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ON THE DIFFERENTIATION OF INTEGRALS

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§0. INTRODUCTION.

In this paper we present some results and open problems in the theory of differentiation. Of the results, some are new and others are known. The latter ones are presented here either because we felt we could offer new or more illuminating proofs of them or because of their relationship with open problems which seem to be of interest.

The first section presents some aspects of the relationship between the Hardy-Littlewood maximal operator and the differentiation properties of a basis. Here we also give a characterization of the space $L(1+\log^+ L)\ (\mathbb{R}^n)$ in terms of an integrability condition of the maximal function. The second section shows the intimate connection of differentiation with various types of covering lemmas. The third section deals with some special types of differentiation bases. In the course of our discussion, when it seems appropriate, we mention some open questions which appear to be of interest either for the new methods one should try to apply in order to answer them or for the possible roles solutions to such problems would have in other areas.

We have sacrificed generality in hopes of achieving clarity of exposition. Most of the time our setting will be $\mathbb{R}^n$ (or $\mathbb{R}^2$) with Lebesgue measure. If $A \subset \mathbb{R}^n$ is measurable, $|A|$ will denote its Lebesgue measure. In $\mathbb{R}^n$, we will call a family of non-empty open bounded sets, $\mathcal{R}$, a *differentiation basis* if for each $x \in \mathbb{R}^n$ there is a sequence of sets $\{R_k\} \subset \mathcal{R}$ which *contract to* x, i.e. given any neighborhood 0 of x, all sets R_k, from some subindex k_o on ,

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are inside **O**.

Given a function $f \in L_{loc}(\mathbb{R}^n)$, we define the *upper and lower derivatives of* $\int f$ with respect to $\mathcal{R}$ at the point $x \in \mathbb{R}^n$ by

$$\overline{D}(\int f, x) = \sup \limsup_{k \to \infty} (1/|R_k|) \int_{R_k} f$$

and

$$\underline{D}(\int f, x) = \inf \liminf_{k \to \infty} (1/|R_k|) \int_{R_k} f$$

where the supremum and the infimum are taken over all the sequences $\{R_k\} \subset \mathcal{R}$ contracting to x.

We say that $\mathcal{R}$ differentiates $\int f$ whenever $\overline{D}(\int f, x) = \underline{D}(\int f, x)$ for almost every $x \in \mathbb{R}^n$. This common value is then called *the derivative of* $\int f$ (with respect to Lebesgue measure and) *with respect to* $\mathcal{R}$ *at* x. The main problem of the theory of differentiation is to find out for which classes of functions and for which differentiation bases $\mathcal{R}$, differentiation of $\int f$ holds. The theory, of course, begins with Lebesgue differentiation theorem which states that if $\mathcal{R}$ is the system of all balls in $\mathbb{R}^n$ and $f \in L_{loc}(\mathbb{R}^n)$, then $\mathcal{R}$ differentiates $\int f$ to f(x) at almost every $x \in \mathbb{R}^n$.

For a detailed account and extensive bibliography on the theory of differentiation of integrals one can consult [1] and [6].

1. DIFFERENTIATION AND THE MAXIMAL OPERATOR.

Let $\mathcal{R}$ be a differentiation basis in $\mathbb{R}^n$ and $f \in L_{loc}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ we consider

$$Mf(x) = \sup (1/|R|) \int_R |f(y)|dy$$

where the supremum is taken over the R's such that $x \in R \in \mathcal{R}$. The function Mf is called *the Hardy-Littlewood maximal function of* f relative to $\mathcal{R}$. The operator M: f $\to$ Mf is called *the Hardy-Littlewood maximal operator* with respect to $\mathcal{R}$. Since $\{x : Mf(x) > \lambda\}$ is open, Mf is a measurable function.

The maximal operator is very closely connected with the differentiation properties of a basis, as the following theorem shows. This theorem was first stated and proved in a slightly weaker form and in terms of a so-called halo condition by Busemann and Feller [2]. It can also be found in this form in [6]. The technique of the proof we present here of the second part belongs to Calderón (cf. [15] vol. II, p. 165). This technique has been used and extended in a more general setting by Stein [12]. It is easy to see that also some other criteria of [2] can be concisely stated in terms of the maximal operator.

In order to state the next theorem, we need some terminology. We say that a differentiation basis $\mathcal{R}$ is *homothecy invariant* if $R \in \mathcal{R}$ implies that any set homothetic to R is also in $\mathcal{R}$. A sublinear operator A defined on $L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$, with values in the set of measurable functions $M(\mathbb{R}^n)$ is said to be of *weak type* (p,q), whenever Af, for $f \in L^p$, satisfies, for any $\lambda > 0$,

$$|\{x : |Af(x)| > \lambda\}| \leqslant \left(\frac{c}{\lambda} \|f\|_p\right)^q$$

where $c > 0$ is independent of f and λ. This is a condition weaker than continuity of A from L^p to L^q. This last situation is expressed by saying that A is *of strong type* (p,q).

1.1. THEOREM. *Let* $\mathcal{R}$ *be a differentiation basis in* $\mathbb{R}^n$ *which is homothecy invariant. Then the two following conditions are equivalent:*

(a) *The maximal operator* M *with respect to* $\mathcal{R}$ *is of weak type* (1,1)

(b) $\mathcal{R}$ *differentiates* $\int f$ *for every* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $D(\int f, x) = f(x)$ *for almost every* $x \in \mathbb{R}^n$

Proof. (a) $\Rightarrow$ (b). Since differentiation is obviously a local property we can shall assume $f \in L^1(\mathbb{R}^n)$ and, changing f, if necessary, on a set of measure zero, $|f(x)| < \infty$ for every x. We then have

$$E = \{x : D(\int f,x) \text{ does not exist or } D(\int f,x) \neq f(x)\} =$$

$$= \{x : |\overline{D}(\int f,x) - f(x)| > 0\} \cup \{x : |\underline{D}(\int f,x) - f(x)| > 0\} = A \cup B$$

We shall prove that for any $a > 0$

$$|A_a| = |\{x : |\overline{D}(\int f,x) = f(x)| > a\}| = 0$$

Hence $|A| = 0$. Similarly $|B| = 0$ and so $|E| = 0$. (Questions of measurability for the sets we are considering are pretty obvious in view of the character of the differentiation basis we consider). Given $\varepsilon > 0$, we can take $f = g+h$, $g \in C(\mathbb{R}^n), h \in L^1(\mathbb{R}^n)$, $\|h\|_1 \leqslant \varepsilon$. Obviously $D(\int g,x) = g(x)$ at every x, and so

$$|A_a| = |\{x : |\overline{D}(\int f,x)-f(x)| > a\}| = |\{x : |\overline{D}(\int h,x)-h(x)| > a\}|$$

$$\leqslant |\{x : Mh(x) > \frac{a}{2}\}| + |\{x : h(x) > \frac{a}{2}\}| \leqslant \frac{2c+1}{a}\varepsilon$$

Since ε can be taken arbitrarily small, $|A_a| = 0$. Hence (a) $\Rightarrow$ (b) is proved.

(b) $\Rightarrow$ (a). As we show later (lemma 1.3), (a) is equivalent to :
(a') *The maximal operator* M *is of weak type* (1,1) *on functions which are in* K, *the set of finite linear combinations of characteristic functions of bounded measurable pairwise disjoint sets* (i.e. for any such function f and for $\lambda > 0$, we have

$$|\{x : Mf(x) > \lambda\}| \leqslant \frac{c}{\lambda} \|f\|_1) .$$

It is therefore sufficient to prove (b) $\Rightarrow$ (a'). Assume that (a') is not true. Then there is a sequence $\{f_k^*\} \subset K$, $f_k^* \geqslant 0$, $\|f_k^*\|_1 = 1$, and numbers $\lambda_k \geqslant 0$, such that

$$|E_k^*| = |\{x : Mf_k^*(x) > \lambda_k\}| > 2^k/\lambda_k$$

If we define, for $H > 0$,

$$M_H f_k^*(x) = \sup(1/|R|) \int_R f_k^*(y)dy$$

where the supremum is taken for $R \in \mathcal{R}$, $x \in R$, and the diameter $\delta(R)$ of R is $< H$, then we can take H_K sufficiently big so that

$$|E_k^*| = |\{x : M_{H_k} f_k^*(x) > \lambda_k\}| > 2^k/\lambda_k$$

If we now consider $f_k(x) = f_k^*(x/\mu_k)$, $\mu_k > 0$, then $\|f_k\|_1 = \mu_k^n$; and if we set $E_k = \mu_k E_k^*$,

then $|E_k| = \mu_k^n |E_k^*| = |\{\mu_k x : M_{H_k} f_k^*(x) > \lambda_k\}| =$

$$= |\{y : M_{H_k} f_k^*(y/\mu_k) > \lambda_k\}| = |\{y : M_{\mu_k H_k} f_k(y) > \lambda_1\}| \geqslant$$

$$\geqslant \mu_k^n 2^k/\lambda_k = 2^k \|f_k\|_1/\lambda_k$$

Now E_k^* is clearly bounded. Hence we can select μ_k small enough so that E_k is contained in the ball $B(0,1/4)$ of center 0 and radius 1/4, and furthermore $\delta(E_k) \to 0$. We can also choose a positive integer r_k so that $2 \geqslant r_k |E_k| \geqslant 1$. Then $\sum_1^\infty r_k |E_k| = \infty$.

We shall now use the following lemma, which is a straight-forward generalization of a result of Calderón, and which can be seen in [15], vol. II, p. 165:

1.2. LEMMA. *Let* $\{A_k\}$ *be a sequence of measurable sets contained in* $B(0,1/4)$ *so that* $\sum |A_k| = \infty$. *Then it is possible to select a sequence of points* $\{x_k\}$ *of* $B(0,1)$ *so that every point of a set of positive measure is in infinitely many of the sets* $A_k(x_k) = x_k + A_k$.

To use this lemma we consider E_k repeated r_k times. Then, since $\sum r_k |E_k| = \infty$ we can apply the lemma. We translate E_k to obtain $E_k(x_1)$, $E_k(x_2)$,..., $E_k(x_{r_k})$ and simultaneously the functions f_k, obtaining the functions f_k^1 , f_k^2 ,..., $f_k^{r_k}$. We then set

$f = \sum_{k=1}^\infty \sum_{i=1}^{r_k} \alpha_k^i f_k^i$ where $\alpha_k^i = \alpha_k > 0$ are to be chosen in a moment. We have $\| f \|_1 \leq \sum \alpha_k r_k \| f_k \|_1$ and since $r_k |E_k| \leq 2$ and $|E_k| > 2^k \| f_k \|_1 / \lambda_k$, we get $\| f \|_1 \leq \sum (2\alpha_k \lambda_k / 2^k)$.

On the other hand, there is a set of positive measure E such that every $x \in E$ belongs to infinitely many $E_k(x_i)$'s, and so there is, for such an x, a sequence $\{R_h\} \subset \mathcal{R}$, $x \in R_h$, contracting to x such that

$$(1/|R_h|) \int_{R_h} f(y)dy \geq (\alpha_h^i / |R_h|) \int_{R_h} f_h^i(y)dy \geq \alpha_k \lambda_k$$

We can choose α_k so that $\sum (2\alpha_k \lambda_k / 2^k) < \infty$ and $\alpha_k \lambda_k \to \infty$. It is enough to take $\alpha_k = 2^{k/2} / \lambda_k$. So we get $f \in L^1$ and for $x \in E$, $\overline{D}(\int f, x) = \infty$, which contradicts (b). So the theorem is proved.

We finally prove the lemma we have stated at the beginning of this proof. It can be seen in [4] in a more general form.

1.3. LEMMA. *The two following conditions are equivalent:*

(a) M *is of weak type* (1,1).

(a') M *is of weak type* (1,1) *on functions in* K.

Proof. We only have to prove (a') ⇒ (a). Let $f \in L^1$, $f \geq 0$. We can take $g_k \in K$, $g_k \geq 0$, $g_k \to f(L^1)$. For any x, $x \in R \in \mathcal{R}$ we have

$$(1/|R|) \int_R f(y)dy = \lim_{k \to \infty} (1/|R|) \int_R g_k(y)dy$$

We denote $E = \{x : Mf(x) > \lambda\}$, $E_k = \{x : Mg_k(x) > \lambda\}$. If $x \in E$ then, for some $R \in \mathcal{R}$, $x \in R$, we have

$$(1/|R|) \int_R f(y)dy > \lambda \quad \text{and so , from some } k_o \text{ on}$$

$$(1/|R|) \int_R g_k(y)dy > \lambda \ .$$

Hence E lim sup E_k and $|E| \leq \limsup |E_k|$, $k \to \infty$. Thus $|E| \leq \limsup c \| g_k \|_1 / \lambda = c \| f \|_1 / \lambda$.

There is an interesting question related to theorem 1.1:

1.4. PROBLEM. *Can one substitute weak type* (1,1) *in theorem* 1.1 *for weak type* (p',p), $1 < p < \infty$, *and simultaneously* L^1 *for* L^p *?*

One could think of applying the techniques of [3] or [12].

Also in relationship with theorem 1.1 and problem 1.4 one can formulate another interesting question. One knows that the space $L \log^+ L(\mathbb{R}^2)$ of measurable functions f in $\mathbb{R}^2$ such that $\int |f| \log^+|f| < \infty$ where $\log^+ a = \log a$ if $a \geq 1$, $\log^+ a = 0$ if $0 < a < 1$, behaves in a special way in the theory of differentiation (cf. for example [7]).

1.5. PROBLEM. *Is there any simple relationship between some property similar to the one in* 1.1, 1.4, *for the maximal operator and the fact that* R *differentiates* $\int f$ *for* $f \in L \log^+ L$?

The following theorem, which is an extension of a result of Stein [13] gives a characterization of the space

$$L(1+\log^+ L)(\mathbb{R}^n) = \{f \in M(\mathbb{R}^n) : \int |f|(1+\log^+|f|) < \infty\}$$

in terms of an integrability condition on the classical Hardy-Littlewood maximal function.

1.6. DEFINITION. For $x \in \mathbb{R}^n$ we consider the system of open balls centered at x, $B(x,r)$, $r > 0$. Define, for $f \in L_{loc}(\mathbb{R}^n)$

$$Mf(x) = \sup_{r>0} (1/|B(x,r)|) \int_{B(x,r)} |f(y)|dy$$

Mf is then called the (classical) *Hardy-Littlewood maximal function*. It is easy to observe that Mf is measurable.

1.7. THEOREM. *Let* $g \in L(\mathbb{R}^n)$. *Then the following two conditions are equivalent:*

(a) $\int_{Mg(x)>1} Mg(x)dx < \infty$

(b) $g \in L \log^+ L$

Proof. We can assume, without loss of generality, $g \geqslant 0$. Define, for $\lambda > 0$, $\omega(Mg,\lambda) = |\{x : Mg(x) > \lambda\}|$. We shall use the following inequalities (cf. [13]).

$$(*) \qquad \frac{c_1}{\lambda}\int_{g(x)>c_3\lambda} g(x)dx \leqslant \omega(Mg,\lambda) \leqslant \frac{c_2}{\lambda}\int_{g(x)>\frac{\lambda}{2}} g(x)dx$$

where $c_1, c_2, c_3 > 0$ are independent of g,λ. Assume first that (a) holds. We can write

$$\int_{Mg(x)>1} Mg(x)dx = [-\lambda\omega(Mg,\lambda)]_1^\infty + \int_1^\infty \omega(Mg,\lambda)d\lambda$$

By the above inequalities $\omega(Mg,1) < \infty$ and $\lambda\omega(Mg,\lambda) \to 0$ as $\lambda \to \infty$. Hence

$$\infty > \int_1^\infty \omega(Mg,\lambda)d\lambda \geqslant c_1 \int_1^\infty \frac{1}{\lambda}\int_{g(x)>c_3\lambda} g(x)dxd\lambda =$$

$$= c_1 \int_{g(x)>c_3} g(x) \int_1^{(1/c_3)g(x)} \frac{1}{\lambda}\, d\lambda dx \;.$$

Since $g \in L$, we see that $g \in L\log^+L$ also and (b) is proved. In a similar fashion, if (b) is true,

$$\int_{Mg(x)>1} Mg(x)dx = \omega(Mg,1) + \int_1^\infty \omega(Mg,\lambda)d\lambda$$

and we have $\omega(Mg,1) < \infty$, $\int_1^\infty \omega(Mg,\lambda)d\lambda \leqslant$

$$\leqslant c_2 \int_1^\infty \frac{1}{\lambda}\int_{g(x)>\frac{\lambda}{2}} g(x)dxd\lambda = c_2 \int_{2g(x)>1} g(x)\int_1^{2g(x)} \frac{1}{\lambda}\, d\lambda dx =$$

$$= c_2 \int g(x) \log^+(2g(x))dx < \infty \text{ , which proves (a) .}$$

Stein's theorem results now as a corollary of 1.7.

1.8. COROLLARY. *Let* $f \in L_o^1(\mathbb{R}^n)$, $f > 0$. *Then, if* M *is defined* *as in* 1.6, *we have that* $f \in L \log^+ L$ *if and only if* $\int_A Mf(x)dx < \infty$, *where* A *denotes the support of* f.

Proof. We have $\int_A Mf(x)dx = \int_{A\cap B} Mf(x)\,dx + \int_{A\cap B'} Mf(x)dx$ where $B = \{x : Mf(x) > 1\}$ and B' is the complement of B.

If $f \in L \log^+ L$, the first integral is bounded by the theorem. The second is obviously bounded, since A is compact.

Assume now $\int_A Mf(x)dx < \infty$. Then, as Stein shows, $Mf \in L_{loc}(\mathbb{R}^n)$. Furthermore, if $f \in L_o(\mathbb{R}^n)$, then B is bounded. So $\int_B Mf(x)dx < \infty$ and by the theorem $f \in L \log^+ L$. Q.E.D.

Another simple corollary of the theorem is the following:

1.9. COROLLARY. *Let* $f \in L^1(\mathbb{R}^n)$ *and* M *defined as in* 1.6. *Then, for every* $A \subset \mathbb{R}^n$, *measurable*, $|A| < \infty$, *one has*

$$\int_A Mf(x)dx \leqslant c_1^*(|A| + \int_A |f(x)|(1+\log^+|f(x)|)dx)$$

and also

$$\int Mf(x)dx \geqslant c_2^* \int |f(x)|(1+\log^+|f(x)|)dx \quad .$$

Proof. The first inequality is well known and results very easily from the theorem:

$$\int_A Mf(x)dx = \int_{A\ \{Mf(x)>1\}} Mf(x)dx + \int_{A\ \{Mf(x)\leq 1\}} Mf(x)dx \leqslant$$

$$\leqslant c_1^*(\int |f(x)|(1+\log^+|f(x)|)dx + |A|)$$

As for the second, we have, recalling (*) in the proof of theorem 1.7. (note that c_3 can be taken $\geqslant 1$)

$$\int Mf(x)dx \geqslant \int_{Mf(x)>1} Mf(x)dx = \omega(Mf,1) + \int_1^\infty \omega(Mf,\lambda)d\lambda \geqslant$$

$$\geq c_1 \int_{|f(x)|<c_3} |f(x)|dx + c_1 \int_1^\infty \frac{1}{\lambda} \int_{|f(x)|>c_3\lambda} |f(x)|dxd\lambda =$$

$$= c_1 \int_{|f(x)|>c_3} |f(x)|dx + c_1 \int_{|f(x)|>c_3} |f(x)|\log(\frac{|f(x)|}{c_3})\, dx.$$

Since we also have $\int Mf(x)dx \geq \int |f(x)|dx$ one easily obtains

$$\int Mf(x)dx \geq c_2^* \int |f(x)|(1+\log^+|f(x)|)dx \quad . \qquad \text{Q.E.D.}$$

If Q is the unit cube this corollary implies that $Mf \in L^1(Q)$ if and only if $f \in L(1+\log^+L)(Q)$.

§2. DIFFERENTIATION AND COVERING LEMMAS.

The connection of covering properties of some differentiation bases with their differentiation properties is well known. The classical Vitali lemma has been the standard tool to prove the Lebesgue differentiation theorem. A way to state and use it for this purpose is as follows:

2.1. THEOREM. (Vitali lemma). *Consider, for every* $x \in \mathbb{R}^n$ *the set of closed cubic intervals centered at* x. *Call it* $\mathcal{R}(x)$. *Let* E *be a set in* $\mathbb{R}^n$ *and assume that for every* $x \in E$ *there is given a sequence* $\{R_k(x)\} \subset \mathcal{R}(x)$ *contracting to* x *as* $k \to \infty$. *Then one can select from all these given sets a sequence* $\{Q_k\}$ *of disjoint closed cubic intervals such that* $|E - Q_k| = 0$.

A way to prove the Lebesgue differentiation theorem of $\int f$, $f \in L^1_{loc}$, with respect to closed cubic intervals centered at the corresponding points is now the following:

Assume $f \in L^1$. Take $g \in C_o$, $h \in L^1$, $\|h\|_1 \leq \varepsilon$, $f = g+h$. Then, assuming $|f(x)| < \infty$ everywhere, if $\alpha > 0$, we have:

$$|\{x : |\overline{D}(\int f,x)-f(x)| > \alpha\}| = |\{x : |\overline{D}(\int h,x)-h(x)| > \alpha\}| \leq$$

$$\leq |\{x : |\overline{D}(\int h,x)| > \frac{\alpha}{2}\}| + |\{x : |h(x)| > \frac{\alpha}{2}\}| = |A| + |B|$$

We clearly have $|B| \leq \frac{2}{\alpha} \|h\|_1$. As for the set A, it is a measurable set and for every $x \in A$ we have a sequence $\{R_k\} \subset R(x)$ contracting to x such that $(1/|R_k|)| \int_{R_k} h(y)dy| > \frac{\alpha}{2}$. Hence, applying Vitali lemma, we get $\{Q_k\}$, $|A-Q_k| = 0$, Q_k pairwise disjoint. Thus:

$$|A| = |A \cap (\cup Q_k)| \leq \sum |Q_k| \leq \frac{2}{\alpha} \sum \left| \int_{Q_k} h(y)dy \right| \leq \frac{2}{\alpha} \int |h(y)|dy \leq \frac{2\varepsilon}{\alpha}$$

Since ε is arbitrarily small, we get $|\{x : |\overline{D}(\int f,x)-f(x)| > \alpha\}=0$ and so $\overline{D}(\int f,x) = f(x)$ for almost every $x \in \mathbb{R}^n$. In the same way $\underline{D}(\int f,x) = f(x)$ a.e. thus $D(\int f,x) = f(x)$ a.e.

This type of proof clearly shows the role played by the covering lemmas in differentiation. Theorem 2.2 emphasizes this role in a slightly more general setting.

For the remainder of this section we shall be considering only functions defined on the closed unit cube Q of $\mathbb{R}^n$. Our differentiation bases will similarly be subsets of Q, with the relative topology. Since differentiation properties are local this does not lead to any lack of generality. We consider Orlicz spaces with Q as their fundamental set. For a detailed account of the theory of these spaces we refer to [8].

We denote the norm of the Orlicz space L_Φ associated to the N-function Φ by $\| \ \|_\Phi$. The Orlicz space dual of L_Φ will be denoted by L_ψ. The space L_ψ is said to satisfy the Δ_2-condition if there is a $k > 0$ such that the N-function ψ satisfies $\psi(2t) \leq k\psi(t)$ for large values of t.

2.2. THEOREM. *Let L_Φ be an Orlicz space whose dual L_ψ satisfies the Δ_2-condition. Let R be a differentiation basis in Q with the following property:*

(a) *If A is a measurable subset of Q and for every $x \in A$ there is given a sequence $\{R_k(x)\} \subset R$, $R_k(x) \to x$ (i.e. contractin to x), then, given $\varepsilon > 0$, one can select a sequence $\{Q_k\}$ among these given sets so that, if $S = \cup Q_k$*

$$(1) \quad |A - S| < \varepsilon$$

(2) $|S - A| < \varepsilon$

(3) $\|\sum \chi_k - \chi_S\|_\Phi < \varepsilon$

where χ_k *is the characteristic function of* Q_k *and* χ_S *that of* S.

Then:

(b) *R differentiates* $\int f$ *for every* $f \in L_\psi$ *and the value of the derivative is* $f(x)$ *almost everywhere in* Q.

2.3. REMARK. Note that conditions (1) and (2) mean that the symmetric difference of A and S can be made arbitrarily small in measure and (3) that the overlapping function can be made arbitrarily small in norm.

Proof of theorem 2.2. Let $f \in L_\psi$. Since L_ψ satisfies the Δ_2-condition, the set of continuous functions on Q is dense in L_ψ and so we can take, given $\varepsilon > 0$, $g \in C$, $h \in L_\psi$, $\|h\|_\psi \leq \varepsilon$, $f = g + h$. We can assume $|f(x)| < \infty$ for almost every x. We have then

$$|\{x \in Q : |\overline{D}(\int f,x) - f(x)| > \alpha > 0\}| \leq |\{x \in Q : |\overline{D}(\int h,x)| > \tfrac{\alpha}{2}\}| +$$

$$+ |\{x \in Q : |h(x)| > \frac{\alpha}{2}\}| = |A| + |B|$$

For $|B|$ we have

$$|B| = \int_B dx \leq \frac{2}{\alpha} \int_Q |h(x)|dx \leq \frac{2}{\alpha} \|h\|_\psi \|\chi_Q\|_\Phi \leq \frac{2\varepsilon}{\alpha\alpha} \|\chi_Q\|_\Phi$$

by the generalized Hölder inequality.

As for $|A|$ we can proceed as we did after theorem 2.1 for the Lebesgue differentiation theorem, using now property (a) to get:

$$|A| \leq |A-S| + |S| \leq \varepsilon + \sum |Q_k| \leq \varepsilon + \frac{2}{\alpha} \sum \int_{Q_k} |h(y)|dy \leq$$

$$\leq \varepsilon + \frac{2}{\alpha} \int (\sum \chi_k(y))|h(y)|dy \leq$$

$$\leq \varepsilon + \frac{2}{\alpha}\int [(\sum \chi_k(y)) - \chi_S(y)]\,|h(y)|\,dy + \frac{2}{\alpha}\int \chi_S(y)\,|h(y)|\,dy \leq$$

$$\leq \varepsilon + \frac{2}{\alpha}\,\|\sum \chi_k - \chi_S\|_\Phi \|h\|_\psi + \frac{2}{\alpha}\,\|\chi_Q\|_\Phi\,\|h\|_\psi \leq$$

$$\leq \varepsilon + \frac{2\varepsilon}{\alpha}\,(\varepsilon + \|\chi_Q\|_\Phi)$$

Since ε is arbitrarily small, we get $|A| = 0$. In the same way we can proceed with $\underline{D}(\int f, x)$. Thus (b) is proved.

There are some known results about the possible equivalence of covering properties and differentiation properties, i.e. of the equivalence of a property of the type (a) and a property of the type (b) in theorem 2.2. This is the main result of the Possel equivalence theorem [10], which can be stated in the following way:

2.4. THEOREM. (de Possel). *Let* R *be a differentiation basis in* Q. *Then the two following statements are equivalent:*

(a) *Given any measurable subset* A *of* Q *and for every* $x \in A$ *a sequence* $\{R_k(x)\} \subset R$, $R_k(x) \to x$ *(i.e. contracting to* x*), then for every* $\varepsilon > 0$, *one can select a sequence* $\{Q_k\}$ *among the given sets* $R_k(x)$*'s, such that, if* $S = \cup Q_k$

(1) $|A - S| < \varepsilon$

(2) $|S - A| < \varepsilon$

(3) $\|\sum \chi_k - \chi_S\|_1 < \varepsilon$

where χ_k *is the characteristic function of* Q_k *and* χ_S *that of* S.

(b) R *differentiates* $\int f$ *for every* $f \in L^\infty$ *and the value of the derivative is* $f(x)$ *almost everywhere in* Q.

Proof. We first show that differentiation of $\int f$, $f \in L^\infty$ to f a.e. is equivalent to differentiation of $\int f$ to f a.e., for f characteristic function of a measurable set. Assume then that R has this last property (density property). Let $f \in L^\infty$. We shall show that if $A_{uv} = \{x : \overline{D}(\int f,x) \geq v > u > f(x)\}$, then $|A_{uv}| = 0$. Consider $E_{rs} = \{x : r \leq f(x) < s\}$. Since $\mathcal{R}$ is a basis with the density property the set $N_{rs} = \{x \in E_{rs} : \underline{D}(\int \chi_{E_{rs}},x) < 1\}$ is a null set. Take $x \in E_{rs} - N_{rs}$. Then, if $R_k \to x$, $R_k \in \mathcal{R}$, we have, if $\|f\|_\infty = C$, and E'_{rs} denotes the complement of E_{rs}:

$$-C|E'_{rs} \cap R_k| + r|E_{rs} \cap R_k| \leq \int_{R_k} f(y)dy \leq s|E_{rs} \cap R_k| + C|E'_{rs} \cap R_k|$$

Dividing by $|R_k|$ and letting $k \to \infty$ we get that

$$r \leq \underline{D}(\int f,x) \leq \overline{D}(\int f,x) \leq s$$

at almost every $x \in E_{rs}$. Hence, if $s-r < v-u$, we clearly get $|A_{uv} \cap E_{rs}| = 0$ and so $|A_{uv}| = 0$. Thus $|\{x : \overline{D}(\int f,x) > f(x)\}| = 0$. In a similar way $|\{x : \underline{D}(\int f,x) < f(x)\}| = 0$. This proves that $d(\int f,x) = f(x)$ at almost every x.

We now prove that (a) implies that $\mathcal{R}$ has the density property. Suppose there exists $M \subset Q$, measurable set, and α such that $0 < \alpha < 1$, so that

$$|A| = |\{x \in M : \underline{D}(\int f_M,x) < \alpha < 1\}| > 0$$

For every $x \in A$ there is then $R_k \to x$, $\{R_k\} \subset \mathcal{R}$, such that

$$|R_k \cap A| < \alpha|R_k|$$

We apply the property (a) of $\mathcal{R}$ and select $\{T_k\} \subset \mathcal{R}$ so that $|T_k \cap A| < \alpha|T_k|$ and for $\varepsilon > 0$,

$$|A - \cup T_k| \leq \varepsilon \quad , \quad \sum |T_k| \leq |A| + \varepsilon$$

Then we have

$$|A| + \varepsilon \geq \sum|T_k| \geq (1/\alpha)\sum |T_k \cap A| \geq (1/\alpha)|(\cup T_k) \cap A| =$$

$$= (1/\alpha)[\,|A| - |A - \cup T_k|\,] \geqslant (1/\alpha)|A| - (\varepsilon/\alpha)$$

Thus $\alpha \geqslant (|A| - \varepsilon)/(|A| + \varepsilon)$ for ε arbitrarily close to zero which is impossible, since $\alpha < 1$. Thus $|A| = 0$. Hence R satisfies the density property.

We now prove (b) $\Rightarrow$ (a) or, what is equivalent, that the fact that R is a basis with the density property implies (a).

Let A be a measurable subset of Q, $|A| > 0$. Suppose that for every $x \in A$ we are given $R_k \to x$, $R_k \in R$. Given $\varepsilon > 0$, take $\alpha = |A|/(|A| + \varepsilon) < 1$, and a relatively open subset G of Q , $G \supset A$, such that $|G-A| < \varepsilon$. Call R^* the sets among the given ones of R which are in G. Define

$$\mu_A = \sup \{|R| : R \in R^* , |A \cap R| > \alpha|R|\}$$

It is clear that $\mu_A > 0$, since otherwise $|A| = 0$, for R is a basis with the density property. Let $R_o \in R^*$ be such that $|R_o| > (3/4)\mu_A$ and $|A \cap R_o| > \alpha|R_o|$. Consider now $A_1 = A-R_o$. We proceed now in a similar way with A_1, i.e. we consider

$$\mu_{A_1} = \sup \{|R| : R \in R^* , |A_1 \cap R| > \alpha|R|\}$$

and take $R_1 \in R^*$ such that $|R_1| > (3/4)\mu_{A_1}$ and $|A_1 \cap R_1| > \alpha|R_1|$. Consider now $A_2 = A_o - \cup_o^1 (R_i \cap A_i)$, $A_o = A$, and so on. If the sequence $\{A_i\}$ is finite we stop this process. Assume $\{A_i\}$ is infinite. The sets $R_i \cap A_i$ are disjoint and so

$$|A| \geqslant |A \cap (\cup R_i)| \geqslant |\cup(A_i \cap R_i)| = \sum|A_i \cap R_i| \geqslant \alpha\sum|R_i|$$

Hence $\sum|R_i| \leqslant |A| + \varepsilon$, i.e. $\int (\sum\chi_i - \chi_A) \leqslant \varepsilon$, if χ_i denotes the characteristic function of R_i. Since $\sum|R_i| < \infty$, we have $|R_i| \to 0$ and so $\mu_{A_i} \to 0$. Let $A_\infty = A - \cup R_i$. Then $\mu_{A_\infty} \leqslant \mu_{A_i}$ for every i . Hence $\mu_{A_\infty} = 0$ and $|A_\infty| = 0$. This proves (1) of (a). Since $R_i \subset G$ for every i, we have $|\cup R_i - A| < \varepsilon$, which proves (2). As for (3), if $S = \cup R_i$, then because of (1),

$$\int (\sum \chi_i - \chi_S) = \int (\sum \chi_i - \chi_A)$$

and this is $\leqslant \varepsilon$ as proved before. This completes the proof of the theorem.

The following theorem, which belongs to Hayes and Pauc, shows another case in which a covering property is equivalent to a differentiation property. Here we present a simplified statement and proof of the theorem, in terms of the differentiation bases we consider. For the result in all its generality we refer to [5] or [6].

2.5. DEFINITION. For brevity we will say that a differentiation basis R in Q is an S^p *basis* for some p, $1 \leqslant p < \infty$ if it satisfies property (a) of theorem 2.4 with $\| \ \|_1$ replaced by $\| \ \|_p$, i.e. the L^p-norm.

2.6. THEOREM. (Hayes, Pauc). *Let R be a differentiation basis in* Q. *Then the two following statements are equivalent:*

(1) R *is an* S^p *basis for all* p, $1 < p < \infty$.

(2) R *differentiates* $\int f$, $f \in L^q$, *for all* q, $1 < q < \infty$ *and the value of the derivative is* f *almost every - where*

Proof. The proof relies on the following lemma, which will be proved later. In order to state this lemma, we introduce first some terminology. For a finite sequence $A = \{M_k\}_{1,\dots,j}$ of measurable sets we will write $\sigma(A) = \cup M_k$, $\omega(A,x) = \sum \chi_k(x) - \chi_S(x)$ where χ_k is the characteristic function of M_k and $S = \sigma(A)$.

2.7. LEMMA. *Let* $1 < z < p < z + 1 < \infty$. *Assume that R is an* S^z *basis but not an* S^p *basis. Then there exists a measurable set* A, *a subcollection* R^* *of* R, *which contains, for every* $x \in A$, *a sequence contracting to* x, *and there exist numbers* ε, $\alpha > 0$, *such that for every finite subcollection of* R^*, $G = \{R_k\}_{1,\dots,N}$, *for* *which we have*

$$\text{(i)} \qquad |A - \sigma(G)| < \varepsilon$$

$$\text{(ii)} \qquad \sum_1^N |R_k| - |A| < \varepsilon$$

$$\text{(iii)} \qquad \|\omega(G,\cdot)\|_z < \varepsilon$$

(such G exists, since R is an S^z *basis) we also have*

$$\text{(iv)} \qquad |\cup\{R_k \in G : \int_{R_k} [\omega(G,x)]^{p-1}dx > \alpha|R_k|\}| \geqslant \varepsilon$$

(The set between bars in (iv) will be denoted $\tau(G,\alpha)$). With this lemma we proceed as follows. Suppose that R is a differentiation basis which differentiates $\int f$ to f at almost every $x \in Q$ for all $f \in L^p$ $(1 < p < \infty)$. Then, by theorem 2.4, R is an S^1 basis. Assume R is not an S^ℓ basis for some $\ell > 1$. Then we may take p,z, $1 \leqslant z < p < z+1$ such that R is an S^z basis but not an S^p basis . We now apply lemma 2.7.

Let A, R^*, ε, α the elements which appear in the lemma. For $h = 1,2,\ldots$ we consider $F_h = \{R_k^h\}_{k=1,2,\ldots,N_h} \subset R^*$ such that

$$|A - \sigma(F_h)| < \varepsilon$$

$$|\sigma(F_h) - A| < \varepsilon$$

$$\int [\omega(F_h,x)]^z dx < \min\ (\varepsilon/2^{h+2}, \varepsilon^z)$$

and finally $\max\{\delta(R_k^h) : k = 1,2,\ldots,N_h\} \to 0$ as $h \to \infty$. It is possible to choose them in this way since R is an S^z basis. Now, because of the lemma, we have $|\tau(F_h,\alpha)| \geqslant \varepsilon$. We try now to construct a function $f \in L^q$, $q = \frac{z}{p-1} > 1$, such that differentiation of $\int f$ to f at almost every point does not hold, reaching so a contradiction.

Let $0_h = \{x : \omega(F_h,x) > 0\}$. We get $|0_h| \leqslant \int \omega(F_h,x)dx \leqslant$

$\leqslant \int [\omega(F_h,x)]^z dx \leqslant \frac{\varepsilon}{2^{h+2}}$. Call $D = \cup O_h$. We have $|D| \leqslant \frac{\varepsilon}{2}$. Let $Q_h = \tau(F_h,\alpha) - D$. Then $|Q_h| \geqslant \frac{\varepsilon}{2}$. If $C = \limsup Q_h$, then $|C| \geqslant \frac{\varepsilon}{2} > 0$. Define now $f(x) = \sum [\omega(F_h,x)]^{p-1}$. We have $\|f\|_q \leqslant \sum \|[\omega(F_h,\cdot)]^{p-1}\|_q = \sum (\int [\omega(F_h,x)]^z dx)^{1/q} \leqslant \varepsilon^{z/q} \sum (1/2^{h+2})^{1/q} < \infty$. Hence $f \in L^q$. If $x \in C$ then $x \notin D = \cup O_h$ and so $f(x) = 0$. On the other hand, if $x \in C$ then $x \in Q_h \subset \tau(F_h,\alpha)$ for infinitely many h's, and so there is a sequence $\{T_k\} \subset R$, $T_k \to x$, such that $\int_{T_k} f > \alpha |T_k|$. Hence $\overline{D}(\int f,x) \geqslant \alpha > 0$ for every $x \in C$, which is a contradiction. This proves that (2) implies (1). The implica - tion (1) $\Rightarrow$ (2) is obtained as in theorem 2.2.

Proof of lemma 2.7. Assume that the lemma is not true. So R is an S^z basis, not an S^p basis, and for every A measurable, for eve ry $R^* \subset R$ such that every $x \in A$ has a sequence of sets in R^* contracting to x, and for every $\varepsilon,\alpha > 0$, there exists a finite collection $\{R_k\}_{k=1,\dots,N} \subset R^*$ satisfying (i) (ii) (iii) and not (iv). We shall prove that R has to be then an S^p basis, reaching a contradiction.

Let O be a given measurable set, $R^* \subset R$ such that every $x \in O$ has a sequence of sets of R^* contracting to x, and $\eta > 0$, $\alpha > 0$. We can assume that all sets of R^* are contained in some open (rela - tively to Q) neighborhood of O , U , such that $|U - O| < \eta$. Select $G = \{R_k\}_{k=1,\dots,N}$, such that (i) (ii) (iii) hold, with ε re placed by η, and also we have $|T(G,\alpha)| < \eta$. Denote by $\rho(G)$ the collection of sets in G which do not constitute $T(G,\alpha)$, i.e.

$$\rho(G) = \{R \in G : \int_R [\omega(G,x)]^{p-1} dx \leqslant \alpha |R|\}$$

We then have

(1) $\quad |O - \sigma(\rho(G))| \leqslant |O - \sigma(G)| + |\tau(G,\alpha)| < 2\eta$

(2) $\quad |\sigma(\rho(G)) - O| \leqslant |U - O| < \eta$

Finally, if $\rho(G) = \{T_h\}_{h=1,\dots,M}$, χ_h denotes the characteristic function of T_h and χ_S that of $S = \cup T_h$, we have

$$(3) \quad \int |\sum \chi_h - \chi_S|^p \leqslant \int |\sum \chi_h - \chi_S|^{p-1} (\sum \chi_h) \leqslant \sum \int_{T_h} [\omega(G,\cdot)]^{p-1} \leqslant$$

$$\leqslant \alpha \sum |T_h| \leqslant \alpha \sum |R_k| \leqslant \alpha (|0| + \eta)$$

Now (1) (2) (3) show that R is an S^p basis. This concludes the proof of the lemma.

Theorems 2.4 and 2.6 suggest the following questions

2.8. PROBLEM. *State simple general conditions on the equivalence of covering properties and differentiation properties of a basis* R.

The interest of such a problem is obvious when one considers the usefulness in many fields of analysis of certain covering lemmas and the difficulty of the methods of obtaining them. In the way suggested in 2.8 one can think of obtaining first differentiation properties and deducing from them covering theorems. For example, one knows that the system of all intervals in R^n satisfies (2) of theorem 2.6 [14]. So one has for this system a covering theorem, expressed by (1) of 2.6. In this context the following problem , which is a particular case of 2.8, seems to be especially interesting.

2.9. PROBLEM. *One knows that the system of all intervals in* R^n *differentiates* $\int f$ *to* f *at almost every point for all* $f \in L(\log^+ L)^{n-1}$ [7]. *Can one obtain from this fact a better covering lemma than the one of the Possel given in* 2.4, *and also better than theorem* 2.6?

§3. SPECIAL DIFFERENTIATION BASES.

In R^2 we fix ℓ different directions $d_1, d_2, \ldots, d_\ell$. Consider the system B of all open convex polygons such that each of their sides is parallel to one of the ℓ fixed directions. When $\ell = 2$ one knows that B differentiates $\int f$ to f at almost every point for all $f \in L \log^+ L$, but not for $f \in L^1$ [7]. Busemann and Feller [2] have shown that $\mathcal{B}$ is a density basis for arbitrary ℓ, i.e. has the density property. We will show that $\mathcal{B}$, for arbitrary ℓ, differentiates $\int f$ to f at almost every point for all $f \in L \log^+ L$. The theorem depends on a geometrical lemma which is very simple in R^2. The theorem admits an extension to R^n, which is proved by the same type of geometrical arguments, although lemma 3.1 is only valid in R^n, for $n > 2$, with altered wording. For simplicity and brevity we will present the theorem only for $n = 2$.

3.1. LEMMA. *Let* $B \in \mathcal{B}$. *Among all parallelograms containing* B *there is a closed one* $P(B)$ *of minimal area such that:*

(a) $|P(B)| \leqslant 2|B|$

(b) *The sides of* $P(B)$ *are parallel to two of the* ℓ *given directions defining the system* $\mathcal{B}$.

Proof. Let K be any bounded convex body in R^2. An easy continuity argument shows that, among all parallelograms containing it, there is at least one of minimal area. Let $P(K)$ be any of them. We show that $|P(K)| \leqslant 2|K|$. To do so, it will be enough to produce a parallelogram $S \supset K$ such that $|S| \leqslant 2|K|$. For this, take any direction d in R^2 and consider the two supporting straight lines t_1, t_2 of K parallel to d. Take now the segment AB joining any $A \in t_1 \cap \bar{K}$ and any $B \in t_2 \cap \bar{K}$. Draw the two supporting lines s_1, s_2 of K parallel to AB. Then it is obvious, from the convexity of K, that the parallelogram S formed by t_1, t_2, s_1, s_2 is such that $|S| \leqslant 2|K|$. So $|P(B)| \leqslant 2|B|$ is proved.

Take now any closed parallelogram U of minimal area circumscribing B and assume it does not satisfy (b). Then it clearly has two opposite sides each with just one point in common with $\bar{B}$. It is then an easy matter to show that those two sides can be simultaneously rotated around these points so to obtain a new parallelogram U^1 circumscribing B and such that $|U^1| \leqslant |U|$. Since U is of minimal are $|U^1| = |U|$. Rotating further we arrive at a parallelogram with two of their sides parallel to one of the ℓ given directions. In the same way we proceed with the other two sides, obtaining a parallelogram $P(B)$ satisfying (b). This proves the lemma.

3.2. THEOREM. *The differentiation basis* $\mathcal{B}$ *differentiates* $\int f$ *at almost every point and the value of the derivative is* f, *for all* $f \in L \log^+ L$.

Proof. The theorem is a trivial corollary of the lemma and of the theorem for $\ell = 2$, ℓ being the number of directions in $\mathcal{B}$. It is enough to observe that for any B, $x \in B \in \mathcal{B}$, we have

$$(1/|B|) \int_B |f(y)|dy \leqslant (2/|P(B)|) \int_{P(B)} |f(y)|dy$$

where P(B) is as in lemma 3.1. Now the possible directions of the sides of P(B) are a fixed number $O(\ell)$. Hence all the usual estimates from which differentiability, in the case $\ell = 2$, is deduced (cf. [7] or [16]) are here valid. Hence differentiability also holds here.

The following statement is rather trivial, but it will be presented here because of its connections with some interesting questions.

3.3. THEOREM. *Let* $d = (d_i)_{i\in N}$ *be a denumerable set of directions in* R^2. *To each point* $x \in R^2$ *assign* $d(x) \in d$, *one of the directions of* d, *and assume that, for each* $i \in N$, $\{x : d(x)=d_i\}$ *is measurable. Consider now for each* x *the set* $\mathcal{R}(x)$ *of all open rectangles containing* x *and having one side parallel to* d(x). *Then there is differentiation of* $\int f$ *to* f *at almost every point with respect to this system of sets for all* $f \in L \log^+ L\ (R^2)$.

On the other hand one knows the following:

3.4. THEOREM. *Let* $\mathcal{R}$ *be the system of all open rectangles in* R^2. *Then* $\mathcal{R}$ *does not satisfy the density property, i.e.,* $\mathcal{R}$ *does not differentiate* $\int f$ *to* f *at almost every point for* f *characteristic function of a measurable set.*

For the proof of this theorem we refer to [9], remark of Zygmund at the end, or [2].

These two theorems lead in a natural way to some problems, which are still open.

3.5. PROBLEM. (Coifman, Guzmán). *Consider in* R^2 *a continuous field of directions, i.e. for every* $x \in R^2$ *there is assigned a direction* $d(x)$*, determined by* $d(x) \in R^2$, $|d(x)| = 1$ *, so that* $d(x)$ *varies continuously. For every* x *we consider* $R(x)$*, the set of all rectangles centered at* x *with one side parallel to* $d(x)$*. Is there differentiation of* $\int f$ *with respect to this system of sets for some class of functions* f*?*

When $d = \{d(x) : x \in R^2\}$ is denumerable and for each $a \in d$ the set $\{x : d(x) = a\}$ is measurable, we have theorem 3.3.

When $d(x)$ is constant, we have differentiability of $\int f$ for all $f \in L \log^+ L(R^2)$ and we have also some inequalities for the maximal operator and the related operators used in similar situations (cf. [16]). One can construct a counterexample to see that the maximal operator in the general case of 3.5 does not satisfy inequalities of the same type.

The following question, also related to 3.3 and 3.4 is due to Zygmung:

3.6. PROBLEM. (Zygmund). *It is known that, once we have fixed a rectangular coordinate system in* R^2*, one can find a function* $f \in L^1$ *such that the system of intervals does not differentiate* $\int f$ *to* f [Saks]. *Suppose now that* $g \in L^1(R^2)$ *is given. Can a pair of rectangular directions be found such that* $\int g$ *is differentiable to* g *at almost every* $x \in R^2$ *with respect to the system of rectangles with those directions? If the answer is affirmative, how is the collection of all such directions?*

The interest of such a question is clear, since in many problems, given $g \in L^1$, one is free to choose an appropriate coordinate system.

The following problem, stated here in very particular terms, seems to be of interest in some questions of harmonic analysis.

3.7. PROBLEM. (Coifman, Guzmán). *Let* $f \in L^p(R^2)$, $1 < p < \infty$ *, and consider*

$$Mf(x) = \sup_{S>0} [1/\ell(S)] \int_{-S}^{S} |f(x_1+t, x_2+t^2)|\,dt$$

where $\ell(S)$ *is the length of the curve section on which one integrates. What are the properties of this maximal operator? Is it of type* (p,p)*?*

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