Revista de la
Unión Matemática Argentina
Volumen 25, 1971.

THE FUNCTORS K^n FOR THE RING OF A CURVE

by M . I . Platzeck and O.E. Villamayor

Dedicado al profesor Alberto González Domínguez

We freely use the notations and terminology of [KV].

In this note we compute the groups $K^n(A)$, $n > 0$, for an affine curve C, where A is the affine ring of C.

As it has been proved by Bass [1], these groups are zero for C non singular, hence the nonzero values depend on the singularities of C.

We prove essentially two results:

1) $K^i(A) = 0$ for $i \geq 2$ (dimension theorem ?)

2) If C has singular points P_α and s_α is the number of branches of C passing through P_α, then $K^1(A) = Z^{\sum_\alpha (s_\alpha - 1)}$, where Z de- the additive group of integers.

Let k be a field, A an affine one dimensional integral k-algebra and $m_1, \ldots, m_h$ those maximal ideals in A such that the local rings A_{m_i} are not regular. Let $\bar{A}$ be the integral closure of A in its quotient field and $c = \mathrm{Ann}(\bar{A}/A)$ the conductor of $\bar{A}$ in A.

LEMMA 1. *If* B *is an affine finite dimensional integral* k-*algebra* m *a maximal ideal, then* B_m *is not integrally closed if and only if* m *contains the conductor of* $\bar{B}$ *in* B.

Proof. See [4], Ch. II, § 2.2d.

Since we are dealing with one dimensional k-algebras, the lemma says that the maximal ideals m such that A_m is not regular are ex actly those containing the conductor c. If m_i is one of such ideals, $m_i\bar{A}$ is an ideal in $\bar{A}$ and we may call $\mathcal{B}_i$ the (finite) set of maximal ideals in $\bar{A}$ containing $m_i\bar{A}$. Hence $\mathcal{B} = \cup\mathcal{B}_i$ is the set of maximal ideals in $\bar{A}$ containing c. In fact, if I is a maximal ideal in $\bar{A}$, $I \supseteq c$, then $J = I \cap A \supseteq c$ and J is maximal in A. Let us call $r_i = \cap_{M\in\mathcal{B}_i} M$, $r = \cap_{M\in\mathcal{B}} M$.

Remark that the radical of $\bar{A}_{m_i} = \bar{A} \otimes_A A_{m_i}$ is $r_i.\bar{A}_{m_i}$, and the conductor of $\bar{A}_{m_i}$ in A_{m_i} is $c \otimes A_{m_i}$.

Let now $r = \cap r_i \subseteq \bar{A}$. Then $r \supseteq c$ and there exists an exponent s_i such that $(r_i \otimes A_{m_i})^{s_i} \subseteq c \otimes A_{m_i}$. If $s = \max(s_i)$, then $r^s \subseteq c$, hence r/c is a nilpotent ideal in $\bar{A}/c$. Since $\cap m_i \subseteq \cap r_i = r$ we also have that $\cap m_i/c$ is a nilpotent ideal in A/c.

LEMMA 2. *Let* B *be a ring with identity and* $J \subseteq B$ *a nilpotent ideal. Then* $K^i(J) = 0$ *for every* $i \geq 0$.

Proof. J^+ (i.e., the ring obtained from J by adding an identity) has a nilpotent ideal J and $J^+/J = k$ is a regular ring. By applying [1], Th. 10.1, Ch. XII we obtain $K^i(J^+) = 0$ for every $i > 0$ and besides, $K^o(J^+) \cong K^o(J^+/J) \cong K^o(k) = Z$. Hence $K^o(J) = 0$.

LEMMA 3. *If* c *is contained in exactly* p *different maximal ideals of* $\bar{A}$, *then* $K^1(r) = Z^{p-1}$, $K^i(r) = 0$ *for every* $i \geq 2$.

Proof. Let $M_1,\dots,M_p$ be the different maximal ideals of $\bar{A}$ which contain c and $J_q = \cap_1^q M_i$, $1 \leq q \leq p$. We shall prove by induction on q that $K^1(J_q) = Z^{q-1}$, $K^i(J_q) = 0$ for $i \geq 2$. This implies the lemma since $J_p = r$.

If q = 1, we have the exact sequence

$$0 \longrightarrow M_1 \longrightarrow \bar{A} \longrightarrow \bar{A}/M_1 \longrightarrow 0$$

which gives us

$$K^o(M_1) \longrightarrow K^o(\bar{A}) \xrightarrow{\alpha} K^o(\bar{A}/M_1) \longrightarrow K^1(M_1) \longrightarrow K^1(\bar{A}) \longrightarrow 0 \longrightarrow$$

$$\longrightarrow \ldots \longrightarrow 0 \longrightarrow K^i(M_1) \longrightarrow K^i(\bar{A}) \longrightarrow 0 \longrightarrow \ldots$$

since $\bar{A}/M_1$ is a field.

Besides, $K^o(\bar{A}/M_1) = Z$ and α is the rank map, i.e., for a finitely generated $\bar{A}$-projective module P, $\alpha(P) = \text{rk } P$, hence α is surjective and, since $\bar{A}$ is regular, $K^i(\bar{A}) = 0$ for $i > 0$, so we obtain $K^i(M_1) = 0$ for $i > 0$.

Assume then to have $K^1(J_q) = Z^{q-1}$, $K^i(J_q) = 0$ for $i > 1$, $1 \leqslant q < p$, and consider the exact sequence

$$0 \longrightarrow J_{q+1} \longrightarrow J_q \xrightarrow{\beta} k' \longrightarrow 0$$

where $k' = \bar{A}/M_{q+1}$ and β is surjective since $J_{q+1} \neq J_q$.

We have then

$$K^o(J_{q+1}) \longrightarrow K^o(J_q) \xrightarrow{\alpha} K^o(k') \longrightarrow K^1(J_{q+1}) \longrightarrow K^1(J_q) \longrightarrow$$

$$\longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow K^i(J_{q+1}) \longrightarrow K^i(J_q) \longrightarrow 0 \longrightarrow \ldots$$

$(i > 1)$ since $K^i(k') = 0$ for $i > 0$ because k' is a field.

So, we have $K^i(J_{q+1}) = 0$ for $i > 1$ and to finish the proof we must show that α is the zero map.

Since $\bar{A}$ is a k-algebra, by taking J_q^+ by adding k to J_q, we have $J_q \subset J_q^+ \subseteq A$ and the natural map $\bar{A} \longrightarrow \bar{A}/M_{q+1} = k'$ induces the commutative diagram

$$\begin{array}{ccccc} J_q & \xrightarrow{i} & J_q^+ & \xrightarrow{\delta} & k \\ & \searrow^{\beta} & \downarrow{\bar{\beta}} & \swarrow_{\gamma} & \\ & & k' & & \end{array}$$

and, obviously, $\bar{\beta}$ induces the rank map $K^o(J_q^+) \longrightarrow K^o(k')$.

Since $K^o(J_q) = \mathrm{Ker}[K^o(J_q^+) \longrightarrow K^o(k)]$ and $K^o(\beta) = K^o(\gamma)K^o(\delta)K^o(i)$, $\alpha = K^o(\beta) = 0$ is the zero map.

LEMMA 4. *Let* B *be a commutative ring with connected spectrum (i.e., without non trivial idempotents),* $M_1,\dots,M_p$ *maximal ideals in* B, $\alpha: K^o(B) \longrightarrow K^o(B/\cap M_i) = \oplus_1^p K^o(B/M_i) \cong Z^p$, $\alpha = K^o(\beta)$, β *the canonical map* $B \longrightarrow B/\cap M_i$. *Then* Im α *is the diagonal* Δ *of* Z^p. *If* P *is a finitely generated projective* B*-module and* [P] *its class in* $K^o(B)$, *then* $\alpha[P] = (h,h,\dots,h)$, $h = \mathrm{rk}\,P$.

Proof. If P is a finitely generated projective module it is obvious that the image of [P] in $K^o(B/M_i)$ is $\mathrm{rk}\,P \otimes_B B/M_i$, but, as it is well known, $\mathrm{rk}\,P \otimes_B B/M_i = \mathrm{rk}\,P \otimes_B B_{M_i}$ where B_{M_i} is the local ring at M_i. Since SpecB is connected, $\mathrm{rk}\,P \otimes_B B_{M_i}$ is independent of M_i and equal to rk P, hence the lemma is proved.

THEOREM. *If* h *is the number of maximal ideals of* A *containing* c *and* p *the number of maximal ideals of* $\bar{A}$ *containing* c, *then*

$$K^1(A) = Z^{p-h} \quad , \quad K^i(A) = 0 \quad \textit{for every} \quad i > 1 .$$

Proof. From the exact sequences

$$0 \longrightarrow c \longrightarrow r \longrightarrow r/c \longrightarrow 0$$

$$0 \longrightarrow c \longrightarrow \cap m_i \longrightarrow \cap m_i/c \longrightarrow 0$$

since r/c and $\cap m_i/c$ are nilpotent, we obtain $K^i(c) = K^i(r) = K^i(\cap m_i)$ for all $i > 0$, and

$$0 \longrightarrow \cap m_i \longrightarrow A \longrightarrow A/\cap m_i \longrightarrow 0$$

gives, using that $K^i(A/\cap m_i) = \oplus K^i(A/m_i) = 0$ if $i > 0$, that $K^i(A) = K^i(\cap m_i)$ for $i \geq 2$, so

$K^i(A) = K^i(r) = 0$ for $i \geqslant 2$, by lemma 3.

To compute $K^1(A)$ we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \cap m_i & \longrightarrow & A & \longrightarrow & A/\cap m_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow & & \\
0 & \longrightarrow & r & \longrightarrow & \bar{A} & \longrightarrow & \bar{A}/r & \longrightarrow & 0
\end{array}$$

induced by the inclusion $i: A \longrightarrow \bar{A}$. Hence

$$\begin{array}{ccccccccccc}
K^o(\cap m_i) & \longrightarrow & K^o(A) & \xrightarrow{\varphi_o} & K^o(A/\cap m_i) & \xrightarrow{\gamma} & K^1(\cap m_i) & \longrightarrow & K^1(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \alpha_o & & \downarrow & & \downarrow & & \\
K^o(r) & \longrightarrow & K^o(\bar{A}) & \xrightarrow{\psi_o} & K^o(\bar{A}/r) & \xrightarrow{\delta} & K^1(r) & \longrightarrow & 0 & &
\end{array}$$

is exact and commutative.

If $\{N_{i,j}\}$ is the set of maximal ideals of $\bar{A}$ containing $\bar{A}.m_i$, the images of $K^o(A/m_i) \longrightarrow \oplus_j K^o(A/N_{i,j})$, φ_o and ψ_o are the diagonals of the codomains Z^s , hence they are direct summands of such codomains.

Since $\mathrm{Im}\ \varphi_o = \mathrm{Im}\ \psi_o = Z$ we have

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Z & \longrightarrow & Z^h & \xrightarrow{\gamma} & Z^{p-1} & \longrightarrow & K^1(A) & \longrightarrow & 0 \\
 & & \updownarrow & & \alpha_o \downarrow & & \updownarrow & & & & \\
0 & \longrightarrow & Z & \longrightarrow & Z^p & \xrightarrow{\delta} & Z^{p-1} & \longrightarrow & 0 & &
\end{array}$$

Hence Im γ is a direct summand in Z^{p-1}, so $K^1(A)$ = Coker γ = Z^{p-h}.

REFERENCES

[1] BASS H., *Algebraic K-theory*, Benjamin Press.

[KV] KAROUBI M. et VILLAMAYOR O.E., *Les focteurs K en Algèbre et Topologie*, C.R. Acad. Sc. Paris, (1969).

[3] -----------------------------, *K-théorie algébrique et K-théorie topologique*, To appear.

[4] SAMUEL P., *Méthodes d'algèbre abstraite en Géométrie algébrique*, Springer-Verlag, (1955).

Universidad Nacional del Sur
Universidad Nacional de Buenos Aires
Argentina

Recibido en abril de 1971.