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BOUNDED GROUPS AND A THEOREM OF GELFAND

by G. Lumer*

Dedicated to Professor Alberto González Domínguez

A well-known theorem of Gelfand [G] asserts the following:

THEOREM. *Let* A *be a complex Banach algebra with unit* 1 *(of norm* 1*). Let* a *be invertible in* A *with* $\|a^n\| \leqslant$ constant *for* $n = 0, \pm 1, \pm 2, \ldots$ *, and spectrum equal to* $\{1\}$. *Then* $a = 1$.

In this paper, considering a more general situation and following a different approach, we obtain a quantitative result (theorem 1, estimate (1), below) which generalizes the preceding theorem.[(1)] Our approach involves so-callled hermitian elements of a general Banach algebra (see [L]), and the fact that for such an element the norm equals the spectral radius ([S]). In turn the latter statement about hermitians can be easily recovered if one assumes theorem 1 to be known, which shows in particular that estimate (1) is sharp in some reasonable sense.

Let us establish some notation. A shall henceforth always denote a complex Banach algebra with unit denoted by 1 (always of norm 1) If A is commutative, M(A) denotes the Gelfand space of A, the points of which are homomorphisms. For $a \in A$, sp(a) designates the spectrum of a, so that when A is commutative we have sp(a) = $= \{\phi(a) : \phi \in M(A)\}$. Given a set $S = A$, we denote by $|S|$ the $\sup \{\|s\| : s \in S\}$; this applies in particular to the case $A = C$,

(1) A new proof of Gelfand's original result was given recently by A. Browder in "States, numerical ranges, etc.", Proceedings of the Brown U. informal analysis seminar, Summer 1969, via an approach having some contact with what we do.

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C the complex field normed by $\|z\|$ = modulus of z for $z \in C$. If $F(\cdot)$ is a function defined on S, F(S) denotes the set $\{F(s):s\in S\}$; again this applies in particular to $A = C$. However, by slight abuse of notation A^{-1} shall denote {invertibles in A}. The expression "S is commutative" of course means that " $s's = ss'$ $\forall$ $s,s' \in S$". A path in A is the continuous image of a (compact) interval; and given a path Γ in A, we shall consider its length $\ell(\Gamma)$ defined in the obvious way via the sup of the lengths of inscribed polygonals (the latter computed from the norm). Finally, given a domain $G \subset C$, by "a determination of the logarithm in G", we mean a single-valued analytic function f defined in G and satisfying $\exp f(z) = z$ for all z in G. What we obtain is as follows:

1. THEOREM. *Let* G *be a bounded group contained in* A. *Suppose that for* $a \in G$ *there is a path* Γ *of finite length, which is commutative and joins* a *to* 1 *in* A^{-1}, *and that there is a determination of the logarithm defined on* $\cup_{g\in\Gamma}\, sp(g)$ *(i.e. defined on some domain containing* $\cup_{g\in\Gamma}\, sp(g)$*), vanishing at* 1. *Then*

$$\|a-1\| \leqslant |G|^2 e^{|\Gamma^{-1}|\ell(\Gamma)}\,|\log\, sp(a)| \tag{1}$$

where "log" denotes the above mentioned determination.

2. REMARK. That Gelfand's theorem follows from (1) above is immediate. In that case, a generates a bounded group G, Γ may be taken to be $\{t1 + (1-t)a : 0 \leqslant t \leqslant 1\}$, and using the usual principal determination of the logarithm for "log", $|\log sp(a)| = 0$, so that $a = 1$.

Proof of theorem 1. First of all notice that since we can always find a commutative subalgebra A_1 of A containing a, Γ, and Γ^{-1} and such that the spectrum of a relative to A_1 is the same as relative to A, (and we can replace G by the group generated by a), it is clear that we may assume without loss of generality that A is commutative.

It is well-known that under our hypothesis a has a logarithm, i.e. can be written in the form $a = e^b$; but here we shall need more precise quantitative information: we need to estimate the norm of the appropriate logarithm and know its spectrum. We proceed in several steps.

Notice that since Γ is compact, given $\delta > 0$ one can find on Γ points $x_o = 1, x_1, x_2, \ldots, x_n = a$ such that $\|x_{j+1} - x_j\| < \delta$ for $0 \leqslant j \leqslant n-1$; let us call such a collection of points a δ-partition of Γ. It follows from the compactness of Γ and the upper-semi-continuity of spectra (see [H-Ph], p. 167) that $E = \cup_{g \in \Gamma} \mathrm{sp}(g)$ is compact, and hence for any $\varepsilon > 0$ there exists a $\delta_1(\varepsilon)$ such that $|\log z - \log w| < \varepsilon$ for $|z-w| < \delta_1(\varepsilon)$ and $z, w \in E$, where "log" denotes the determination of the logarithm mentioned in the hypothesis. We consider now a δ-partition of Γ, $x_o = 1, x_1, \ldots, x_n = a$, with a $\delta > 0$ satisfying (for reasons that shall become clear later) the following:

$$\delta < \text{minimum of} \left\{ \frac{\pi}{|\Gamma^{-1}|(1+\pi)}, \ \delta_1(\pi) \right\}. \tag{2}$$

For $x \in A$, $\|x\| < 1$, write $\mathrm{Log}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ (use also the same notation for $z \in C$, $|z| < 1$). Define

$$\begin{aligned} y_o &= 0 \\ y_{j+1} &= y_j + \mathrm{Log}(1 + x_j^{-1}(x_{j+1} - x_j)) \qquad 0 \leqslant j \leqslant n-1 \\ y_n &= b \end{aligned} \tag{3}$$

This makes sense because of (2), and $e^b = a$. (Verify by finite induction that $e^{y_j} = x_j$; the induction step is:
$e^{y_{j+1}} = x_j(1 + x_j^{-1}(x_{j+1} - x_j)) = x_{j+1}$). We have

$$\begin{aligned} \|y_{j+1} - y_j\| &= \|\mathrm{Log}(1 + x_j^{-1}(x_{j+1} - x_j))\| \leqslant \textstyle\sum_{k=1}^{\infty} \dfrac{\|x_j^{-1}(x_{j+1} - x_j)\|^k}{k} \\ &= \mathrm{Log} \dfrac{1}{1 - \|x_j^{-1}(x_{j+1} - x_j)\|} \leqslant \dfrac{\|x_j^{-1}(x_{j+1} - x_j)\|}{1 - \|x_j^{-1}(x_{j+1} - x_j)\|} \leqslant \dfrac{|\Gamma^{-1}| \ \|x_{j+1} - x_j\|}{1 - \delta|\Gamma^{-1}|} \end{aligned} \tag{4}$$

Hence we have

$$(5) \qquad \|b\| \leq \frac{|\Gamma^{-1}|}{1-\delta|\Gamma^{-1}|}\sum_{j=0}^{n-1}\|x_{j+1}-x_j\| \leq \frac{|\Gamma^{-1}|\ell(\Gamma)}{1-\delta|\Gamma^{-1}|}$$

If we knew that b does not depend on the δ-partition used, for small $\delta > 0$, we would get from (5) the estimate we really want, $\|b\| \leq |\Gamma^{-1}|\ell(\Gamma)$, by letting $\delta \to 0$. The latter estimate is indeed correct (as follows easily from what we show later), but we shall refine our estimate later on, and for the moment (5) will do.

Next we show that

$$(6) \qquad sp(b) = \log sp(a)$$

For this purpose we proceed by finite induction to show that for $0 \leq j \leq n$, we have $\forall\ \phi \in M(A)$, $\phi(y_j) = \log \phi(x_j)$. The latter is obvious for $j = 0$; assume it is true for a $j < n$, then for any $\phi \in M(A)$, we have

$$\phi(y_{j+1}) = \phi(y_j) + \mathrm{Log}(1+(\phi(x_j))^{-1}(\phi(x_{j+1}) - \phi(x_j)))$$

$$= \log \phi(x_j) + \mathrm{Log}(\ldots) = \log \phi(x_{j+1}) + 2k\pi i$$

where k is an integer. This together with $\phi(y_j) = \log \phi(x_j)$ yields,

$$|2\pi k| \leq |\phi(y_{j+1}-y_j)| + |\log \phi(x_{j+1}) - \log \phi(x_j)|$$

$$(7) \qquad < \frac{\delta|\Gamma^{-1}|}{1-\delta|\Gamma^{-1}|} + \pi < 2\pi$$

since $|\phi(x_{j+1})-\phi(x_j)| \leq \|x_{j+1}-x_j\| < \delta_1(\pi)$, by (2), and since also by (2) the other summand is $< \pi$. Hence $k = 0$, and the induction step is proved. Hence $\phi(b) = \log \phi(a)\ \ \forall\ \phi \in M(A)$ which is what we wanted to show.

Now for any t real, write $t = k+\tau$ where k is an integer ,

$0 \leqslant |\tau| \leqslant \frac{1}{2}$, then

$$\|e^{tb}\| \leqslant \|e^{kb}\| \ \|e^{\tau b}\| = \|a^k\| \ \|e^{\tau b}\| \leqslant |G| \ e^{\frac{1}{2}\|b\|}$$

(8)

$$\leqslant |G| \ e^{|\Gamma^{-1}|\ell(\Gamma)/2(1-\delta|\Gamma^{-1}|)}$$

Next we want to make use of what is known about hermitian elements in general Banach algebras. For this purpose let us first simplify notation by calling the last quantity in (8) K , i.e. writing $\|e^{tb}\| \leqslant K$ for all t real: The bounded group $\{e^{tb}\}$ induces a renorming on A considered as a Banach space, by defining the new norm $\| \ \|'$ as follows:

$$\forall \ x \in A \ , \ \|x\|' = \sup \{\|e^{tb}x\| \ : \ -\infty < t < +\infty\} \quad .$$

Clearly one has $K^{-1}\|x\| \leqslant \|x\|' \leqslant K\|x\|$. Call the renormed space A'. Now renorm A as an algebra by giving to $x \in A$, the operator norm as multiplication operator on A'. Call that norm $\| \ \|''$ and the renormed algebra A''. We have

(9) $$\frac{1}{K^2}\|x\| \leqslant \|x\|'' \leqslant K^2\|x\| \quad \forall \ x \in A$$

and spectra are unaffected by this equivalent renorming of A. For any t real, $\|e^{tb}x\|' = \sup\limits_{-\infty<s<+\infty} \|e^{(t+s)b}x\| = \|x\|'$, so that $\|e^{tb}\|''=1$. The latter implies that ib is hermitian as an element of A'' (see [L-Ph], [L] lemma 12, or [B-D]), and consequently by Sinclair's result [S], and (6), $\|b\|'' = |sp(b)| = |\log sp(a)|$.

On the other hand

$$\|a-1\|'' = \left\|\int_0^1 be^{sb}ds\right\|'' \leqslant \|b\|'' \int_0^1 \|e^{sb}\|'' \ ds = \|b\|''$$

hence by (9), and what precedes,

$$\|a-1\| \leqslant K^2|\log sp(a)| = |G|^2 \ e^{|\Gamma^{-1}|\ell(\Gamma)/1-\delta|\Gamma^{-1}|}|\log sp(a)| \ .$$

Finally, notice that the previous inequality holds for any $\delta > 0$ satisfying (2), hence by letting $\delta \to 0$ in that inequality, we obtain inequality (1) and the proof comes to an end.(2)

3. REMARK. The fact that for any hermitian element h of a Banach algebra A,

$$(10) \qquad \|h\| = |sp(h)|$$

(which was used in the proof of (1)) can in turn be recovered in its general form from the previous theorem.

Proof of remark 3. Given h hermitian in A, let $G = \{e^{ith}: -\infty < t < +\infty\}$, then G is a bounded group with $|G| = 1$. Let $a = e^{ih}$ and let $\Gamma = \{e^{ith}: 0 \leqslant t \leqslant 1\}$. If $\|h\|$ is small enough we can use Log z as our determination of the logarithm and $|\log sp(a)| = |sp(h)|$. Since $\|e^{it_2h} - e^{it_1h}\| = \|\int_{t_1}^{t_2} he^{ish}ds\| \leqslant |t_2 - t_1|\|h\|$, we have $\ell(\Gamma) \leqslant \|h\|$. We apply now these observations replacing the given h by th, $0 < t$ small, and using estimate (1) we obtain

$$\|e^{ith} - 1\| \leqslant te^{t\|h\|}|sp(h)|$$

Hence,

$$\|h\| = \lim_{t\to 0} \left\|\frac{e^{ith}-1}{t}\right\| \leqslant |sp(h)| \leqslant \|h\|$$

which is what we wanted to prove.

(2) For our proof of theorem 1 we do not have to know that b is independent of the δ-partition used (δ small enough). On the other hand it is easy to check the independence of b, using what we establish during the proof of theorem 1. Indeed, assume we have two partitions for δ, δ' both satisfying inequality (2), and giving rise to b and b'. From our proof we see that $\forall\phi \in M(A), \phi(b') = \log\phi(a) = \phi(b)$, so $sp(b-b') = \{0\}$.

But $e^{b-b'} = 1$, so that with the notations of our proof, $i(b-b')$ is hermitian in the corresponding A" and hence $= 0$, i.e. $b' = b$.

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University of Washington.
Seattle, Washington.

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