Revista de la
Unión Matemática Argentina
Volumen 26, 1972.

THE P-MODEL AND THE CONSISTENCY AND THE INDEPENDENCE OF THE AXIOM OF REGULARITY

Alexander Abian

In this self-contained paper, by virtue of the Completeness theorem, we let $(K,\in)$ be a model for the six axioms of Extensionality, Replacement (which is in fact an axiom scheme), Powerset, Sumset, Infinity and Choice of Zermelo-Fraenkel set-theory. Then, based on $(K,\in)$, we construct the corresponding P-model for the six axioms and we show that the axiom of Regularity is valid in the P-model. Clearly, this proves the consistency of the axiom of Regularity with the six axioms. Moreover, we show that if the axiom of Regularity is valid in $(K,\in)$ then $(K,\in)$ coincides with the P-model. Furthermore, using $(K,\in)$ we construct the model $(K',\in')$ in which the axiom of Regularity is not valid whereas the six axioms mentioned above, are valid. Clearly, this proves the independence of the axiom of Regularity from the six axioms.

In a way, the P-model mentioned above, can be considered as an inner model for set-theory [1]. However, its construction is simpler than those which exist in the literature [2,3]. In particular, the notion of the rank of a set [4] in the P-model has many desirable properties which simplify the proofs of various assertions. Our proof of the independence of the axiom of Regularity is a direct generalization of that of Bernays [5] for the case of Ackermann's model [6] which (in contrast to the P-model) does not satisfy the axiom of Infinity.

In what follows *equality* "=" is not borrowed from Logic and is taken by its usual set-theoretical definition (i.e., $x = y$ if and only if every element of x is an element of y and vice versa). The set-theoretical indistinguishability between equal sets is secured by the axiom of *Extensionality*, which states that equal sets are elements of the same sets. The Powerset axiom states that every set s has a powerset (i.e., a set whose elements are exactly the subsets of s). If s has a powerset then it is unique and it is denoted by $P(s)$. The axiom of *Sumset* states that every set s has a sumset (i.e., a set whose elements are exactly the elements of the elements of s). If s has a sumset then it is unique and, as usual, it is denoted by $\cup s$. The axiom of *Replacement*, which is in fact an axiom scheme, states that for every set s and every binary predicate $F(x,y)$ functional in x on s there exists the set whose elements are exactly the mates, under $F(x,y)$, of the elements of s. The axiom of *Infinity* states that there exists a set I such that the empty set (whose existence is ensured by the axiom of Replacement) is an element of I and if x is an element of I then $x \cup \{x\}$ is also an element of I, where the existence of $x \cup \{x\}$ is ensured by the axioms of Replacement, Powerset and Sumset. If a set c has a unique element in common with every nonempty element of a nonempty set s, and if c has no other elements, then c is called a choice-set of s. The choice-set of the empty set is defined to be the empty set. With this in mind, the axiom of *Choice* states that every disjointed set none of whose element is the empty set, has a choice-set, where a set is called *disjointed* if no two distinct elements of it have an element in common. Finally, the axiom of *Regularity* states that every nonempty set s has an element which has no element in common with s.

THE P-MODEL CORRESPONDING TO $(K,\in)$. As mentioned above, we let $(K,\in)$ be a model for the six axioms of Extensionality, Replacement, Powerset, Sumset, Infinity and Choice. For every ordinal v (naturally, of K), let the set A_v (naturally, of K) be given by

(1) $$A_v = P(\bigcup_{u<v} A_u)$$

where $P(x)$ is the powerset (naturally, in K) of x.

Thus, for instance,

$$A_0 = \{\phi\}, \qquad A_1 = \{\phi,\{\phi\}\}, \ \ldots$$

where ϕ is the empty set (naturally, of K).

From (1) it follows that if $x \in A_v$ and $y \in x$ then $y \in \bigcup_{u<v} A_u$. Hence $y \in A_k$ for some $k < v$, which, again by (1) implies $y \subset \bigcup_{u<k} A_u$. But since $k < v$, we have $y \subset \bigcup_{u<v} A_u$ which by (1) implies $y \in A_v$. Consequently, we have

(2) $$x \in A_v \quad \textit{implies} \quad x \subset A_v$$

DEFINITION 1. *By the* P-*model corresponding to* $(K,\in)$ *we mean a collection* P *of sets* x *of* K *such that* x *is a set of the* P-*model if and only if*

(3) $$x \in A_v \quad \textit{for some ordinal} \quad v$$

and where the membership relation in the P-*model is the* $\in$-*relation of* K. *Thus, we may denote the* P-*model by* $(P,\in)$.

LEMMA 1. *Let* x *be a set of* $(P,\in)$. *Then* $y \in x$ *in* $(P,\in)$ *if and only if* $y \in x$ *in* $(K,\in)$.

PROOF. Let $y \in x$ in $(P,\in)$. Then since the membership relation in the P-model is the $\in$-relation in K, we see that $y \in x$ in $(K,\in)$. Conversely, let $y \in x$ in $(K,\in)$. Then since $x \in A_v$ for some v, by (2) we see that $y \in A_v$. Thus, from (3) it follows that y is a set of $(P,\in)$ and therefore, $y \in x$ in $(P,\in)$.

From Lemma 1 we obtain

COROLLARY 1. *Let* x *be a set of* $(P,\in)$. *Then* $y = x$ *in* $(P,\in)$ *if and only if* $y = x$ *in* $(K,\in)$.

PROPOSITION 1. *The axiom of Extensionality is valid in* $(P,\in)$.

PROOF. Let $x = y$ and $x \in z$ in $(P,\in)$. Then by Corollary 1 and Lemma 1 and the fact that the axiom of Extensionality is valid in $(K,\in)$ we see that $y \in z$ in $(K,\in)$. But then by Lemma 1 we have $y \in z$ in $(P,\in)$. Thus, the axiom of Extensionality is valid in $(P,\in)$.

DEFINITION 2. *For every set* x *of* P *the ordinal number* r(x) *is called the rank of* x *if and only if* r(x) *is the smallest ordinal such that* $x \in A_{r(x)}$.

NOTATION 1. *In what follows* M(x) *denotes the formula written in the language of* $(K,\in)$ *and which is equivalent to:*

$$\text{"}\ x \in A_v \ \textit{for some ordinal} \ v\ \text{"}$$

From (1) it is easily proved that for every ordinal v and w

(4) $$v < w \quad \textit{if and only if} \quad A_v \in A_w$$

which by (2) implies

(5) $$v \le w \quad \textit{if and only if} \quad A_v \subset A_w$$

But then from (1) and (5) we obtain

(6) $$A_{v+1} = P(\bigcup_{u\leq v} A_u) = P(A_v)$$

PROPOSITION 2. *The axiom of Replacement is valid in* $(P,\in)$.

PROOF. Let s be a set of $(P,\in)$ and let $F(x,y)$ be a binary predicate which is functional in x on s in $(P,\in)$. Based on Notation 1, we let $F'(x,y)$ be the formula obtained from $F(x,y)$ by replacing each occurrence of $(\forall z)(\cdots)$ in $F(x,y)$ by $(\forall z)(M(z)\to\cdots)$ and each occurrence of $(\exists z)(\cdots)$ in $F(x,y)$ by $(\exists z)(M(z) \wedge \cdots)$ for every bound variable z of $F(x,y)$. Moreover, let

$$F^*(x,y) \equiv F'(x,y) \wedge M(x) \wedge M(y)$$

Clearly,

(7) $F(a,b)$ *is true in* $(P,\in)$ *if and only if*

$F^*(a,b)$ *is true in* $(K,\in)$

By Corollary 1, we see that $F^*(x,y)$ is functional in x on s in $(K,\in)$. Since the axiom of Replacement is valid in $(K,\in)$, the set t whose elements are precisely the mates of the elements of s under $F^*(x,y)$, exists in K. To prove that the axiom of Replacement is valid in $(P,\in)$, in view of Lemma 1 and (7), it is enough to prove that t is a set of $(P,\in)$. In view of (7) it is clear that the elements y of t are sets of $(P,\in)$. Based on Definition 1, we let $w = \operatorname{lub}_{y\in t} r(y)$. But then from (5) it follows that $y \in A_w$ for every $y \in t$ which implies that $t \subset A_w$. But then (6) implies that $y \in A_{w+1}$. Thus, by (3) we see that t is a set of $(P,\in)$. Hence the axiom of Replacement is valid in $(P,\in)$.

Next we observe that for every ordinal v we have

(8) $x \in A_v$ *implies* $P(x) \in A_{v+1}$

(9) $x \in A_v$ *implies* $(\cup x) \in A_v$

(10) $v \in A_v$

PROPOSITION 3. *The axiom of Powerset is valid in* $(P,\in)$.

PROOF. Let x be a set of $(P,\in)$. Hence by (3) we have $x \in A_v$ for some v. Therefore, by (8) we have $P(x) \in A_{v+1}$.

But then by (3) and Lemma 1 we see that $P(x)$ is the powerset of x in $(P,\in)$.

PROPOSITION 4. *The axiom of Sumset is valid in* $(P,\in)$.

PROOF. Let x be a set of $(P,\in)$. Hence by (3) we have $x \in A_v$ for some v. Therefore by (9) we have $\cup x \in A_v$. But then by (3) and Lemma 1 we see that $\cup x$ is the sumset of x in $(P,\in)$.

PROPOSITION 5. *The axiom of Infinity is valid in* $(P,\in)$.

PROOF. The set of all natural numbers ω is an ordinal number such that $\phi \in \omega$ and if $x \in \omega$ then $(x \cup \{x\}) \in \omega$. But then by (10) we have $\omega \in A_\omega$ and hence by (3) we see that ω is a set of $(P,\in)$. Thus, the axiom of Infinity is valid in $(P,\in)$.

REMARK 1. In view of Propositions 4 and 5 we see that if x is a set of $(P,\in)$ then the powerset as well as the sumset of x in $(P,\in)$ coincides respectively with the powerset and the sumset of x in $(K,\in)$.

PROPOSITION 6. *The axiom of Choice is valid in* $(P,\in)$.

PROOF. Let d be a disjointed set of $(P,\in)$ such that the empty set is not an element of d. Then by Lemma 1 we see that d has the same properties in $(K,\in)$. Since the axiom of Choice is valid in $(K,\in)$ it follows that d has a choice-set c in $(K,\in)$. Clearly, $c \subset \cup d$ and hence $c \in P(\cup d)$ in $(K,\in)$. But then by Remark 1 we see that $P(\cup d) \in A_v$ for some v and thus by (2) we have $c \in A_v$. Hence the axiom of Choice is valid in $(P,\in)$.

Let us observe that in view of (1) and Definition 2, for every set x and y of $(P,\in)$ we have

(11) $$x \in y \quad \textit{implies} \quad r(x) < r(y)$$

PROPOSITION 7. *The axiom of Regularity is valid in* $(P,\in)$.

PROOF. Let s be a nonempty set of $(P,\in)$ and let $r = \min_{x \in s} r(x)$, which exists. Clearly, s has an element t whose rank is r. But then from (11) it follows that s cannot have an element in common with t.

REMARK 2. In view of Propositions 1 to 7 we see that the axiom of Regularity is consistent with the remaining six axioms.

REMARK 3. Let us observe that if a set s of $(K,\in)$ is such that every element x of s is an element of $(P,\in)$ then s itself is an element of $(P,\in)$. Indeed, in view of (1) we see that $s \in A_v$ with $v = (\operatorname{lub}_{x \in s} r(x)) + 1$.

PROPOSITION 8. *If the axiom of Regularity is valid in* $(K,\in)$ *then* $(K,\in)$ *coincides with* $(P,\in)$, *i.e., if* s *is a set of* $(K,\in)$ *then* s *is a set of* $(P,\in)$.

PROOF. Assume on the contrary that s is a set of $(K,\in)$ and s is not a set of $(P,\in)$. Then from Remark 3 it follows that s has an element q such that q is not a set of $(P,\in)$. Similarly, q has an element p such that p is not a set of $(P,\in)$. Thus, the finite descending $\in$-chain $p \in q \in s$ starting with s is such that none of the sets occuring in it is a set of $(P,\in)$. Let C be the set of all those sets which occur in such finite chains. Clearly, C has an element in common with every element of itself. But this contradicts the hypothesis of Proposition 8. Hence our assumption is false and Proposition 8 is proved.

Next we prove that the axiom of Regularity is independent of the six axioms of Extensionality, Replacement, Powerset, Sumset, Infinity and Choice. To this end, we construct a model $(K',\in')$ in which the axiom of Regularity is not valid whereas the above-mentioned six axioms are valid.

THE MODEL $(K',\in')$. As mentioned at the beginning of this paper, let $(K,\in)$ be a model for the six axioms of Extensionality, Replacement, Powerset, Sumset, Infinity and Choice.

The model $(K',\in')$ is defined as follows

(12) *the symbols for the sets in* $(K',\in')$ *are exactly the symbols for sets in* $(K,\in)$

and $(x \in' y)$ *if and only if*

(13) $((x = 0)\wedge(1 \in y))\vee((x = 1)\wedge(0 \in y))\vee(x \in (y-\{0,1\}))$

In other words,

(14) $0 \in' y$ *if and only if* $1 \in y$

(15) $1 \in' y$ *if and only if* $0 \in y$

(16) *Otherwise,* $x \in' y$ *if and only if* $x \in y$

The usual definitions of $\subset'$ and $='$ are assumed in $(K',\in')$.

Now let us suppose that $x \subset' y$. Then by (14), (15) and (16) we have

$$(0 \in x) \to (1 \in' x) \to (1 \in' y) \to (0 \in y)$$

$$(1 \in x) \to (0 \in' x) \to (0 \in' y) \to (1 \in y)$$

$$(z \in (x-\{0,1\})) \to (z \in' x) \to (z \in' y) \to (z \in (y-\{0,1\}))$$

But then from the above it follows that

(17) $x \subset' y$ *if and only if* $x \subset y$

and

(18) $x =' y$ *if and only if* $x = y$

Thus, *inclusion* and *equality* in $(K',\in')$ are the same as in $(K,\in)$. Hence instead of $\subset'$ and $='$ we may use respectively $\subset$ and $=$.

PROPOSITION 9. *The axiom of Regularity is not valid in* $(K',\in')$.

PROOF. Since $x \in 1$ if and only if $x = 0$, we see by (15) that $x \in' 1$ if and only if $x = 1$. Thus, in $(K',\in')$ the set 1 has one and only one element 1. Therefore, in $(K',\in')$ the nonempty set 1 has an element in common with every element of itself. Hence,the axiom of Regularity is not valid in $(K',\in')$.

Next, we show that the remaining axioms are valid in $(K',\in')$. However, we first prove the following lemma.

LEMMA 2. *For every set* s *there exists a set* s' *such that*

$$x \in s \quad \textit{if and only if} \quad x \in' s'$$

and conversely.

PROOF. Let us observe that in view of the axioms of Replacement, Powerset and Sumset for every set t of $(K,\in)$, the sets

$$(t-\{0\}) \cup \{1\} \quad \text{and} \quad (t-\{1\}) \cup \{0\}$$

exist. But then the proof of the lemma follows immediately from (14), (15), and (16).

PROPOSITION 10. *The axiom of Extensionality is valid in* $(K',\in')$.

PROOF. In view of (18) and the fact that the axiom of Extensionality is valid in $(K,\in)$, it follows from (14), (15) and (16) that the axiom of Extensionality is also valid in $(K',\in')$.

NOTATION 2. In what follows we let $L(x,y)$ denote the formula given by (13).

Clearly,

(19) $L(x,y)$ in $(K,\in)$ if and only if $(x \in' y)$ in $(K',\in')$.

PROPOSITION 11. *The axiom of Replacement is valid in* $(K',\in')$.

PROOF. Let s' be a set of $(K',\in')$ and let $F'(x,y)$ be a binary predicate which is functional in x on s' in $(K',\in')$. Let $F(x,y)$ be the predicate obtained from $F'(x,y)$ by replacing each occurrence of $(x \in' y)$ by $L(x,y)$. From (19) it follows that $F(x,y)$ in $(K,\in)$ if and only if $F'(x,y)$ in $(K',\in')$. However by Lemma 2 there exists a set s such that $x \in s$ if and only if $x \in' s'$, implying that $F(x,y)$ is functional in x on s in $(K,\in)$. Since the axiom of Replacement is valid in $(K,\in)$, there exists a set m such that for all y,

$(y \in m)$ if and only if $(\exists x)((x \in s) \wedge (F(x,y)))$ in $(K,\in)$

Hence,

$(y \in m)$ if and only if $(\exists x)((x \in' s') \wedge (F'(x,y)))$ in $(K',\in')$

But then, by Lemma 2, there exists a set m' such that $(y \in' m')$ if and only if $(y \in m)$. Consequently, $(y \in' m')$ if and only if $(\exists x)((x \in' s') \wedge (F'(x,y)))$ in $(K',\in')$. Hence, m' is the desired set of the mates under $F'(x,y)$ of the elements of s'. Thus, the axiom of Replacement is valid in $(K',\in')$.

PROPOSITION 12. *The axiom of Powerset is valid in* $(K',\in')$.

PROOF. Let x be a set of $(K',\in')$. Since the Powerset axiom is valid in $(K,\in)$ the powerset $P(x)$ of x exists in $(K,\in)$. But $(y \in P(x))$ if and only if $(y \subset x)$. Thus, by Lemma 2 and (17) there exists a set $P'(x)$ such that $(y \in' P'(x))$ if and only if $(y \subset' x)$. But then $P'(x)$ is the powerset of x in $(K',\in')$. Therefore the axiom of Powerset is valid in $(K',\in')$.

PROPOSITION 13. *The axiom of Sumset is valid in* $(K',\in')$.

PROOF. Let x be a set of $(K',\in')$ and let $w = x-\{0,1\}$. Since the axiom of Sumset is valid in $(K,\in)$ the sumset $\cup w$ of w exists. We prove that the axiom of Sumset is valid in $(K',\in')$ by showing that $\cup' x$ is the sumset of x in $(K',\in')$ where

$$(20) \qquad (z \in' \cup' x) \leftrightarrow (z \in' \cup w) \quad \textit{provided} \quad (1 \notin' x)$$

and

$$(21) \qquad (z \in' \cup' x) \leftrightarrow ((z \in' \cup w) \vee (z = 1)) \ \textit{provided} \ (1 \in' x)$$

First, we show that for every set z

$$(22) \qquad (z \in' \cup w) \leftrightarrow (\exists y)((z \in' y) \wedge (y \in' w))$$

Clearly, since $0 \notin w$ and $1 \notin w$ we see that

$$(23) \qquad (0 \in' \cup w) \leftrightarrow (1 \in \cup w) \leftrightarrow (\exists y)((1 \in y) \wedge (y \in w)) \leftrightarrow$$
$$\leftrightarrow (\exists y)((0 \in' y) \wedge (y \in' w))$$

and

$$(24) \qquad (1 \in' \cup w) \leftrightarrow (0 \in \cup w) \leftrightarrow (\exists y)((0 \in y) \wedge (y \in w)) \leftrightarrow$$
$$\leftrightarrow (\exists y)((1 \in' y) \wedge (y \in' w))$$

Also, if $z \neq 0$ and $z \neq 1$ then

$$(25) \qquad (z \in' \cup w) \leftrightarrow (z \in \cup w) \leftrightarrow (\exists y)((z \in y) \wedge (y \in w)) \leftrightarrow$$
$$\leftrightarrow (\exists y)((z \in' y) \wedge (y \in' w))$$

Thus, in view of (23) to (25) we have established (22).

We note further that since 0 is the void set of $(K',\in')$ we have

$$(26) \qquad (\exists y)((z \in' y) \wedge (y \in' x)) \leftrightarrow$$
$$\leftrightarrow ((\exists y)((z \in' y) \wedge (y \in' w)) \vee ((z \in' 1) \wedge (1 \in' x)))$$

But (20) follows immediately from (22) and (26), and (21) follows from (22), (26) and the observation that $z \in' 1$ if and only if $z = 1$. Hence, the axiom of Sumset is valid in $(K',\in')$.

PROPOSITION 14. *The axiom of Infinity is valid in* $(K',\in')$.

PROOF. Since the axiom of Infinity is valid in $(K,\in)$ the set ω of natural numbers exists in $(K,\in)$. We define a function f from ω by $f(0) = 0$ and

$$(27) \qquad (z \in' f(n+1)) \leftrightarrow ((z \in' f(n)) \vee (z = f(n)))$$

Let $D = f[\omega]$ (i.e. $x \in D$ if and only if $x = f(n)$ for some element n of ω), and let D' be the set such that $x \in' D'$ if and only if $x \in D$. Clearly, by Lemma 2, the set D' exists. But $0 \in' D'$ since $0 \in D$. Moreover, if $x \in' D'$ then $x = f(n)$ for some element n of ω and therefore $x \cup' \{x\}' = f(n) \cup' \{f(n)\}' = f(n+1)$ where $\{x\}'$ and $\cup'$ have their usual meaning in $(K',\in')$. But $f(n+1) \in D$ so that $x \cup' \{x\}' = f(n+1) \in' D'$. Thus, the axiom of Infinity is valid in $(K',\in')$.

PROPOSITION 15. *The axiom of Choice is valid in* $(K',\in')$.

PROOF. Let d be a disjointed set in $(K',\in')$ such that $0 \notin' d$. Then d is a disjointed set in $(K,\in)$ and $1 \notin d$. We assume, first, that $0 \notin d$. Then, by (13) we have that $y \in d$ if and only if $y \in' d$. Since the axiom of Choice is valid in $(K,\in)$ there is a choice-set c of d in $(K,\in)$. But if $0 \in c$ and $0 \in y$ for some element y of d then $1 \in' c$ and $1 \in' y$; if $1 \in c$ and $1 \in y$ for some element y of d then $0 \in' c$ and $0 \in' y$; and if $z \neq 0$ and $z \neq 1$ and $z \in c$ and $z \in y$ for some element y of d then $z \in' c$ and $z \in' y$. Thus, c is a choice-set for d in $(K',\in')$.

Next, let $0 \in d$ and let c be a choice-set in $(K,\in)$ for the set $k = d - \{0\}$. By the above, c is a choice-set in $(K',\in')$ for k. We claim that the set m given by

$$(x \in m) \leftrightarrow ((x \in c) \vee (x = 0)) \tag{28}$$

is a choice-set for d in $(K',\in')$. To this end we need only show that $0 \notin c$ or that $1 \notin' c$. Suppose that $1 \in' c$. Then $1 \in' y$ for some element y of k. But $1 \in' 1$ and $1 \in' d$ since $0 \in 1$ and $0 \in d$. Hence $1 \notin' y$ for every element y of $k = d - \{0\}$, since d is disjointed. Consequently, $1 \notin' c$ and therefore the set m given by (28) is a choice-set of d in $(K',\in')$. Hence, the axiom of Choice is valid in $(K',\in')$.

REFERENCES

[1] J.C. SHEPHERDSON, *Inner models for set-theory*, Journ. Symb. Logic 16(1951)p.161-190.

[2] P.J. COHEN, *A minimal model for set-theory*, Bull. Amer. Math. Soc. 69(1963)p.537-540.

[3] R. MONTAGUE and R.L. VAUGHT, *Natural models of set-theories*, Fund. Math.47(1959)p.119-242.

[4] A. TARSKI, *The notion of rank in axiomatic set-theory and some of its applications*, Bull. Amer. Math. Soc. 61(1955) p.443.

[5] P. BERNAYS, *A system of axiomatic set-theory*, Part VII, Journ. Symb. Logic 19(1954)p.81-96.

[6] W. ACKERMANN, *Die Widerspruchsfreiheit der allgemeinen Mengenlehre*, Math. Ann. 114(1937)p.305-315.

Iowa State University
Ames, Iowa

Recibido en abril de 1970.