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A HAHN-BANACH THEOREM FOR DISTRIBUTIVE LATTICES

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The familiar Hahn-Banach theorem for real linear spaces ([5],p 9) is still valid if the scalar field R is replaced by a conditionally complete ordered linear space C, and the functionals are replaced by linear and sublinear transformations with values in C ([5], p 105).

M. Cotlar has given in [3] (see also [4], p 7) a theorem on extension of additive functionals defined on certain subsets of a semigroup and dominated in such subset by a subadditive functional. When the semigroup in question is the additive group of a real linear space, M. Cotlar's theorem gives the classical Hahn-Banach theorem.

On the other hand, A. Monteiro has shown in [6] that given a semi-homomorphism d defined on a Boolean algebra A and with values in a complete Boolean algebra B (i.e., a map $d: A \longrightarrow B$ such that $d(x \vee y) = d(x) \vee d(y)$ and $d(1) = 1$), any homomorphism h with values in B, defined on a subalgebra S of A and dominated by d on S can be extended to an homomorphism from A into B dominated by d on the whole algebra A. This result is the analogue of the Hahn-Banach theorem (in the generalized form mentioned above) for Boolean algebras, and A. Monteiro has shown that some important facts in the theory of Boolean algebras can be derived from it. Furthermore, he observed that Stone's theorem asserting that given an ideal I disjoint from a filter F, there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \phi$ could be derived from a certain extension property for homomorphisms fulfilling the condition $d'(x) \leq h(x) \leq d(x)$, where $d': A \longrightarrow B$ is a map having the dual properties of a semi-homomorphism.

This remark motivated the present paper, in which a theorem of this kind is proved for distributive lattices. This theorem is similar to M. Cotlar's theorem mentioned above, and when the distributive lattice is a Boolean algebra, it gives A. Monteiro's theorem, in the same way that Cotlar's result gives the Hahn-Banach theorem when applied to linear spaces. From this point of view, the extension theorem stated here may be considered as the analogue of the Hahn-Banach theorem for distributive lattices . Moreover, it is proved that some classical results of G. Birkhoff and M.H. Stone on ideals in distributive lattices, as well as a theorem of R. Balbes asserting that complete Boolean algebras are injective objects in the category of distributive lattices, can be derived from the theorem proved here.

The proof of the main theorem is based on a proof of M. Cotlar's theorem due to E. Oklander.

In what follows, L will denote a distributive lattice with 0 and 1, B a complete Boolean algebra and B_o the two-element Boolean algebra. -x will denote the Boolean complement of x.

An homomorphism from L into B is a map $h: L \longrightarrow B$ fulfilling the conditions:

(1) $h(0) = 0$ (2) $h(1) = 1$

(3) $h(x \wedge y) = h(x) \wedge h(y)$ (4) $h(x \vee y) = h(x) \vee h(y)$

The set of all homomorphisms from L into B will be denoted by H(L,B).

A map $m: L \longrightarrow B$ fulfilling conditions (1) , (2) and (3) is called a meet-homomorphism, and a map $j: L \longrightarrow B$ satisfying (1),(2) and (4) is called a join-homomorphism. M(L,B) (J(L,B)) will denote the set of all meet (join) homomorphisms from L into B.

$h \in H(L,B)$ is said to be between $m \in M(L,B)$ and $j \in J(L,B)$ if $m(x) \leqslant h(x) \leqslant j(x)$ for any x in L.

LEMMA. *Let* $m \in M(L,B)$ *and* $j \in J(L,B)$. *In order that a map* $h: L \longrightarrow B$ *be an homomorphism which lies between* m *and* j *it is necessary and sufficient that the following conditions hold:*

H 1) $h(0) = 0$ *and* $h(1) = 1$ *and* ;

H 2) *If* $x_1, \ldots, x_n, y_1, \ldots, y_m, w, z$ *are elements in* L, *the relation* $x_1 \wedge \ldots \wedge x_n \wedge w \leqslant y_1 \vee \ldots \vee y_m \vee z$ *implies that* $h(x_1) \wedge \ldots \wedge h(x_n) \wedge m(w) \leqslant h(y_1) \vee \ldots \vee h(y_m) \vee j(z)$.

Proof. It is plain that an homomorphism which lies between m and j must fulfill the conditions H 1) and H 2) . To prove the sufficiency, observe first that since $m(1) = 1$ and $j(0) = 0$, the conditions $x \wedge y \wedge 1 \leqslant (x \wedge y) \vee 0$, $(x \wedge y) \wedge 1 \leqslant x \vee 0$ and $(x \wedge y) \wedge 1 \leqslant y \vee 0$ imply that $h(x \wedge y) = h(x) \wedge h(y)$. It follows analogously that $h(x \vee y) = h(x) \vee h(y)$. Hence, h is an homomorphism. Moreover, from $x \wedge 1 \leqslant 0 \vee x$ it follows that $h(x) = h(x) \wedge m(1) \leqslant h(0) \vee j(x) = j(x)$, and similarly $m(x) \leqslant h(x)$.

The above result suggests the following definition: Let S be a subset of L containing the elements 0 and 1, and let $m \in M(L,B)$ and $j \in J(L,B)$. A map $f: S \longrightarrow B$ is said to be between m and j in S if $f(0) = 0$, $f(1) = 1$ and if for any elements $x_1, \ldots, x_n, y_1, \ldots, y_m$ in S and any elements w,z in L, the relation $x_1 \wedge \ldots \wedge x_n \wedge w \leqslant y_1 \vee \ldots \vee y_m \vee z$ implies that $f(x_1) \wedge \ldots \wedge f(x_n) \wedge m(w) \leqslant f(y_1) \vee \ldots \vee f(y_m) \vee j(z)$.

THEOREM. *Let* $m \in M(L,B)$, $j \in J(L,B)$, S *a subset of* L *containing the elements* 0 *and* 1, *and* f *a map from* S *into* L. *Then there exists an* $h \in H(L,B)$ *which lies between* m *and* j *and such* $h(x) = f(x)$ *for any* x *in* S *if and only if* f *lies between* m *and* j *in* S .

Proof. The necessity of the condition follows at once from the above lemma. The sufficiency will be proved in the following steps:

1) Suppose u is not in S and let $S_1 = S \cup \{u\}$. Then there exists an extension g of f that lies between m and j in S_1.

Let:

$a = \bigvee \{f(x_1) \wedge \ldots \wedge f(x_n) \wedge f(w) \wedge -f(y_1) \wedge \ldots \wedge -f(y_m) \wedge -j(z)\}$

where the supremum is taken over all the elements $x_1, \ldots, x_n, y_1, \ldots, y_m$ in S and w, z in L such that :

$$x_1 \wedge \ldots \wedge x_n \wedge w \leqslant y_1 \vee \ldots \vee y_m \vee u \vee z$$

and let:

$b = \bigwedge \{-f(x_1) \vee \ldots \vee -f(x_n) \vee -m(w) \vee f(y_1) \vee \ldots \vee f(y_m) \vee j(z)\}$

where the infimum is taken over all the elements $x_1, \ldots, x_n, y_1, \ldots, y_m$ in S and w, z in L such that :

$$x_1 \wedge \ldots \wedge x_n \wedge u \wedge w \leqslant y_1 \vee \ldots \vee y_m \vee z$$

Since $x \wedge 0 \leqslant y \vee u \vee 1$ and $x \wedge u \wedge 0 \leqslant y \vee 1$ for any pair x, y in S, a and b are well defined as elements of the complete Boolean algebra B. It will now be proved that $a \leqslant b$. To this end it will be enough to verify that if $x_1, \ldots, x_n, y_1, \ldots, y_m$, $x'_1, \ldots, x'_p, y'_1, \ldots y'_q$ are elements in S and w, z, w', z' are elements in L fulfilling the conditions:

(5) $$x_1 \wedge \ldots \wedge x_n \wedge w \leqslant y_1 \vee \ldots \vee y_m \vee u \vee z$$

and

(6) $$x'_1 \wedge \ldots \wedge x'_p \wedge u \wedge w' \leqslant y'_1 \vee \ldots \vee y'_q \vee z'$$

then the following relation holds:

(7) $$f(x_1) \wedge \ldots \wedge f(x_m) \wedge m(w) \wedge -f(y_1) \wedge \ldots \wedge -f(y_m) \wedge -j(z) \leqslant$$
$$\leqslant -f(x'_1) \vee \ldots \vee -f(x'_p) \vee -m(w') \vee f(y'_1) \vee \ldots \vee f(y'_q) \vee j(z')$$

Joining of $y_1 \vee \ldots \vee y_m \vee z$ to both members of (6) we obtain:

(8) $$(x'_1 \wedge \ldots \wedge x'_p \wedge u \wedge w') \vee y_1 \vee \ldots \vee y_m \vee z \leqslant$$
$$\leqslant y'_1 \vee \ldots \vee y'_q \vee y_1 \vee \ldots \vee y_m \vee z \vee z'$$

It follows from (5) that:

$$(x_1 \wedge \ldots \wedge x_n \wedge w) \wedge (x_1' \wedge \ldots \wedge x_p' \wedge w') \leqslant$$

$$\leqslant (y_1 \vee \ldots \vee y_m \vee u \vee z) \wedge (x_1' \wedge \ldots \wedge x_p' \wedge w') =$$

$$= [(y_1 \vee \ldots \vee y_m \vee z) \wedge (x_1' \wedge \ldots \wedge x_p' \wedge w')] \vee (u \wedge x_1' \wedge \ldots \wedge x_p' \wedge w') \leqslant$$

$$\leqslant (x_1' \wedge \ldots \wedge x_p' \wedge w' \wedge u) \vee y_1 \vee \ldots \vee y_m \vee z$$

Hence (8) implies that:

$$(9) \qquad x_1 \wedge \ldots \wedge x_n \wedge x_1' \wedge \ldots \wedge x_p' \wedge w \wedge w' \leqslant$$

$$\leqslant y_1 \vee \ldots \vee y_m \vee y_1' \vee \ldots \vee y_q' \vee z \vee z'$$

Since $m(w \wedge w') = m(w) \wedge m(w')$ and $j(z \vee z') = j(z) \vee j(z')$, and f lies between m and j in S, it follows from (9) that:

$$f(x_1) \wedge \ldots \wedge f(x_n) \wedge f(x_1') \wedge \ldots \wedge f(x_p') \wedge m(w) \wedge m(w') \leqslant$$

$$\leqslant f(y_1) \vee \ldots \vee f(y_m) \vee f(y_1') \vee \ldots \vee f(y_q') \vee j(z) \vee j(z')$$

and it is easy to see that the above relation is equivalent with (8).

Let c be any element of B such that $a \leqslant c \leqslant b$, and define $g(x) = f(x)$ if x is in S and $g(u) = c$. It follows at once from the choice of c that g lies between m and j in S_1.

2) Consider the pairs (S_t, f_t) such that $S \subseteq S_t$ and f_t is an extension of f which lies between m and j in S_t. Define $(S_t, f_t) \leqslant (S_s, f_s)$ if $S_t \subseteq S_s$ and f_s is an extension of f_t. From the Zorn's lemma it follows that there exists a pair (S_o, f_o) which is maximal for the above order relation, and 1) implies that $S_o = L$. So f_o lies between m and j in L, and it follows from the lemma that $h = f_o \in H(L,B)$ and that $m(x) \leqslant h(x) \leqslant j(x)$ for any x in L.

COROLLARY 1. *Let* $m \in M(L,B)$, $j \in J(L,B)$, S *a sublattice of* L *and* $f \in H(S,B)$. *There exists an* $h \in H(L,B)$ *such that* h *lies* *between* m *and* j *in* L *and* $h(x) = f(x)$ *for any* x *in* S *if and only if for any pair* x , y *of elements of* S *and any pair* w , z *of elements of* L , *the condition* $x \wedge w \leqslant y \vee z$ *implies* $f(x) \wedge m(w) \leqslant f(y) \vee j(z)$.

COROLLARY 2. *Let* $m \in H(L,B)$ *and* $j \in J(L,B)$. *There exists an* $h \in H(L,B)$ *such that* $m(x) \leqslant h(x) \leqslant j(x)$ *for any* x *in* L *if and only if* $m(x) \leqslant j(x)$ *for any* x *in* L.

Proof. Define $S = \{0,1\}$, $f(0) = 0$, $f(1) = 1$ and apply Corollary 1.

The following result is a classical one in the theory of distributive lattices, and was first proved by G. Birkhoff ([2], Th.21.1). A proof given by M.H. Stone ([7], Th.6, second proof) had a strong influence in the establishment of the main theorem of this paper.

COROLLARY 3. *If* I *is an ideal in* L *and* F *a filter disjoint from* I, *there exists a prime ideal* P *of* L *such that* $I \subseteq P$ *and* $P \cap F = \phi$.

Proof. Set $m(x) = 1$ if x is in F and $m(x) = 0$ otherwise, and $j(x) = 0$ if x is in I and $j(x) = 0$ otherwise. It is well known that $m \in M(L,B_o)$ and $j \in J(L,B_o)$. The condition $I \cap F = \phi$ implies that $m(x) \leqslant j(x)$ for every x, and so, by the Corollary 2, there exists $h \in H(L,B_o)$ such that $m(x) \leqslant h(x) \leqslant j(x)$. Then $P=h^{-1}(\{0\})$ is a prime ideal of L, and it is plain that $I \subseteq P$ and $P \cap F = \phi$.

REMARK. In case L does not have 0 and 1, consider $L' = L \cup \{0,1\}$, with $0 < x < 1$ for any x in L. Then L' is a distributive lattice with 0 and 1. If I is an ideal in L disjoint from the filter F , then $I' = I \cup \{0\}$ is an ideal of L' disjoint from the filter $F' = F \cup \{1\}$, So, by the above result, there exists a prime ideal P' such that $I' \subseteq P'$ and $P' \cap F' = \phi$. But $P = P' \cap L$ is a prime ideal in L, and it is clear that $I \subseteq P$ and $P \cap F = \phi$. So in Corollary 3 the condition that L has 0 and 1 can be dropped.

The next result was first proved by R. Balbes ([1], Lemma 3.1) but using some deep theorems of the theory of Boolean algebras.

COROLLARY 4. *Let* L *be a distributive lattice,* S *a sublattice of* L *and* $f \in H(S,B)$. *Then there exists an* $h \in H(L,B)$ *such that* $h(x) = f(x)$ *for* x *in* S.

Proof. Define $m(1) = 1$ and $m(x) = 0$ otherwise, and $j(0) = 0$ and $j(x) = 1$ otherwise. It is plain that $m \in M(L,B)$ and $j \in J(L,B)$. Suppose that x, y are in S, w, z are in L and $x \wedge w \leqslant y \vee z$. If $w \neq 1$, then $0 = f(x) \wedge m(w) \leqslant f(y) \vee j(z)$, if $z \neq 0$, then $f(x) \wedge m(w) \leqslant f(y) \vee j(z) = 1$. In case $w = 1$ and $z = 0$, it follows that $f(x) \wedge m(1) = f(x) \leqslant f(y) = f(y) \vee j(0)$. Therefore the thesis follows from Corollary 1.

REMARK. Considering the lattice $L' = L \cup \{0,1\}$ it is easy to extend the above result to cover the case of a distributive lattice without 0 or 1 (cf. [1], Th.3.2).

Let A be a Boolean algebra. A semi-homomorphism ([6]) is a map $d: A \longrightarrow B$ fulfilling conditions (2) and (4). Observe that $j(x) = d(x)$ for any $x \neq 0$ and $j(0) = 0$ is a join-homomorphism from A into B, and that if $h: A \longrightarrow B$ is any homomorphism such that $h(x) \leqslant d(x)$ for any x, then $h(x) \leqslant j(x) \leqslant d(x)$. j is called the join-homomorphism associated with the semi-homomorphism d.

THEOREM (A. Monteiro [6]). *Let* S *be a subalgebra of the Boolean algebra* A, $d: A \longrightarrow B$ *a semi-homomorphism and* $f: S \longrightarrow B$ *an homomorphism such that* $h(x) \leqslant d(x)$ *for any* x *in* S. *Then there exists an homomorphism* $h: A \longrightarrow B$ *such that* $h(x) \leqslant d(x)$ *for any* x *in* A *and* $h(x) = f(x)$ *for* x *in* S.

Proof. Define $m(1) = 1$ and $m(x) = 0$ otherwise, and let j be the join-homomorphism associated with d. Suppose that x, y are in S, w, z are in L and that $x \wedge w \leqslant y \vee z$. If $w \neq 1$, then $0 = f(x) \wedge m(w) \leqslant f(y) \vee j(z)$. If $w = 1$, then $x \leqslant y \vee z$, so $x \wedge -y \leqslant z$, and since f is a (Boolean algebra) homomorphism, and x, y are in S, it follows that $f(x) \wedge -f(y) = f(x \wedge -y) \leqslant j(z)$, therefore $f(x) \wedge m(1) = f(x) \leqslant f(y) \vee j(z)$. So by Corollary 1 there exists a map $h: A \longrightarrow B$ fulfilling the conditions (1)-(4) and such that $h(x) \leqslant j(x) \leqslant d(x)$ for any x in A and $h(x) = f(x)$ for any x in S. But since $1 = h(x \vee -x) = h(x) \vee h(-x)$ and $0 = h(x \wedge -x) = h(x) \wedge h(-x)$ it follows that $h(-x) = -h(x)$, so h is a (Boolean algebra) homomorphism.

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