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RESIDUES OF FORMS WITH LOGARITHMIC SINGULARITIES

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Let $(X,\mathcal{O}_X)$ be a reduced complex space of pure dimension n, and Y a nowhere dense closed subspace, not necessarily reduced, locally defined by one equation. Both X and Y may be singular.

The residue $\mathrm{Res}[\tilde{\omega}]$ of a meromorphic p-form $\tilde{\omega} \in \Gamma(X,\Omega^p_X(*Y))$ with poles on Y is in general a (2n-p-1) - current on X with support in Y (cf. [5] and [6]).

Suppose that $\tilde{\omega}$ has only logarithmic singularities on Y. This means that $\tilde{\omega} = \frac{d\varphi}{\varphi} \wedge \psi + \theta$ on each member W of some open covering of X, where φ is a defining equation of Y in W, and ψ and θ are (Grauert-Grothendieck) holomorphic forms on W.

The purpose of this note is to show that in such case $\mathrm{Res}[\tilde{\omega}]$ can be identified, in a sense precised in theorem 2.1, with a unique reduced holomorphic form $\mathrm{res}[\tilde{\omega}] \in \Gamma(Y,\Omega^{p-1}_{r,Y})$.

This property has been announced in [5], and it complements results in [6]. We use without reference notations from this last paper.

1. THE CYCLE OF ZEROS OF AN INVERTIBLE IDEAL.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H^p(X;\mathbb{C}) & \longrightarrow & H^p(U;\mathbb{C}) & \xrightarrow{\delta} & H^{p+1}_Y(X;\mathbb{C}) & \longrightarrow \\
(1) & \uparrow I & & \uparrow I(*Y) & & \uparrow \mu & \\
\longrightarrow & \mathbb{H}^p(X;\Omega^\bullet_X) & \longrightarrow & \mathbb{H}^p(X;\Omega^\bullet_X(*Y)) & \longrightarrow & \mathbb{H}^p(X;Q^\bullet_X) & \longrightarrow
\end{array}$$

associated in [6], 2.3, to the exact sequence

$$0 \longrightarrow \Omega^{\bullet}_X \longrightarrow \Omega^{\bullet}_X(*Y) \longrightarrow Q^{\bullet}_X \longrightarrow 0$$

of holomorphic forms on X, meromorphic forms with poles on Y and their quotient $Q^{\bullet}_X$; here $U = X - Y$. Denote by I the (invertible) ideal in $\mathcal{O}_X$ that defines Y.

The local forms $\frac{d\varphi_W}{\varphi_W}$, where φ_W is a generator of I on each member W of some open covering of X, determine a section $s(I) \in \Gamma(X;Q^1_X)$ that only depends on I: If φ_W and $\psi_W = f\varphi_W$ are generators, then $\frac{d\psi_W}{\psi_W} = \frac{d\varphi_W}{\varphi_W} + \frac{df}{f}$, with f invertible.

This section $s(I)$ is closed; we denoted by $\mathfrak{s}(I)$ its image in $\mathbb{H}^1(X;Q^{\bullet}_X)$, and by

$$c(I) = \mu(\mathfrak{s}(I)) \cap [X] \in H^Y_{2n-2}(X;\mathbb{C})$$

the cap product of $\mu(\mathfrak{s}(I)) \in H^2_Y(X;\mathbb{C})$ with the fundamental class $[X] \in H_{2n}(X;\mathbb{C})$ of X. We identify $H^Y_{2n-2}(X;\mathbb{C}) \cong H_{2n-2}(Y;\mathbb{C})$ and define the cycle $[Y]$ associated to I as the couple (cf. [1], n. 2)

(2) $$[I] = [Y, c(I)]$$

Ocasionally, we will also denote with $[\varphi]$ the cycle associated to the ideal generated by a holomorphic function φ.

This notion of analytic cycle coincides with the classical one [2]. If I_1 and I_2 are invertible ideals of $\mathcal{O}_X$, it is clear that $c(I_i \cdot I_2) = c(I_1) + c(I_2)$ and $[I_1 \cdot I_2] = [I_1] + [I_2]$, as follows locally from $\frac{d(\varphi_1 \cdot \varphi_2)}{\varphi_1 \cdot \varphi_2} = \frac{d\varphi_1}{\varphi_1} + \frac{d\varphi_2}{\varphi_2}$, for generators $\varphi_i \in I_i$. The homomorphism $I \longrightarrow [I]$ is compatible with restriction to open subspaces of X.

Moreover, diagram (1) is functorial with respect to proper morphisms $g: X' \longrightarrow X$ of (paracompact) complex spaces such that $Y' = g^{-1}(Y)$ is 1-codimensional. This follows because, except for the sign, μ is composition of the connexion homomorphism $\mathbb{H}^p(X;Q^\bullet_X) \cong \mathbb{H}^p_Y(X;Q^\bullet_X) \longrightarrow \mathbb{H}^{p+1}_Y(X;\Omega^\bullet_X)$ with the integration map $I: \mathbb{H}^{p+1}_Y(X;\Omega^\bullet_X) \longrightarrow H^{p+1}_Y(X;\mathbb{C})$ (cf. [1], 3.5), both compatible with morphisms (cf. [1], 3.11; the other vertical maps in (1) are also compatible, being defined by integration.

Consequently, If $I' = g^{-1}(I)$ is the inverse image ideal of I and $g^\bullet: H^\bullet_Y(X;\mathbb{C}) \longrightarrow H^\bullet_{Y'}(X';\mathbb{C})$ is the mapping induced by g, we have

(3) $$\mu_{X'}(\mathcal{s}(I')) = g^\bullet(\mu_X(\mathcal{s}(I))) .$$

In this conditions, suppose that $g_\bullet[X'] = [X]$, where $g_\bullet: H_{2n}(X';\mathbb{C}) \longrightarrow H_{2n}(X;\mathbb{C})$ is the mapping in (Borel-Moore) homology induced by g, and $[X']$ is the fundamental class of X'; this is the case, for instance, when g is a resolution.

Then

(4) $$g_\bullet[I'] = [I] \quad ,$$

as follows from (3) and the equality $g_\bullet(a \cap g^\bullet(b)) = g_\bullet(a) \cap b$, for $a \in H_\bullet(X';\mathbb{C})$ and $b \in H^\bullet_Y(X;\mathbb{C})$. ([3], p. 214).

LEMMA 1.1. *Let* $z = (z_1,\dots,z_n)$ *be a coordinate system for* $\mathbb{C}^n$. *Then* $[z_j] = 2\pi i (z_j)^{-1}_{\mathbf{z}}(0)$, *where* $(z_j)^{-1}_{\mathbf{z}}(0)$ *denotes the hyperplane* $z_j = 0$, *oriented canonically.*

Proof. In this case $X = \mathbb{C}^n$ and $Y = (z_j = 0)$. Consider the exact sequence of semianalytic cochains

$$0 \longrightarrow S^{\bullet}(X,U;\mathbb{C}) \longrightarrow S^{\bullet}(X;\mathbb{C}) \longrightarrow S^{\bullet}(U;\mathbb{C}) \longrightarrow 0$$

used in [6], 1.5, to construct the upper sequence in (1).

The class $s(z_j) \in \mathbb{H}^1(X;Q^{\bullet}_X)$ in (1) is image of the class $t(z_j)$ defined by dz_j/z_j in $\mathbb{H}^1(X;\Omega^{\bullet}_X(*Y)) \cong H^1\Gamma(X;\Omega^{\bullet}_X(*Y))$. Thus $\mu(s(z_j)) = \delta I(*Y)(t(z_j))$, and we identify $I(*Y)(t(z_j))$ with the class in $H^1(U;\mathbb{C})$ represented by the semianalytic cocycle of $S^1(U;\mathbb{C})$ associated to dz_j/z_j by integration on the compact chains in U. The class $\delta I(*Y)(t(z_j))$ is represented by the boundary $\delta a \in S^2(X,U)$ of any cochain $a \in S^1(X)$ such that $a = dz_j/z_j$ on U.

Let $B_j \in S_2(X;\mathbb{C})$ be the chain ($z_k = 0$ if $k \neq j$, $|z_j| < \epsilon$ for some $\epsilon > 0$), with the complex orientation of the j-coordinate plane. Necessarily $\delta a(B_j) = \frac{dz_j}{z_j}(\partial B_j) = 2\pi i$, and this implies that the cap product of the class of δa in $H^2_Y(X;\mathbb{C})$ with the fundamental class of $\mathbb{C}^n$ is the class in $H^Y_{2n-2}(X;\mathbb{C})$ represented by the cycle $2\pi i(z_j)^{-1}_{\boldsymbol{z}}(0)$ "transversal" to B_j (cf. [6], 1.6 and 1.7). This is clear in the case $n = 1$ and in our case can be checked using the product structure of $\mathbb{C}^n = \mathbb{C} \times Y$.

COROLLARY 1.2. *In the conditions of the introduction, suppose that no irreductible component of the set of zeros of* $\varphi \in \Gamma(W,\mathcal{O}_X)$ *is contained in the singular set of* X. *Then*

$$[\varphi] = 2\pi i \varphi^{-1}_{\boldsymbol{z}}(0) \tag{5}$$

where $\varphi^{-1}_{\boldsymbol{z}}(0)$ *denotes the inverse image cycle of* 0 *by* φ ([2], 4.12)

Proof. It suffices to prove the equality on the simple points of $(\varphi = 0)$. Locally at one such point, φ can be expressed as a product $z_1^p\psi$, with ψ invertible and z_1 a coordinate. Therefore $[\varphi] = p[z_1]$ and also $\varphi^{-1}_{\boldsymbol{z}}(0) = p(z_1)^{-1}_{\boldsymbol{z}}(0)$ ([2], 4.10), so that (5) is implied by last lemma.

2. RESIDUES OF FORMS WITH LOGARITHMIC POLES.

Let M be a complex subspace of an open set U in $\mathbb{C}^n$. Define the subsheaf $N^{\bullet}_{U,M_r} \subset \Omega^{\bullet}_U$ of germs of holomorphic forms α with the property: $i^*(\alpha) = 0$, where i denotes the embedding in U of the set of simple points of the reduced space M_r associated to M.

The quotient $\Omega^{\bullet}_{r,M} = \Omega^{\bullet}_U / N^{\bullet}_{U,M_r}$ is called the sheaf of reduced holomorphic forms on M. This sheaf behaves functorially with respect to local morphisms of complex spaces ([1], 3.3), which allows to define the sheaf $\Omega^{\bullet}_{r,Y}$ of reduced forms for any complex space Y (cf. also [9]).

There is a canonical epimorphism $j: \Omega^{\bullet}_Y \longrightarrow \Omega^{\bullet}_{r,Y}$, where $\Omega^{\bullet}_Y$ is the sheaf of Grauert-Grothendieck forms on Y. The image by j of a form $\psi \in \Omega^{\bullet}_Y$ will be denoted by $\psi|Y_r$, if no confusion can arise. Observe that $\Omega^q_{r,Y} = 0$ if $q > \dim Y$, a property not shared in general by Ω^q_Y, even when Y is reduced.

THEOREM 2.1. *Let $\tilde{\omega}$ be a meromorphic p-form on X that has only logarithmic singularities on Y. There exists a unique reduced holomorphic (p-1)-form $\mathrm{res}(\tilde{\omega}) \in \Gamma(Y, \Omega^{p-1}_{r,Y})$ such that*

$$\mathrm{Res}[\tilde{\omega}] = I[\mathcal{I}] \wedge \mathrm{res}(\tilde{\omega}) ,$$

where $\mathcal{I} \in \mathcal{O}_X$ is the invertible ideal that defines Y, $I[\mathcal{I}]$ is the integration current on the cycle $[\mathcal{I}]$ and $I[\mathcal{I}] \wedge \mathrm{res}(\tilde{\omega})$ is the current $\alpha \longrightarrow I[\mathcal{I}](\mathrm{res}(\tilde{\omega}) \wedge \alpha)$.

Proof. Given a point $x \in X$, choose a neighborhood W_X of x in X such that a representation

$$\tilde{\omega} \,|\, W_X = \frac{d\varphi}{\varphi} \wedge \psi + \theta$$

as described in the introduction can be found (φ is a generator of I on W). One can suppose that a function $\rho \in \Gamma(W_x, \mathcal{O}_x)$ exists whose set of zeros contains the singular set of W_x and is nowhere dense.

Then $Y_o = (\varphi\rho = 0)$ is nowhere dense in W_x, and $W_x - Y_o$ has only simple points. By Hironaka's resolution of singularities ([7], p. 158), W_x can be chosen small enough to assure the existence of a proper holomorphic mapping $\pi: W' \longrightarrow W_x$ from a complex manifold W' onto W_x, such that $Y_o' = \pi^{-1}(Y_o)$ has only normal crossing and the induced mapping $W'-Y_o' \longrightarrow W_x-Y_o$ is an isomorphism.

Under these conditions, the following equality is contained implicitly in the proof of theorem 7.1 of [6] (cf. n° 7(4)):

$$(6) \qquad \mathrm{Res}[\omega](\alpha) = \mathrm{Res}[\frac{d\varphi}{\varphi} \wedge \psi](\alpha) = \mathrm{Res}[\frac{d\varphi^*}{\varphi^*} \wedge \psi^*](\alpha^*) \quad ,$$

for all smooth forms α with compact support in W_x; here $\varphi^* = \varphi \cdot \pi$, and α^* and ψ^* denote the reciproque image of φ, α and ψ by π.

The variety $Y_o' = (\varphi^*\rho^* = 0)$ has only normal crossings, so that $Y' = (\varphi^* = 0)$ has also this property. Therefore, we can choose a coordinate neighborhood Q in W', centered at any given point in Y', together with coordinates $\omega = (\omega_1, \ldots, \omega_n)$ with the property that $\varphi^*(\omega) = \omega^\nu = \prod_{i=1}^{i=n} \omega_i^{\nu_i}$ on Q, for some vector $\nu = (\nu_1, \ldots, \nu_n) \in Z^n$ (cf. [6], 7.3).

The restriction of $[\varphi^*]$ to Q is equal then to

$$(7) \qquad [\omega^\nu] = \sum_{k=1}^n \nu_k [\omega_k] \quad .$$

Moreover, $\frac{d\varphi^*}{\varphi^*} = \sum_{k=1}^n \nu_k \frac{d\omega_k}{\omega_k}$ on Q, which implies ([6], 7.2)

$$(8) \qquad \mathrm{Res}[\frac{d\varphi^*}{\varphi^*} \wedge \psi^*] = \sum_{k=1}^n \nu_k \, \mathrm{Res}[\frac{d\omega_k}{\omega_k} \wedge \psi^*]$$

By the definition of Res and lemma 1.1,

$$\operatorname{Res}\left[\frac{d\omega_k}{\omega_k}\wedge\psi^*\right](\beta) = \lim_{\delta\to 0}\int_{|\omega_k|=\delta}\frac{d\omega_k}{\omega_k}\wedge\psi^*\wedge\beta$$

$$= 2\pi i\int_{\omega_{k=0}}\psi^*\wedge\beta = I[\omega_k]\wedge\psi^*(\beta) \qquad ,$$

for any smooth form β with compact support in Q.

By (7) and (8), we deduce that

$$\text{(9)} \qquad \operatorname{Res}\left[\frac{d\varphi^*}{\varphi^*}\wedge\psi^*\right] = I[\varphi^*]\wedge\psi^*$$

on Q, and consequently on the whole W'. Then, by (6), (9) and next lemma,

$$\operatorname{Res}[\omega^*](\alpha) = I[\varphi^*](\psi^*\wedge\alpha^*) = I[\pi[\varphi^*]](\psi\wedge\alpha) \; .$$

Moreover, $\pi[\varphi^*] = [\varphi]$ by (4), so that we conclude

$$\text{(10)} \qquad \operatorname{Res}[\tilde{\omega}] = I[\varphi]\wedge\psi$$

on W_x, which is the local version of the theorem.

We show now that the support of the cycle $[\varphi]$ (in the sense of [1], 2.2) is exactly $W_x \cap Y_r$ or, equivalently, that the multiplicity in $[\varphi]$ of each (n-1)-dimensional irreducible component of $W_x \cap Y_r$ is not zero (in fact, is $2\pi i \cdot k$, with k integer > 0).

This is elementary for those components not contained in the singular set sX of X; it can be deduced, for instance, from 1.1 and 1.2. Consider, therefore, a (n-1)-component Γ of $W_x \cap Y_r$ contained in sX . The class $c(\varphi) \in H_{2n-2}(W_x \cap Y_r;\mathbb{C})$ of $[\varphi]$ determines a class $c(\Gamma) \in H_{2n-2}(\Gamma;\mathbb{C})$ (as in [2], 4.3), and the class $c(\varphi^*) \in H_{2n-2}(Y';\mathbb{C})$ of $[\varphi^*]$ determines $c(\pi^{-1}(\Gamma)) \in H_{2n-2}(\pi^{-1}(\Gamma);\mathbb{C})$. Since $\pi_*(c(\varphi^*)) = c(\varphi)$ the image of $c(\pi^{-1}(\Gamma))$ by $\pi'_\bullet: H_\bullet(\pi^{-1}(\Gamma);\mathbb{C}) \longrightarrow H_\bullet(\Gamma;\mathbb{C})$ is $c(\Gamma)$, where $\pi'_\bullet$ is the mapping induced by the restriction $\pi': \pi^{-1}(\Gamma) \longrightarrow \Gamma$ of π.

The support of $[\varphi^*]$ is Y', because W' is a manifold, so that the support of $c(\pi^{-1}(\Gamma))$ is $\pi^{-1}(\Gamma)$. Define $K = \pi(A \cup s(\pi^{-1}(\Gamma)) \cup s\Gamma$, where A is the critical set of π'; A and K are analytic sets (by the proper mapping theorem), dim $K < n-1$ and π' induces a proper local isomorphism $\pi^{-1}(\Gamma - K) \longrightarrow \Gamma - K$.

This implies that the class $c(\Gamma)$, which is image of $c(\pi^{-1}(\Gamma))$, is a non-zero multiple of the fundamental class of Γ, as wanted.

It follows from this property of $[\varphi]$ that the restriction $\psi | W_x \cap Y_r$ is uniquely determined by (10), since $I[\varphi] \wedge \psi = 0$ if and only if the restriction of ψ to the set of simple points of $W_x \cap Y_r$ is zero.

One deduces immediately that the local residue forms $\psi | W \cap Y_r$ associated by (10) to $\mathrm{Res}[\tilde{\omega}]$ define a global form $\mathrm{res}[\tilde{\omega}] \in \Gamma(Y, \Omega^{p-1}_{r,Y})$ with the wanted properties.

LEMMA 2.2. *Let* $\pi: X' \longrightarrow X$ *be a proper morphism of paracompact analytic spaces, and let* $[M,c] \in S_p(X;\mathbb{C})$ *and* $[M',c'] \in S_p(X';\mathbb{C})$ *be semianalytic cycles such that* $\pi[M',c'] \equiv [M,c]$ *(i.e.:* $\pi(M') = M$ *and* $\pi_\bullet(c') = c$, *where* $\pi_\bullet: H_p(M') \longrightarrow H_p(M)$ *is the induced map in homology; cf.* [1], 2.2*). Then*

$$I[M,c](\gamma) = I[M',c'](\pi^*\gamma) ,$$

for each smooth p-*form* γ *with compact support in* X.

Proof. Let sM' and sM be the singular sets of M' and M, and A the critical set of the restriction $M' - (sM' \cup \pi^{-1}(sM)) \longrightarrow M - sM$

of π. Then $K = \pi(A \cup sM') \cup sM$ is not semianalytic, in general ([8], p. 135), but has p-dimensional Haussdorf measure equal zero. The currents $\pi I[M',c']$ and $I[M,c]$ being locally flat, it suffices to show that they are equal on $M - K$ ([4], 4.1.20 and 4.2.28). This is obvious, since the restriction $M' - \pi^{-1}(K) \longrightarrow M - K$ of π is a proper local isomorphism.

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