant a_{3,2}\}$$

which is non-empty since $p_2 > p$.

Now, consider the imputation

$$a_3 = (p,a_{2,2}+ \frac{p_2-p}{2} ,a_{3,2}+ \frac{p_2-p}{2}) \in L(a^2) .$$

If $a_3 \in L_{\alpha_3}$, the assertion is satisfied, and if $a_3 \notin L_{\alpha_3}$, by the definition of the solution L_{α_3} there exists an imputation

$$a^4 = (p_4,a_{2,4},a_{3,4}) \in L_{\alpha_3}$$

so that α_3-dominates a_3, i.e.,

$$p_4 > p \quad \text{and} \quad a_{2,4} > a_{2,2}+ \frac{p_2-p}{2} \quad \text{or} \quad a_{3,4} > a_{3,2}+ \frac{p_2-p}{2} .$$

We will prove that $p_2 > p_4$. Suppose that $p_2 < p_4$, then a^4 α_3-dominates a^2 because

$$a_{2,4} > a_{2,2} \quad \text{or} \quad a_{3,4} > a_{3,2}$$

which is impossible. Suppose that $p_2 = p_4$, then because $a^2 \neq a^4$, the set of imputations

$$M = \{(q,b_2,b_3) : q>p_2,\min(a_{2,2},a_{2,4})<b_2<\max(a_{2,2},a_{2,4})$$
$$\text{and} \min(a_{3,2},a_{3,4})<b_3<\max(a_{3,2},a_{3,4})\}$$

is non-empty, and therefore there is an imputation $b \in M$.

This imputation cannot be a member of the solution L_{α_3}, since in such a case the imputation b would α_3-dominate both imputations a^2 and a^4, which is impossible. Therefore, there exists an imputation $c \in L_{\alpha_3}$ which α_3-dominates $b \in M$. Hence, the imputation c α_3-dominates at least one of the imputations a^2 and a^4, which is absurd.

Thus we have $p_2 > p_4$. Since a^2 and a^4 are members of the solution L_{α_3} we obtain

$$a_{2,4} > a_{2,2} \quad \text{and} \quad a_{3,4} > a_{3,2} \quad .$$

Evidently we have

$$p_4 < p_2 - \frac{p_2-p}{2} < \frac{p_2-p}{2} < \frac{1-3p}{4} \quad .$$

Again, the imputation a^4 determines the non-empty set $L(a^4)$. Consider the imputation

$$a_5 = (p, a_{2,4}+ \frac{p_4-p}{2}, a_{3,4}+ \frac{p_4-p}{2}) \; .$$

If $a_5 \in L_{\alpha_3}$, the assertion is satisfied, and in the contrary case $a_5 \notin L_{\alpha_3}$, by the definition of the solution L_{α_3} there exists an imputation

$$a^6 = (p_6, a_{2,6}, a_{3,6}) \in L_{\alpha_3}$$

which α_3-dominates to a_5, i.e.,

$$p_6 > p \quad \text{and} \quad a_{2,6} > a_{2,4}+ \frac{p_4-p}{2} \quad \text{or} \quad a_{3,6} > a_{3,4}+ \frac{p_4-p}{2} \quad .$$

By an analogous way as before, we have $p_4 > p_6$. Moreover,

$$a_{2,6} > a_{2,4} \quad \text{and} \quad a_{3,6} > a_{3,4}$$

since the imputations a^6 and a^4 are in L_{α_3}.

Again, we evidently have

$$p_6 < p_4 - \frac{p_4 - p_2}{2} < \frac{p_4 - p}{2} < \frac{1-7p}{8} \quad .$$

By repeating this procedure, we obtain a sequence of imputation

$$a^2 \ , \ a^4 \ , \ \ldots \ , \ a^{2n} \ , \ \ldots$$

of the form

$$a^{2n} = (p_{2n}, a_{2,2n}, a_{3,2n})$$

such that for all n = 1,...

$$a_{2,2n} > a_{2,2(n-1)}, a_{3,2n} > a_{3,2(n-1)}$$

and

$$p < p_{2n} < \frac{1-(2^n-1)p}{2^n} \quad .$$

by taking $n \to \infty$ we obtain from

$$p_{2n} \to p, a_{2,2n} \to a_2 \quad \text{and} \quad a_{3,3n} \to a_3$$

the imputation

$$a = (p, a_2, a_3)$$

which belongs to the set of imputations $\cap_{n=1}^{\infty} L(a^{2n})$.

The imputation a is a member of the solution L_{α_3}. Suppose that

$a \notin L_{\alpha_3}$, then there exists an imputation $b \in L_{\alpha_3}$ which α_3-dominates to a, i.e.,

$$b_1 > p \quad \text{and} \quad b_2 > a_2 \quad \text{or} \quad b_3 > a_3 \ .$$

Suppose that the first two equalities are true, then there exists an n_o such that for all $n > n_o$

$$b_1 > p_{2n} \quad \text{and} \quad b_2 > a_2 > a_{2,2n}$$

which is absurd because $a^{2n} \in L_{\alpha_3}$. In an analogous way , obtain a contradiction for the case in which the first and last inequalities hold.

Therefore, the imputation $a \in L_{\alpha_3}$, and with this result we have demonstrated for each solution the existence of an imputation

$$(p,a_2,a_3) \in L_{\alpha_3}$$

for each $p \in [0,1)$.

On the other hand, by an analogous argument to that used above for the case $p_2 = p_4$, one can easily see that there cannot exist two different imputations

$$(p,a_2,a_3) \quad \text{and} \quad (p,b_2,b_3)$$

for the same $p \in [0,1]$ for a given solution L_{α_3}.

Now, consider the imputations

$$(p,a_2,a_3) \quad \text{and} \quad (q,b_2,b_3)$$

in the solution L_{α_3} corresponding to $p,q \in [0,1]$ with $p > q$, then clearly $a_2 \leqslant b_2$ and $a_3 \leqslant b_3$. Hence L_{α_3} has the form

$$(p,f_2(p),f_3(p))$$

with f_2 and f_3 monotonically non-increasing, non-negative func -

tions in the variable $p \in [0,1]$ such that

$$f_2(p) + f_3(p) = (1-p)$$

for each $p \in [0,1]$.

Clearly such functions must be continuous. The solutions for the last support function, as is well known, correspond to the theory of von Neumann-Morgenstern. Q.E.D.

Define the core with respect to α as the set of all the imputations undominated ,in the sense of domination with respect to α.

It is very easy to see that the core with respect to α_i: C_{α_i} for the above example with i = 1,2,3,4 are the following sets:

$$C_{\alpha_1} = \Phi \ , \ C_{\alpha_2} = L_{\alpha_2} \ , \ C_{\alpha_3} = \{(1,C,0)\} \ , \ C_{\alpha_4} = \Phi \ .$$

In our case Shapley's conjecture would be formulated by: for each support function α the intersection of all the solutions with respect to α. It is interesting to note that in each case above Shapley's conjecture is satisfied.

Universidad de Cuyo
San Luis, Argentina

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