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ON A CLASS OF NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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1. THE THOMAS-FERMI EQUATION. The point of departure of this investigation if the author's recent work on the Thomas-Fermi and the Emden equations, items [6,7,8] of the References at the end of this paper. After the work was finished and partly in print, my attention was drawn to the existence of a number of papers in the physical literature which have a bearing on my results. Thus most of the series expansions appear to be known with a profusion of numerical results to replace the scanty convergence proofs. On the other hand, the qualitative results may still be new as well as the methods in general.

Among the many papers unknown to me, the most important seem to be E.B. Baker [1], L. Brillouin [2], C.A. Coulson and N.H. March [3], N.H. March [10,11,12] and, in particular, P.J. Rijnierse [13]. In trying to evaluate the situation I have seen that some of my results for the Thomas-Fermi equation could be extended in the light of previous results. But it also became clear that the methods I had used could be applied to large class of equations and it is to this extension that the present paper is devoted. Actually the methods apply to a still larger class, but the greatest possible generality and clear exposition do not always go hand in hand.

The TF (= Thomas-Fermi) equation reads

$$(1.1) \qquad y''(x) = x^{-\frac{1}{2}}[y(x)]^{\frac{3}{2}} \quad .$$

Here the variables x and y are supposed to be non-negative. We list some of the main properties of the solutions most of which extend to the more general class to be studied.

(1) If $y(x)$ is a solution and c a constant (real or complex!) then $c^3\, y(cx)$ is also a solution.

(2) There is an elementary solution of the form

$$x^3\, y = \text{constant } (= 144) \; . \tag{1.2}$$

(3) This solution may be embedded in a one-parameter family ot solutions represented by the Coulson-March (CM) series of the first kind

$$y(x;p) = 144\, x^{-3}\{1 + \textstyle\sum_{n=1}^{\infty} c_n\, p^n\, x^{-n\sigma}\} \; . \tag{1.3}$$

Here p is an arbitrary parameter and the coefficients c_n are so chosen that $y(x;-1)$ is the Thomas-Fermi solution which satisfies the boundary condition

$$y(0;-1) = 1 \; . \tag{1.4}$$

The series converges for large values of $|x|$ and

$$\sigma = \frac{1}{2}\,[\sqrt{73} - 7] \; . \tag{1.5}$$

(4) The elementary solution (1.2) may also be embedded in the one-parameter family of CM series of the second kind

$$y_o(x;q) = 144\, x^{-3}\{1 + \textstyle\sum_{n=1}^{\infty} d_n\, q^n\, x^{n\tau}\} \tag{1.6}$$

converging for small values of $|x|$. Here

$$\tau = \frac{1}{2}\,[\sqrt{73} + 7] \tag{1.7}$$

The two CM families have only the elementary solution (1.2) in common.

(5) At $x = 0$ there is also a convergent series solution of the form

$$x + \frac{1}{6}\, x^3 + \frac{1}{80}\, x^5 + \dots . \tag{1.8}$$

involving only odd powers of x. This series is possibly of no interest to the physicists. On the other hand the Baker series

$$(1.9) \qquad 1 + Bx + \frac{4}{3} x^{\frac{3}{2}} + \frac{2}{5} B x^{\frac{5}{2}} + \frac{1}{3} x^3 + \dots$$

has been intensely studied. The TF solution $y(x;-1)$ corresponds to a value of B equal to -1.58807 10226 ... according to Rijnierse.

(6) In addition to the fixed singularities $x = 0$ and $x = \infty$ there are movable singularities of two distinct types. (i) Any point $c \neq 0$ where a solution $y(x)$ is zero is a branch point of $y(x)$ with a convergent representation of the form

$$(1.10) \qquad y(x) = b(x - c) + (x - c)^{\frac{7}{2}}[b_o + \textstyle\sum_{n=1}^{\infty} b_n (x - c)^{\frac{1}{2}n}] ,$$

$b_o \neq 0$ and the b_n's are uniquely determined by b and c.
(ii) There are also movable points where $y(x)$ becomes infinite as

$$(1.11) \qquad 400\, c\, (x - c)^{-4} .$$

Though this is the leading term of the local expansion, $x = c$ is not really a pole of order four but the true nature of the singularity is still unknown.

(7) Through every point (a,b) in the first quadrant passes one and only one solution $y(x;a,b)$ which has the positive x-axis as an asymptote. This solution satisfies

$$(1.12) \qquad \lim_{x\to\infty} x^3\, y(x;a,b) = 144 .$$

If (a,b) lies on S, the graph of (1.2), then $y(x;a,b)$ coincides with (1.2). If (a,b) lies below S, then $y(x;a,b)$ exists for $0 \leqslant x \leqslant \infty$ and coincides with one of the solutions $y(x;p)$ with $p < 0$. If (a,b) lies above S, then $y(x;a,b)$ coincides with one of the solutions $y(x;p)$ with $p > 0$. Such a solution has a graph with a vertical asymptote $x = c > 0$ beyond which the solution ceases to be real. As x decreases to c

$$(1.13) \qquad y(x;p) < 400\, c\, (x - c)^{-4} .$$

(8) Through every point (a,b) in the first quadrant passes one and only one solution $y_o(x;a,b)$ having the positive y-axis as an asymptote. It satisfies

$$\lim_{x\downarrow 0} x^3\, y_o(x;a,b) = 144 \tag{1.14}$$

Each such solution coincides with one of the solutions $y_o(x;q)$, q real. The interval (0,c) in which the solution exists and is real is finite and x = c is a singular point of the solution. If (a,b) lies below S case (6:i) holds, if (a,b) lies above S case (6:ii).

(9) Every solution which is real and increasing in some interval (α,β) has a singularity of type (6:ii) and as x increases to c the inequality (1.13) holds.

Most of the results stated here extend in one form or another to the solutions of the Emden equation

$$y''(x) = x^{1-m}\,[y(x)]^m \tag{1.15}$$

for $1 < m < 3$. Also for this equation there are possible extensions along the lines indicated below.

The class of differential equations to be considered here is made up of equations of the form

$$y''(x) = x^{-5}\, F[x^3\, y(x)] \tag{1.16}$$

where

$$F(u) = \sum_{n=0}^{\infty} F_n\, u^{\mu_n} \tag{1.17}$$

with

$$F_n \geqslant 0\ ,\ \forall\, n\ ,\quad \tfrac{4}{3} < \mu_o < \mu_1 < \dots < \mu_n < \dots \tag{1.18}$$

and the series (1.17), if infinite, converges for all $u > 0$. The TF equation corresponds to $F_o = 1$, $F_n = 0$, $n > 0$, $\mu_o = \frac{3}{2}$.

2. PRELIMINARIES ; THE CM SERIES. The assumptions have been made with a view of preserving property (1) of the TF solutions intact and so that property (2) will also hold though with a constant different from 144. We see that $y = \alpha x^{-3}$ is a solution provided

$$(2.1) \qquad F(\alpha) = 12\,\alpha$$

This equation has a unique positive solution $\alpha = \gamma$ since

$$u^2 \frac{d}{du}\left[u^{-1} F(u)\right] = \sum_{n=0}^{\infty} (\mu_n - 1) F_n u^{\mu_n} > 0 \quad \text{for} \quad u > 0$$

and $u^{-1} F(u)$ is strictly increasing from 0 to ∞ as u goes from 0 to ∞.

The solution curve

$$(2.2) \qquad S: \quad x^3 y = \gamma$$

plays an important role in the following.

We start by considering property (7) for the generalized TF equation (1.16). It is required to find a solution $y(x;a,b)$ of this equation with the properties

$$(2.3) \qquad y(a) = b \quad , \quad \lim_{x\to\infty} y(x) = 0 \quad .$$

The existence of a unique solution of this singular boundary value problem follows from a theorem due to A. Mambriani [9] for equations of the form

$$(2.4) \qquad y'' = f(x)\, g(x,y) \quad .$$

Here f is continuous for $0 < x$ and belongs to $L(0,\omega)$ for any finite ω but not to $L(0,\infty)$. Further f and g are positive for $x > 0$, $y > 0$, while $g(x,0) = 0$ for $x > 0$. Moreover, $g(x,y)$ is continuous for $x \geqslant 0$, $y \geqslant 0$, strictly increasing with respect to y for fixed $x > 0$ and bounded on compact sets. If k is the least subcript for which $F_n > 0$, we can take

$$f(x) = x^{3\mu_k - 5} \quad , \quad g(x,y) = x^{-3\mu_k} F(x^3 y) \tag{2.5}$$

and obtain an equation to which Mambriani's theorem applies. The corresponding solution will be denoted by $y(x;a,b)$.

As in the TF case these solutions fall into three sub-classes according as (a,b) lies above, on or below S, the curve defined by (2.2). The second class reduces to a single element, namely S itself. No two curves of the system $\{y(x;a,b)\}$ can intersect unless the curves coincide. This means that all curves with (a,b) above S stay above S as long as they exist, while all those with (a,b) below S stay below S. In the first case, each curve has a vertical asymptote $x = c > 0$; in the second case the integral curves can be extended all the way to the y-axis.

For a closer examination of these curves and the corresponding generalized CM series, we introduce

$$v = x^3 y \tag{2.6}$$

as a new dependent variable. This leads to the equation

$$x^2 v'' - 6 x v' + 12 v = F(v) \quad . \tag{2.7}$$

This equation is obviously satisfied by $v = \gamma$ as is to be expected. But it has other solutions which are merely asymptotic to γ. The equation shows that

$$[x^{-6} v'(x)]' = x^{-8}\{F[v(x)] - 12v(x)\} \quad . \tag{2.8}$$

Here $F[v(x)] - 12v(x)$ is positive, zero or negative at $x = a$ according as (a,b) lies above, on or below S. Moreover the sign is preserved for all $x > a$ if $v(x)$ corresponds to $y(x) = y(x;a,b)$. Hence

$$v'(x) = -x^6 \int_x^\infty s^{-8}\{F[v(s)] - 12v(s)\}ds \quad , \tag{2.9}$$

This integral will exist if $v(x)$ is a bounded solution of (2.7). Since the sign of the integrand stays constant as long as $v(s)$ exists, it is seen that $v'(x) > 0$ for (a,b) below S and $v(x)$ is increasing but bounded above by γ, while for (a,b) above S, v(x) is decreasing and bounded below by γ. In either case $\lim_{x\to\infty} v(x)$ exists and a simple argument shows that the limit equals γ.

For the further exploration of v(x) we revert to (2.7). Since we are interested in solutions that tend to γ when $x \to \infty$ it is natural to substitute

$$v = \gamma + u \quad .$$

This leads to an equation which may be written

$$x^2 u'' - 6x u' - Qu = G(u) \tag{2.10}$$

where

$$Q = F'(\gamma) - 12 \quad , \quad G(u) = F(\gamma+u) - F(\gamma) - u F'(\gamma) \quad . \tag{2.11}$$

Since $F'(\gamma) > 12$, it is seen that $Q > 0$. Further G(u) is a holomorphic function of u for $|u| < \gamma$ and its Maclaurin series starts with

$$\frac{1}{2} F''(\gamma) u^2 \quad .$$

The homogeneous equation

$$x^2 U'' - 6x U' - Q U = 0$$

has two linearly independent solutions

$$x^{-\sigma} , x^{\tau} , \sigma = \frac{1}{2} [(49+4Q)^{\frac{1}{2}} - 7] , \tau = \frac{1}{2} [(49 + 4Q)^{\frac{1}{2}} + 7] . \tag{2.12}$$

This fact suggests solving (2.10) by means of a power series in $x^{-\sigma}$ for large $|x|$ and by a power series in x^{τ} for small values of $|x|$. This will give the analogues of the CM series for the generalized TF equation.

We set

$$u(x;p) = \sum_{n=1}^{\infty} c_n p^n x^{-n\sigma} \tag{2.13}$$

where we normalize the solution in such a manner that

(2.14) $$y(x;p) = \gamma\, x^{-3}[1 + u(x;p)]$$

coincides with $y(x;0,1)$ for $p = -1$. This we can do by virtue of property (1) and the fact that $y(x;0,1)$ exists and is necessarily one of the solutions $y(x;p)$. If

(2.15) $$G(u) = \sum_{m=2}^{\infty} g_m\, u^m \quad , \quad |u| < \gamma \quad ,$$

we obtain

$$\sum_{n=2}^{\infty} [(n\,\sigma)^2 + 7\,n\,\sigma - Q]\,(-1)^n\, c_n\, x^{-n\sigma} =$$

$$= \sum_{m=2}^{\infty} g_m \left[\sum_{k=1}^{\infty} (-1)^k\, c_k\, x^{-k\sigma}\right]^m \quad .$$

Equating coefficients we obtain recurrence relations of the form

$$[(n\,\sigma)^2 + 7\,n\,\sigma - Q]\,c_n = p_n\,[c_1, c_2, \ldots, c_{n-1}; g_2, g_3, \ldots, g_n]$$

where p_n is a multinomial in the arguments indicated linear in the c's. These relations define the coefficients uniquely in terms of c_1. A convergence proof may be given using variants of E. Lindelöf's majorant method. See the proof of Theorem 5 in [7] where the TF case is treated.

The same method applies to the CM series of the second kind

(2.16) $$y_o(x;q) = \gamma\, x^{-3}\left[1 + \sum_{n=1}^{\infty} d_n\, q^n\, x^{n\tau}\right]$$

which converges for small values of $|x|$.

3. GENERALIZED BAKER SERIES.

The initial conditions

(3.1) $$y(0) = 1 \quad , \quad y'(0) = B$$

determine uniquely a solution $y_o(x;B)$ of (1.16). This solution may be expanded in ascending powers of x of the form

$$(3.2) \qquad y_o(x;B) = 1 + B\,x + \sum_{k=1}^{\infty} B_k\, x^{\lambda_k} \quad .$$

Here the exponents belong to the semi-module generated by

$$(3.3) \qquad M:\ 1\ ,\ \mu_o\ ,\ \mu_1\ ,\ \mu_2\ ,\ \ldots\ ,\ \mu_n\ ,\ \ldots$$

If $2\mu_n$ is an integer for all n, then so is $2\lambda_n$ and something resembling the classical Baker series results for which a fairly simple convergence proof can be given along the lines of that of Theorem 1 in [7]. In the general case a more sophisticated argument is needed but we shall not take up time with it.

A solution of type (1.8) exists for certain choices of the exponents μ_n. We must have $2\mu_n$ integral for all n and, in particular $2\mu_o = 3$.

In the Emden case the Baker series takes on the form (3.2) but the semi-module M has only two generators, namely 1 and m.

4. FINITE POSITIVE SINGULARITIES. It is clear that a solution defined by the initial conditions

$$(4.1) \qquad y(c) = 0 \quad , \quad y'(c) = b \neq 0$$

will admit x = c as a branch point. An expansion of type (1.10) is present if again $2\mu_n$ is an integer for all n and $2\mu_o = 3$. In the general case expansions of the form

$$(4.2) \qquad b(x-c) + \sum_{k=1}^{\infty} b_k\,(x-c)^{\lambda_k}$$

arise where again the λ's belong to the semi-module (3.3).

Of greater interest are the movable infinitudes where the function F affects the nature of the singularity for every choice of the exponents μ_n. In fact, the faster F grows with u, the weaker are the singularites. Thus, if $F(u) = u^{3/2}$, then (1.11) holds and the singularity masquerades as a pole of order four. More generally, take

$$(4.3) \qquad F(u) = u^{\mu} \quad , \quad \tfrac{4}{3} < \mu \quad .$$

Then a solution which becomes infinite as $x \uparrow c$ equals

$$(4.4) \qquad \left\{ \frac{2(\mu+1)}{(\mu-1)^2} \right\}^{1/(\mu-1)} c^{(5-3\mu)/(\mu-1)} (c-x)^{-2/(\mu-1)}$$

up to terms of lower order. The exponent of (c-x) is a negative integer for infinitely many values of μ but only five of these satisfy the restriction $\frac{4}{3} < \mu$ which is required for the applicability of Mambriani's theorem to the equation in question.

To verify (4.4) we assume that

$$(4.5) \qquad y(x) = A(\mu)(c-x)^{-a}$$

up to terms of lower order and that y''(x) is obtainable by formal differentiation so that

$$y''(x) = A(\mu)\ a(a+1)(c-x)^{-a-2} \quad ,$$

again up to terms of lower order. We have then $x \sim c$ and

$$[y(x)]^{\mu} \sim [A(\mu)]^{\mu}\ (c-x)^{-a\mu} \ .$$

Equating dominating terms in the equation gives

$$A(\mu)\ a(a+1)(c-x)^{-a-2} = c^{3\mu-5}[A(\mu)]^{\mu}\ (c-x)^{-\mu}$$

and (4.4) follows.

This argument is open to various objections. How do we know, in the first place, that there is a solution that becomes infinite as x increases towards a finite limit c? If there should exist such a solution, why should its rate of growth obey a law like (4.5)? Finally, do we have the right to differentiate an asymptotic expression not only once but twice? As to the first question, it may be observed that, by property (1), either all real solutions stay bounded for $x \geqslant \delta > 0$ or for any given $c > 0$ there exists a solution which becomes infinite as x approaches c from one side or the other. We shall see that the first alternative is untenable and in so doing we shall obtain inequalities which are consistent with the estimate (4.4).

Again we take the equation

$$y''(x) = x^{3\mu-5}[y(x)]^{\mu} \tag{4.6}$$

but assume $\mu \geqslant \frac{5}{3}$ to simplify matters. Since $y''(x) > 0$ for a non-negative solution, it is seen that $y'(x)$ is increasing. We may disregard the case where $y'(x)$ stays negative in the interval of existence of $y(x)$ as a real solution since this implies either that $y(x)$ is one of the solutions $y(x;p)$ or the right endpoint of the interval is a branch point. In any other case we can find an x_o such that

$$y(x_o) = y_o > 0 \quad , \quad y'(x_o) = y_1 \geqslant 0 \quad . \tag{4.7}$$

We now multiply both sides of (4.6) by $y'(x)$ and integrate to obtain

$$[y'(x)]^2 = y_1^2 + 2\int_{x_o}^{x} s^{3\mu-5}[y(s)]^{\mu}\, y'(s)\, ds \tag{4.8}$$

and hence

$$[y'(x)]^2 > 2x_o^{3\mu-5}\int_{x_o}^{x}[y(s)]^{\mu}\, y'(s)\, ds \quad ,$$

where we have used the assumption that $3\mu \geqslant 5$. This gives

$$\int_{x_o}^{x}\{[y(s)]^{\mu+1} - y_o^{\mu+1}\}^{-\frac{1}{2}}\, y'(s)\, ds \tag{4.9}$$

$$> \left(\frac{2}{\mu+1}\right)^{\frac{1}{2}} x_o^{\frac{1}{2}(3\mu-5)} \quad (x-x_o) \quad .$$

Here we have assumed that $y(x)$ exists for the values of x under consideration. It is required to show that the interval of existence, $[x_o,c)$ say, is finite and that $y(x)$ and $y'(x)$ become infinite as $x \uparrow c$. Let us first note that $y'(x)$ is positive so that

$$\lim_{x \uparrow c} y(x) \equiv L$$

exists. If c and L are finite, then (4.8) shows that $y'(x)$ also has a finite limit, L_1 say. But the existence theorems show that equation (4.6) has a unique solution satisfying the initial conditions

$$y(c) = L \quad , \quad y'(c) = L_1 \quad ,$$

and this solution exists in some neighborhood of $x = c$. To the left of $x = c$ it must coincide with the solution defined by (4.7), to the right of $x = c$ it gives the continuation of this solution. It follows that we have only two alternatives: (i) $y(x)$ exists for all $x > x_o$ or (ii) the interval of existence as a real-valued solution is finite and $y(x)$ and $y'(x)$ become infinite as $x \uparrow c$. We shall show that the first alternative is excluded.

This follows from (4.9) for the left member is dominated by the convergent integral

$$\int_{y_o}^{\infty} [\, t^{\mu+1} - y_o^{\mu+1}]^{-\frac{1}{2}} \, dt$$

for all admissible x while the right member becomes infinite with x. Hence c is finite and $y(x)$ and $y'(x)$ become infinite as $x \uparrow c$. The inequality also gives us some information about the rate of growth of $y(x)$.

To see this we note that we may change the limits of integration in (4.9) replacing x_o by $y(x)$ and x by c. This gives

$$J\,[y(x)] > \left(\frac{2}{\mu+1}\right)^{\frac{1}{2}} x^{\frac{1}{2}(3\mu-5)} \quad (c-x) \tag{4.10}$$

where

$$J(y) = \int_{y}^{\infty} [t^{\mu+1} - y_o^{\mu+1}]^{-\frac{1}{2}} \, dt. \tag{4.11}$$

Since $J(y)$ is monotone decreasing as y increases, we can invert inequality (4.10). It is more instructive to observe that for large values of y

$$J(y) = \frac{2}{\mu-1} y^{-\frac{1}{2}(\mu-1)} [1 + O(y^{-\mu-1})] \tag{4.12}$$

so that

$$\frac{2}{\mu-1} [y(x)]^{-\frac{1}{2}(\mu-1)} > \left(\frac{2}{\mu+1}\right)^{\frac{1}{2}} x_o^{\frac{1}{2}(3\mu-5)} (1-\epsilon)(c-x)$$

where $\epsilon \downarrow 0$ as $x \uparrow c$. This gives an upper bound for y(x) of the type (4.4) but with c replaced by x_o and an extra factor $1 + \eta$ where $\eta \downarrow 0$ as $x \uparrow c$. Since y(x) is independent of x_o, $c > x_o$, we can replace x_o by its upper bound c. The result is

$$y(x) < A(\mu)(1+\eta)(c-x)^{-a} \tag{4.13}$$

where a and $A(\mu)$ have the values given in (4.4) and $\eta \downarrow 0$ as $x \uparrow c$. At the expense of some additional labor we could estimate η in terms of powers of c-x.

This we shall not do but it is desirable to show that the same method leads to an upper bound for y(x) consistent with (4.4). To simplify the argument, suppose that

$$y_1 = y'(x_o) = 0 \quad , \tag{4.14}$$

so that the solution has a positive minimum. Since $x < c < \infty$, formula (4.8) gives

$$[y'(x)]^2 < 2c^{3\mu-5} \int_{x_o}^{x} [y(s)]^{\mu} y'(s)\, ds \quad .$$

Proceeding as above we get

$$J[y(x)] < \left(\frac{2}{\mu+1}\right)^{\frac{1}{2}} c^{\frac{1}{2}(3\mu-5)} (c-x)$$

and, using (4.12), ultimately

(4.15) $$y(x) > A(\mu)\ (1-\eta)\ (c-x)^{-a}$$

where a and $A(\mu)$ have the meaning as above and $\eta \downarrow 0$ as $x \uparrow c$.

These two inequalities (4.13) and (4.15) serve in complete justification of (4.4).

The same type of argument applies to real-valued solutions which become infinite as x decreases to some positive number c. We illustrate this on solutions $y(x;p)$ with $p > 0$. Such a solution has negative slope for $c < x$; moreover the corresponding function $v(x)$ which satisfies (2.9) also has negative slope. This implies that as $x \downarrow c$ the function $v(x)$ tends to a limit $> \gamma$ and this excludes the possibility that $c = 0$ and $y(x;p) = y_o(x;q)$ for some choice of q. Thus $c > 0$ and $y(x)$ as well as $y'(x)$ are unbounded as $x \downarrow c$. The analogue of (4.8) now reads

(4.16) $$[y'(x)]^2 = -2\int_x^{\infty} s^{3\mu-5}[y(s)]^{\mu}\ y'(s)\ ds$$

Since for $y(x) = y(x;p)$ the integrand tends to zero as s^{-9} when $s \to \infty$, the integral certainly exists and is greater than

$$-2x^{3\mu-5}\int_x^{\infty}[y(s)]^{\mu}\ y'(s)\ ds = \frac{2}{\mu+1}\ x^{3\mu-5}[y(x)]^{\mu+1} \quad .$$

This gives

$$-[y(x)]^{\frac{1}{2}(\mu+1)}\ y'(x) > \left(\frac{2}{\mu+1}\right)^{\frac{1}{2}}\ c^{\frac{1}{2}(3\mu-5)}$$

since $c < x$ and

$$\frac{2}{\mu-1}\ [y(x)]^{-\frac{1}{2}(\mu-1)} > \left(\frac{2}{\mu+1}\right)^{\frac{1}{2}}\ c^{\frac{1}{2}(3\mu-5)}\ (x-c) \quad .$$

Solving this for $y(x)$ we obtain

(4.17) $$y(x) < A(\mu)\ (c-x)^{-a}$$

with a and $A(\mu)$ as above. The argument can be adjusted to yield an inequality running in the opposite direction involving a factor $1-\eta$ with $\eta \downarrow 0$ as $x \downarrow c$.

One final remark: the excluded case $\frac{4}{3} < \mu < \frac{5}{3}$ can be handled by a preliminary integration by parts in (4.8).

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