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DISTRIBUTION OF POTENTIAL IN A SPHERICAL RING

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ABSTRACT. As an illustration of the use of Legendre transforms in mathematical physics, Churchill [1] has studied the distribution of potential interior to the unit sphere with boundary conditions of third type at the surface. In this paper the distribution of potential interior to a spherical ring with boundary conditions of third type at the surface is found, by means of the theory of integral transforms. The ring is defined by:

$$0 < r_1 < r < r_2 \ , \quad 0 < \theta_1 < \theta < \theta_2 < \pi \quad , \quad 0 \leqslant \phi \leqslant 2\pi$$

INTRODUCTION. Our purpose is to find the distribution of potential in a spherical ring, without sources in the interior and with symmetry with respect to the angle ϕ. In this case the Laplace's equation:

$$(1)\quad \nabla^2 V(r,x) = \frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}(r,x)\right] + \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial V}{\partial x}(r,x)\right] = 0 \quad \text{where} \quad x = \cos\theta$$

must be solved with the following boundary conditions:

$$(2)\quad \left.\begin{aligned} \left|\alpha_1 \frac{\partial V}{\partial x}(r,x) + \alpha_2 V(r,x)\right|_{x=x_1} &= f_1(r) \\ \left|\beta_1 \frac{\partial V}{\partial x}(r,x) + \beta_2 V(r,x)\right|_{x=x_2} &= f_2(r) \end{aligned}\right\} \quad \text{where } -1 < x_2 < x_1 < 1$$

$$(3)\quad \left.\begin{aligned} &\left|\gamma_1 \frac{\partial V}{\partial r}(r,x) + \gamma_2 V(r,x)\right|_{r=r_1} = g_1(x) \\ &\left|\delta_1 \frac{\partial V}{\partial r}(r,x) + \delta_2 V(r,x)\right|_{r=r_2} = g_2(x) \end{aligned}\right\} \quad \text{where} \quad 0 < r_1 < r_2$$

An integral transform which satisfies the boundary condition (2) can be obtained.

The general solution of the equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] y(x) + \lambda\, y(x) = 0$$

is the function:

$$(4)\qquad y(x;\lambda) = C\, u(x;\lambda) + D\, v(x;\lambda)$$

where:

$$u(x;\lambda) = \sum_{n=o}^{\infty} A_{2n}(\lambda)\, x^{2n}$$

$$v(x;\lambda) = \sum_{n=o}^{\infty} A_{2n+1}(\lambda)\, x^{2n+1}$$

being:

$$A_{2n}(\lambda) = \frac{1}{(2n)!} \prod_{\ell=1}^{n} [(2\ell-2)(2\ell-1)-\lambda]$$

$$A_{2n+1}(\lambda) = \frac{1}{(2n+1)!} \prod_{\ell=1}^{n} [(2\ell-1)2\ell-\lambda]$$

Using the convention that for $n = 0$ is: $A_{2n}(\lambda) = A_{2n+1}(\lambda) = 1$.

The constants C and D can be determined applying to the solution (4) the boundary conditions:

$$\alpha_1\, y'(x_1) + \alpha_2\, y(x_1) = 0$$

$$\beta_1\, y'(x_2) + \beta_2\, y(x_2) = 0$$

where the primes denote differentiation with respect to x. So that:

$$C_i = \alpha_1 v'(x_1;\lambda_i) + \alpha_2 v(x_1;\lambda_i) + \beta_1 v'(x_2;\lambda_i) + \beta_2 v(x_2;\lambda_i)$$

$$D_i = -[\alpha_1 u'(x_1;\lambda_i) + \alpha_2 u(x_2;\lambda_i) + \beta_1 u'(x_2;\lambda_i) + \beta_2 u(x_2;\lambda_i)]$$

being λ_i the positive roots of the transcendental equation:

$$\Delta = \begin{vmatrix} \alpha_1 u'(x_1;\lambda) + \alpha_2 u(x_1,\lambda) & \alpha_1 v'(x_1;\lambda) + \alpha_2 v(x_1;\lambda) \\ \beta_1 u'(x_2;\lambda) + \beta_2 u(x_2;\lambda) & \beta_1 v'(x_2;\lambda) + \beta_2 v(x_2;\lambda) \end{vmatrix} = 0$$

We now define the finite integral transform:

$$(5) \qquad T\{F(x)\} = \bar{F}(\lambda_i) = \int_{x_1}^{x_2} F(x)\ S(x;\lambda_i)\ dx$$

whose kernel is:

$$S(x;\lambda_i) = C_i\ u(x;\lambda_i) + D_i\ v(x;\lambda_i)$$

Because of the orthogonality of the functions $S(x;\lambda_i)$ in the closed interval $[x_1,x_2]$ we have the inversion theorem:

$$(6) \qquad F(x) = \sum_i \frac{\bar{F}(\lambda_i)}{\tau_i}\ S(x;\lambda_i)$$

where:

$$\tau_i = \Big| C_i^2\ Z(x,2m,2n,1) + 2C_iD_iZ(x,2m,2n+1,2) + D_i^2\ Z(x,2m+1,2n+1,3) \Big|_{x_1}^{x_2}$$

being:

$$Z(x,am+b,cn+d,s) = \sum_{m,n} \frac{A_{am+b}(\lambda_i)\ A_{cn+d}(\lambda_i)}{am+cn+s}\ x^{am+cn+s}$$

Integrating by parts, we find that:

$$(7) \qquad T\{\frac{d}{dx}[(1-x^2)\ \frac{d}{dx}]\ F(x)\} = \frac{1-x_2^2}{\beta_1}\ S(x_2;\lambda_i)[\beta_1 F'(x_2) + \beta_2 F(x_2)] -$$

$$- \frac{1-x_1^2}{\alpha_1}\ S(x_1;\lambda_i)[\alpha_1 F'(x_1) + \alpha_2 F(x_1)] - \lambda_i T\{F(x)\}$$

SOLUTION OF THE PROBLEM.

By means of (5), and taking into account (7), equation (1) is transformed into:

$$(8) \qquad r^2 \frac{d^2\bar{V}}{dr^2}(r;\lambda_i) + 2r\frac{d\bar{V}}{dr}(r;\lambda_i) - \lambda_i \bar{V}(r;\lambda_i) = \bar{h}(r;\lambda_i)$$

where:

$$\bar{h}(r;\lambda_i) = \frac{1-x_1^2}{\alpha_1} S(x_1;\lambda_i)\, f_1(r) - \frac{1-x_2^2}{\beta_1} S(x_2;\lambda_i)\, f_2(r)$$

The solution of (8) with the boundary conditions (3) is:

$$\bar{V}(r;\lambda_i) = H(r,\lambda_i) + k_1(\lambda_i)\, r^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}} + k_2(\lambda_i)\, r^{-\frac{1}{2}-\sqrt{\lambda_i+\frac{1}{4}}}$$

being:

$$H(r,\lambda_i) = \frac{1}{2\sqrt{\lambda_i+\frac{1}{4}}}\left\{ r^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}} \int \bar{h}(r,\lambda_i)\, r^{-\frac{1}{2}-\sqrt{\lambda_i+\frac{1}{4}}}\, dr - \right.$$
$$\left. - r^{-\frac{1}{2}-\sqrt{\lambda_i+\frac{1}{4}}} \int \bar{h}(r,\lambda_i)\, r^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}}\, dr \right\}$$

$$k_1(\lambda_i) = \frac{Q(\gamma_1,\gamma_2,r_1)P(\delta_1,\delta_2,r_2,-) - Q(\delta_1,\delta_2,r_2)P(\gamma_1,\gamma_2,r_1,-)}{P(\gamma_1,\gamma_2,r_1,+)P(\delta_1,\delta_2,r_2,-) - P(\gamma_1,\gamma_2,r_1,-)P(\delta_1,\delta_2,r_2,+)}$$

$$k_2(\lambda_i) = \frac{Q(\delta_1,\delta_2,r_2)P(\gamma_1,\gamma_2,r_1,+) - Q(\gamma_1,\gamma_2,r_1)P(\delta_1,\delta_2,r_2,+)}{P(\gamma_1,\gamma_2,r_1,+)P(\delta_1,\delta_2,r_2,-) - P(\gamma_1,\gamma_2,r_1,-)P(\delta_1,\delta_2,r_2,+)}$$

with:

$$P(p,q,r_j,\pm) = p\left(-\frac{1}{2} \pm \sqrt{\lambda_i+\frac{1}{4}}\right) r_j^{-\frac{3}{2}\pm\sqrt{\lambda_i+\frac{1}{4}}} + q\, r_j^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}}$$

$$Q(p, q, r_j) = \bar{g}_j(\lambda_i) - p\, H'(r_j;\lambda_i) - q\, H(r_j,\lambda_i)$$

where the primes denote differentiation with respect to r.

Finally, using the inversion theorem (6), we obtain the solution:

$$V(r,x) = \sum_i \frac{S(x;\lambda_i)}{\tau_i} \left\{ H(r,\lambda_i) + k_1(\lambda_i)\, r^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}} + \right.$$
$$\left. + k_2(\lambda_i)\, r^{-\frac{1}{2}-\sqrt{\lambda_i+\frac{1}{4}}} \right\}$$

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