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AN EXTENSION OF GRONWALL'S INEQUALITY

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1. INTRODUCTION.

The purpose of this paper is to extend Gronwall's inequality (see [1]) to the case where the function depends on n variables. There are some results in that direction. See, for example, [2], [3], [4], [5]. [3] is a refinement of the results in [2], following the same line of reasoning. [4], [5] deal with the case n = 2, and we are only aware of the final results. Our method of proof was suggested by [6], and has the advantage over [3] in that it provides a much easier computation of the bounds involved, while rendering estimates of the same order of magnitude.

Our interest in the problem arose from fruitful conversations with Prof. J.B. Díaz, which we gratefully acknowledge.

2. NOTATION.

Given n non-negative numbers (a_i) $(i = 1,2,\dots,n)$, let $\sigma_i(a_1,a_2,\dots,a_n)$ $(i = 1,2,\dots,n)$ be the elementary symmetric function of order i, i.e. $\sigma_1 = \sum_{i=1}^n a_i$, $\sigma_2 = \sum_{\substack{i,j=1\\ i\neq j}}^n a_i a_j \dots$ $\sigma_n = \prod_{i=1}^n a_i$.

3. PRELIMINARY RESULTS.

LEMMA 3.1. *Let* $f(x)$ *be a non negative continuous real-valued function defined in* $[0,A]$. *Furthermore, let the continuously differentiable functions* $a(x)$, $b(x)$ *be defined in* $[0,A]$, *satisfying:* $a(x) \geq 0$, $a(0) > 0$, $b(x) > 0$, $(a(x)/b(x))' \leq 0$ *and let* $h(x)$ *be continuous and non negative in* $[0,A]$. *Then, if:*

$$0 \leq f(x) \leq a(x) + b(x) \int_0^x f(s)h(s)ds \quad , \text{ we have:}$$

$$f(x) \leqslant a(0)(b(0))^{-1} b(x) \exp\left(\int_0^x b(s)h(s)ds\right)$$

REMARK 3.1. We note that $a(x) \equiv a > 0$, $b(x) \equiv b > 0$, $h(x) \equiv 1$, satisfy the requirements of the Lemma. They provide, precisely , Gronwall's known result.

REMARK 3.2. Although in [7] there are estimates for the case $b(x) \equiv 1$, giving more information about the way in which $a(x)$ affects the bound, it is our result that will be useful in proving our Theorem.

Proof of the lemma. Let $g(x) = a(x) + b(x) \int_0^x f(s)h(s)ds$. Hence:

$$g'(x) = \left[\frac{b'(x)}{b(x)} + b(x)h(x)\right] g(x) + (a(x)/b(x))' b(x)$$

$$\leqslant \left[\frac{b'(x)}{b(x)} + b(x)h(x)\right] g(x)$$

Integration from 0 to x (we observe that $g(0) = a(0) > 0$) yields the final result.

LEMMA 3.2. *With the notations previously introduced,*

$$\max_{x_1+\ldots+x_n=r} \sigma_i(x_1,x_2,\ldots,x_n) = \sigma_i\left(\frac{r}{n},\frac{r}{n},\ldots,\frac{r}{n}\right) = \binom{n}{i}\left(\frac{r}{n}\right)^i$$

Proof. Let $f_n^i(r)$ be the first member of the equality. An application of dynamic programming techniques (see [8]) gives:

$$f_n^i(r) = \max_{0 \leq x_n \leq r} [x_n f_{n-1}^{i-1}(r-x_n) + f_{n-1}^i(r-x_n)] \quad , \quad i = 0,1,\ldots,n \ .$$

Furthermore, the substitution $x_j = ry_j$ $(1 \leqslant j \leqslant n)$ yields $f_n^i(r) = r^i f_n^i(1)$. The case $n = 2$ is obvious. The rest of the Proof follows by induction.

4. THEOREM. *Let* $f(x_1,x_2,\dots,x_n)$ *be continuous in* $\prod_{i=1}^{n}[0,a_i]$, *and such that* $0 \leqslant f(x_1,\dots,x_n) \leqslant A + B\sum_{i=1}^{n}\binom{n}{i} I_i$, *where*

$$I_h = \int_0^{x_{j_1}} \dots \int_0^{x_{j_h}} f(x_1,\dots,s_{j_1},\dots,s_{j_h},\dots,x_n)ds_{j_1}\dots ds_{j_h} .$$

Clearly, there are $\binom{n}{h}$ *such possible arrangements.*

Then: $f(x_1,\dots,x_n) \leqslant \frac{A}{n}\sum_{i=1}^{n}\binom{n}{i}^2 \frac{n^{1-i}}{n-i+1}(x_1+\dots+x_n)^{i-1}g(x_1,\dots,x_n)$

with $g(x_1,\dots,x_n) = \exp\left(B\sum_{i=1}^{n}\binom{n}{i}^2 \frac{n^{1-i}}{n-i+1}(x_1+\dots+x_n)\right)$

REMARK 4.1. The case n = 1 provides precisely the original Gronwall inequality.

Proof of the theorem. The trick consists in transforming the multidimensional problem into a one-dimensional one, and use Lemma 3.1. For that purpose, we define:

$$\phi(r) = \max_{0\le x_1+\dots+x_n\le r} f(x_1,\dots,x_n) .$$

Clearly, $\phi(r) \geqslant 0$, and it does not decrease with increasing r . Furthermore: $f(x_1,\dots,x_n) \leqslant \phi(x_1+\dots+x_n)$.

Take a typical element I_i, i.e. an integral of order i. Then

$$I_i \leqslant \int_0^{x_{j_1}} \dots \int_0^{x_{j_i}} \phi(x_1+\dots+s_{j_1}+\dots+s_{j_i}+\dots+x_n)ds_{j_1}\dots ds_{j_i}$$

By replacing (i-1) s_j's at a time by the corresponding x_j's we certainly don't diminish the value of the integral. Moreover, we thus reduce it to a one-dimensional one, multiplied by products of i-1 elements of the set $(x_{j_1},\dots,x_{j_i})$. Hence, I_i will be majorized by i integrals of order one, of the form

$$J=\frac{1}{i}\, x_{j_1}\cdots x_{j_{h-1}} x_{j_{h+1}}\cdots x_{j_i}\int_0^{x_{j_h}}\phi(x_1+\dots+x_{j_{h-1}}+s_{j_h}+x_{j_{h+1}}\dots+x_n)\,ds_{j_h}$$

Therefore, for this particular I_i, we have the upper bound

$$I \leqslant \frac{1}{i}\,\sigma_{i-1}(x_{j_1},\dots,x_{j_i}) \int_0^{\sum_{h=1}^{i} x_{j_h}} \phi(t)dt$$

$$\leqslant \frac{1}{i}\,\sigma_{i-1}(x_1,\dots,x_n) \int_0^{\sum_{i=1}^{n} x_i} \phi(t)dt$$

In this form we obtain a uniform upper bound for all integrals of the same order. Hence:

$$0 \leqslant \phi(r) \leqslant A + B \sum_{i=1}^{n} \binom{n}{i} \max_{0\leq x\leq r} \frac{1}{i}\,\sigma_{i-1}(x_1,\dots,x_n) \int_0^{x} \phi(t)dt$$

$$\leqslant A + B \sum_{i=1}^{n} \binom{n}{i} \frac{1}{i} \binom{n}{i-1} \left(\frac{r}{n}\right)^{i-1} \int_0^{r} \phi(t)dt \quad ,$$

where we used Lemma 3.2.

Finally, Lemma 3.1 yields the assertion of the Theorem.

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