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# SOME THEOREMS ON NEAR CONSERVATIVE FIELDS OF FORCE [1]

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## 1. NORMAL FAMILIES OF $\infty^{n-1}$ CURVES C ORTHOGONAL TO SIMPLE FAMILIES OF $\infty^1$ HYPERSURFACES IN RIEMANNIAN SPACE $V_n$.

Let $p_i$ be an absolute covariant vector in a Riemannian space $V_n$, which is never the zero vector. A family of $\infty^{n-1}$ curves C in $V_n$ is given as the set of integral solutions of the system of n ordinary differential equations, each of the first order, namely

$$(1.1) \qquad \frac{dx^i}{ds} = \frac{g^{ij}\ p_j}{|p|},$$

where $|p| = (p,p)^{1/2} = [\varepsilon g^{ij} p_i p_j]^{1/2} > 0$, and s is the arc length measured along such a curve C.

Such a family is said to be *normal* if and only if it is orthogonal to a simple family of $\infty^1$ hypersurfaces $V(x) =$ constant, each of deficiency one, in $V_n$.

The family of curves given by (1.1) is normal if and only if the Pfaffian equation

$$(1.2) \qquad p_i\ dx^i = 0,$$

is integrable. For this there must exist two absolute scalar functions $f(x)$ and $\sigma(x)$, with $\frac{\partial f}{\partial x^i}$ not the zero vector, such that

$$(1.3) \qquad f_{,i} = \frac{\partial f}{\partial x^i} = e^{\sigma}\ p_i\ .$$

Then the second order covariant derivative of $f(x)$ is given by

$$(1.4) \qquad f_{,ij} = e^{\sigma}\ (p_{i,j} + p_i\ \sigma_{,j}).$$

THEOREM 1.1. *If* $T_{ij}$ *is the curl of the covariant vector* $p_i$, *then the modified integrating factor* $\sigma$ *obeys the equations*

(1.5) $$T_{ij} = p_{i,j} - p_{j,i} = p_j \sigma_{,i} - p_i \sigma_{,j}.$$

From (1.5), the following symmetric relations are found

(1.6) $$p_i T_{jk} = p_i p_k \sigma_{,j} - p_i p_j \sigma_{,k} ,$$
$$p_j T_{ki} = p_j p_i \sigma_{,k} - p_j p_k \sigma_{,i} ,$$
$$p_k T_{ij} = p_k p_j \sigma_{,i} - p_k p_i \sigma_{,j} .$$

THEOREM 1.2. *The family of* $\infty^{n-1}$ *curves* C, *given as the integral solutions of the system of n first order ordinary differential equations*

(1.7) $$\frac{dx^i}{ds} = \frac{g^{ij} p_j}{|p|} ,$$

*forms a normal family if and only if the curl* $T_{ij}$ *of the covariant vector* $p_i$ *obeys the identities*

(1.8) $$p_i T_{jk} + p_j T_{ki} + p_k T_{ij} = 0 ,$$

*for* $i,j,k = 1,2,\dots,n$.

This is an immediate consequence of (1.6).

## 2. SOME ADDITIONAL PROPERTIES OF THE MODIFIED INTEGRATING FACTOR $\sigma$.

In the preceding section it was shown that if (p) is the covariant vector of the system of differential equations (1.1), which form a normal family, then

(2.1) $$g^{ab} p_a T_{ib} = g^{ab} p_a (p_b \sigma_{,i} - p_i \sigma_{,b}) .$$

Define a covariant vector $u_i$ such that

(2.2) $$u_i = g^{ab} p_a T_{ib} .$$

Thus the differential equations of the modified integrating factor $\sigma$ may be written in the form

(2.3) $$\sigma_{,i} = \frac{(p, \frac{\partial \sigma}{\partial x})}{(p,p)} p_i + \frac{u_i}{(p,p)} .$$

Now the second order covariant partial derivative of $\sigma$ is

$$(2.4) \qquad \sigma_{,ij} = \frac{(p,\frac{\partial\sigma}{\partial x})}{(p,p)} p_{i,j} + \frac{u_{i,j}}{(p,p)} + \frac{p_i \frac{\partial}{\partial x^j}(p,\frac{\partial\sigma}{\partial x})}{(p,p)} - \frac{p_i (p,\frac{\partial\sigma}{\partial x}) \frac{\partial}{\partial x^j}(p,p)}{(p,p)^2} - \frac{u_i \frac{\partial}{\partial x^j}(p,p)}{(p,p)^2} .$$

Hence upon forming the conditions $\sigma_{,ij} - \sigma_{,ji} = 0$, there are obtained the conditions

$$(2.5) \qquad (p,p)\,(p, \frac{\partial\sigma}{\partial x})\, T_{ij} + (p,p)\,[u_{i,j} - u_{j,i}] + (p,p)\,[p_i \frac{\partial}{\partial x^j} - p_j \frac{\partial}{\partial x^i}]\,(p, \frac{\partial\sigma}{\partial x}) - (p, \frac{\partial\sigma}{\partial x})\,[p_i \frac{\partial}{\partial x^j} - p_j \frac{\partial}{\partial x^i}]\,(p,p) - [u_i \frac{\partial}{\partial x^j} - u_j \frac{\partial}{\partial x^i}]\,(p,p) = 0 .$$

Also from the definition of the covariant vector $u_i$ it is seen that

$$(2.6) \qquad p_j u_i - p_i u_j = g^{ab} p_a (p_j T_{ib} - p_i T_{bj}) = g^{ab} p_a p_b T_{ij} = (p,p)\, T_{ij} .$$

THEOREM 2.2. *The covariant vector* $u_i$ *and the modified integrating factor* $\sigma$ *obey the identities*

$$(2.7) \qquad p_j u_i - p_i u_j = (p,p)\, T_{ij} ,$$

*and*

$$(2.8) \qquad (p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j})\,(p, \frac{\partial\sigma}{\partial x}) = (p, \frac{\partial\sigma}{\partial x}) + (u_{i,j} - u_{j,i}) + \frac{(p, \frac{\partial\sigma}{\partial x})}{(p,p)}\,[p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j}]\,(p,p) + \frac{1}{(p,p)}\,[u_j \frac{\partial}{\partial x^i} - u_i \frac{\partial}{\partial x^j}]\,(p,p) .$$

These identities will be important in succeeding sections.

## 3. SOME CONDITIONS FOR AN ISOTHERMAL NORMAL FAMILY.

In this section conditions will be found under which a family of $\infty^{n-1}$ curves C are normal to a simple family of $\infty^1$ hypersurfaces which form an isothermal family. First, if these hypersurfaces are given by $V(x) = \text{constant}$, then the Lamé differential parameters of first and second orders [2] are given by

$$(3.1) \qquad \Delta_1 = \Delta_1(V) = g^{ij} V_{,i} V_{,j} \quad , \quad \Delta_2 = \Delta_2(V) = g^{ik} V_{,ik}$$

From preceding results, a family of $\infty^{n-1}$ curves given as the integral solutions of the system of n first order ordinary differential equations $\frac{dx^i}{ds} = \frac{g^{ij} p_j}{|p|}$, will be normal to the family of $\infty^1$ hypersurfaces $V(x) = \text{constant}$ if and only if there exists a scalar function $\sigma(x)$ such that

$$(3.2) \qquad V_{,i} = e^{\sigma} p_i \quad , \quad V_{,ij} = e^{\sigma} (p_{i,j} + p_i \sigma_{,j}) .$$

Thus it is seen that

$$(3.3) \qquad \frac{\Delta_2(V)}{\Delta_1(V)} = \frac{e^{-\sigma} (\operatorname{div}(p) + (p, \frac{\partial\sigma}{\partial x}))}{(p,p)} .$$

Now it is known that if a family of $\infty^1$ hypersurfaces $V(x) = \text{constant}$ is isothermal, then $\frac{\Delta_2}{\Delta_1}$, and $V(x)$ are functionally dependent [3]. Thus $\frac{\Delta_2}{\Delta_1} = c$, itself defines a member of the family $V(x) = \text{constant}$. Thus the gradient of $\frac{\Delta_2}{\Delta_1}$, is parallel to the covariant vector $p_i$. Thus

$$(3.4) \qquad - \frac{\partial\sigma}{\partial x^i} \left[ \frac{\operatorname{div}(p) + (p, \frac{\partial\sigma}{\partial x})}{(p,p)} \right] + \frac{\frac{\partial}{\partial x^i} \operatorname{div}(p) + \frac{\partial}{\partial x^i} (p, \frac{\partial\sigma}{\partial x})}{(p,p)} -$$

$$- [\operatorname{div}(p) + (p, \tfrac{\partial\sigma}{\partial x})] \frac{\frac{\partial}{\partial x^i}(p,p)}{(p,p)^2} = \lambda p_i ,$$

where $\lambda$ is a scalar function. By equations (2.3), this becomes

$$(3.5) \qquad - u_i [\operatorname{div}(p) + (p, \tfrac{\partial\sigma}{\partial x})] + (p,p) \frac{\partial}{\partial x^i} [\operatorname{div}(p) + (p, \tfrac{\partial\sigma}{\partial x})] -$$

$$- \operatorname{div}(p) \frac{\partial}{\partial x^i} (p,p) - (p, \tfrac{\partial\sigma}{\partial x}) \frac{\partial}{\partial x^i} (p,p) = \mu p_i ,$$

where $\mu$ is a scalar function.

From the foregoing relations it is clear that

$$(3.6)\qquad - [\operatorname{div}(p) + (p, \tfrac{\partial\sigma}{\partial x})]\ (p_j\, u_i - p_i\, u_j) +$$
$$+ (p,p)\ [p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j}]\ \operatorname{div}(p) +$$
$$+ (p,p)\ [p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j}]\ (p, \tfrac{\partial\sigma}{\partial x}) -$$
$$- \operatorname{div}(p)\ [p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j}]\ (p,p) -$$
$$- (p, \tfrac{\partial\sigma}{\partial x})\ [p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j}]\ (p,p) = 0\ .$$

Define two linear differential operators $A_{ij}$ and $B_{ij}$, by

$$(3.7)\qquad A_{ij} = p_j \frac{\partial}{\partial x^i} - p_i \frac{\partial}{\partial x^j}\quad ,\quad B_{ij} = u_j \frac{\partial}{\partial x^i} - u_i \frac{\partial}{\partial x^j}\ .$$

Using the operators (3.7), and applying equations (2.7), these relations of (3.6) become

$$(3.8)\qquad (p,p)\ [-\operatorname{div}(p)\ T_{ij} + A_{ij}\ \operatorname{div}(p) + (u_{i,j} - u_{j,i})] +$$
$$+ B_{ij}\ (p,p) - \operatorname{div}(p)\ A_{ij}(p,p) = 0\ .$$

Now consider the expression

$$(3.9)\qquad S_{ij} = \operatorname{curl}(u_i) = u_{i,j} - u_{j,i}\ .$$

Since

$$(3.10)\qquad u_i = (p,p)\ \sigma_{,i} - (p, \tfrac{\partial\sigma}{\partial x})\ p_i\ ,$$

it is easily seen that

$$(3.11)\qquad S_{ij} = - (p, \tfrac{\partial\sigma}{\partial x})\ T_{ij} + [\sigma_{,i} \frac{\partial}{\partial x^j} - \sigma_{,j} \frac{\partial}{\partial x^i}]\ (p,p) +$$
$$+ A_{ij}\ (p, \tfrac{\partial\sigma}{\partial x})\ .$$

Solve equations (3.10) for the $\sigma_{,i}$, and substitute these into (3.11).

The result is

$$(3.12)\qquad S_{ij} = -\left(p,\frac{\partial\sigma}{\partial x}\right) T_{ij} - \frac{\left(p,\frac{\partial\sigma}{\partial x}\right)}{(p,p)} A_{ij}\,(p,p) - \frac{B_{ij}\,(p,p)}{(p,p)} + A_{ij}\left(p,\frac{\partial\sigma}{\partial x}\right).$$

Now substitute this expression for $S_{ij}$ into the relations (3.8). After simplification it results that

$$(3.13)\qquad (p,p)\,[-T_{ij} + A_{ij}]\left[\operatorname{div}(p) + \left(p, \frac{\partial\sigma}{\partial x}\right)\right] - \left[\operatorname{div}(p) + \left(p, \frac{\partial\sigma}{\partial x}\right)\right] A_{ij}\,(p,p) = 0 \,.$$

Recall the definition of the second order Lamé differential parameter. The equations (3.13) take on the form

$$(3.14)\qquad [-T_{ij} + A_{ij}]\; e^{-\sigma}\,\Delta_2(V) = e^{-\sigma}\,\Delta_2(V)\,A_{ij}\,\log(p,p).$$

If $\Delta_2(V) \neq 0$, these can be written as

$$(3.15)\qquad A_{ij}\,\log e^{-\sigma}\,\Delta_2(V) = A_{ij}\,\log(p,p) + T_{ij}\,.$$

THEOREM 3.1. *The set of* $\infty^{n-1}$ *curves* C *defined as the integral solutions of the system of* n *ordinary differential equations*

$$(3.16)\qquad \frac{dx^i}{ds} = \frac{g^{ij}\,p_j}{|p|}\,,$$

*is orthogonal to an isothermal family of* $\infty^1$ *hypersurfaces, each of deficiency one, if and only if the conditions* (1.8) *and*

$$(3.17)\qquad A_{ij}\,\log\frac{e^{\sigma}\,\Delta_2(V)}{\Delta_1(V)} = T_{ij}$$

*are identically satisfied whenever* $\Delta_2(V) \neq 0$.

This follows from (3.15) and use of the fact that $(p,p) = e^{-2\sigma}\,\Delta_1(V)$.

It is noted that in case $p_i$ is always a unit covariant vector this condition (3.17) becomes

$$(3.18)\qquad A_{ij}\,\log e^{-\sigma}\,\Delta_2(V) = T_{ij}\,.$$

For, in this case $(p,p) = +1$.

THEOREM 3.2. *A set of* $\infty^{n-1}$ *geodesics* C *is transversal to an isothermal family of* $\infty^1$ *hypersurfaces* $V(x) =$ *constant if and only if the conditions* (1.8) *and*

(3.19) $$A_{ij} \log e^{-\sigma} \Delta_2(V) = T_{ij} ,$$

*are valid for each such geodesic* C.

For, as p is displaced parallel to itself along such a geodesic, $(p,p)$ = constant. Thus (3.19) holds and the result follows.

## 4. NEAR CONSERVATIVE FIELDS OF FORCE AND ISOTHERMAL FAMILIES.

Consider a positional field of force $\Phi$, conservative or not. The simple family of $\infty^{n-1}$ Faraday lines of force C is composed of the $\infty^{n-1}$ integral solutions C of the system of n ordinary differential equations

(4.1) $$\frac{dx^i}{ds} = \frac{g^{ij}\,\Phi_j}{|\Phi|} = \frac{\Phi^i}{|\Phi|} ,$$

where s denotes the arc length along such a Faraday line of force C.

This family of $\infty^{n-1}$ Faraday lines of force C is a normal family if and only if it is orthogonal to a family of $\infty^1$ hypersurfaces, each of dimension (n-1), in $V_n$. In this case the conditions of Theorem 1.2, hold. Such a field of force $\Phi$, is termed a *near conservative field of force* $\Phi$.

Thus if $\Phi$ is a near conservative field of force $\Phi$, there exists an associated function $W^*(x)$, and an absolute scalar function $\sigma(x)$, such that

(4.2) $$\Phi_i = e^{-\sigma} \frac{\partial W^*}{\partial x^i} ,$$

and

(4.3) $$\Phi_i\, T_{jk} + \Phi_j\, T_{ki} + \Phi_k\, T_{ij} = 0 ,$$

the expressions (4.3) being identities. Here $T_{ij}$ is the curl of the force vector $\Phi$.

It is noted that the work W in a near conservative field of force, is given by

(4.4) $$W = W(x;C) = \int_1^2 e^{-\sigma} \frac{\partial W^*}{\partial x^i}\, dx^i ,$$

and is in general a functional depending on the particular path C of integration.

THEOREM 4.1. *A field of force $\Phi$ is near conservative if and only if it obeys the identities* (4.3).

A field of force is said to be *isothermal near conservative* if and only if its associated family of $\infty^1$ hypersurfaces $W^*(x)$ = constant, form an isothermal family.

THEOREM 4.2. *A near conservative field of force $\Phi$ is isothermal near conservative if and only if the force vector $\Phi$ and the associated function $W^*(x)$ obey the conditions*

$$A_{ij} \log \frac{e^{\sigma}\, \Delta_2(W^*)}{\Delta_1(W^*)} = T_{ij}\,, \tag{4.5}$$

*where $A_{ij}$ is the linear differential operator defined by* (3.7), *where $p_i$ has been replaced by $\Phi_i$.*

This follows from previous results.

A field of force $\Phi$ is *ultra solenoidal* if and only if

$$\frac{\partial}{\partial x^i} \frac{\operatorname{div}(\Phi)}{(\Phi,\Phi)} = \lambda\, \Phi_i\,, \tag{4.6}$$

where $\lambda$ is an absolute scalar. That is $\Phi$ is ultra solenoidal if and only if the gradient of the absolute scalar $\operatorname{div}(\Phi)/(\Phi,\Phi)$, is parallel to the force vector $\Phi$ .

It follows immediately that a field of force is ultra solenoidal if and only

$$A_{ij} \frac{\operatorname{div}(\Phi)}{(\Phi,\Phi)} = 0. \tag{4.7}$$

Upon expanding this condition it is easily seen that another form is

$$(\Phi,\Phi)\, A_{ij} \operatorname{div}(\Phi) = \operatorname{div}(\Phi)\, A_{ij}\, (\Phi,\Phi). \tag{4.7}$$

However it is known that the condition for an isothermal near conservative field of force may be written in the form

$$(\Phi,\Phi)\, [- T_{ij} + A_{ij}]\, [\operatorname{div}(\Phi) + (\Phi, \tfrac{\partial\sigma}{\partial x})] - \tag{4.9}$$

$$- [\operatorname{div}(\Phi) + (\Phi, \tfrac{\partial\sigma}{\partial x})]\, [A_{ij}\, (\Phi,\Phi)] = 0.$$

Now a field of force is said to be *ultra Laplacean* if and only if it is both near conservative and ultra solenoidal.

THEOREM 4.2. *A field of force* $\Phi$ *is ultra Laplacean if and only if obeys the identities*

$$\Phi_i T_{jk} + \Phi_j T_{ki} + \Phi_k T_{ij} = 0, \tag{4.10}$$

*and*

$$A_{ij} \frac{(\Phi, \frac{\partial\sigma}{\partial x})}{(\Phi,\Phi)} = \frac{e^{\sigma} \Delta_2(V)}{\Delta_1(V)} T_{ij} \tag{4.11}$$

Thus it follows that near conservative fields of force which are solenoidal or near solenoidal are *improper* ultra Laplacean fields of force.

## 5. NEAR CONSERVATIVE FIELDS OF FORCE AND PARALLEL FAMILIES.

Consider a near conservative field of force $\Phi$. Suppose that the associated family of $\infty^1$ hypersurfaces $W^*(x)$ = constant forms a parallel family [4]. If

$$\Phi_i = e^{-\sigma(x)} W^*_{,i} , \tag{5.1}$$

the condition for this is

$$\Delta_1(W^*) = g^{ij} W^*_{,i} W^*_{,j} = H(W^*) \tag{5.2}$$

This means that the gradient of $\Delta_1(W^*)$, must be parallel to the force vector $\Phi$. Hence

$$\frac{\partial}{\partial x^i} e^{2\sigma} (\Phi,\Phi) = \lambda_1 \Phi_i , \tag{5.3}$$

where $\lambda_1$ is an absolute scalar.

Expanding the condition (5.3), it is seen that

$$\frac{\partial}{\partial x^i} (\Phi,\Phi) + 2 \frac{\partial\sigma}{\partial x^i} (\Phi,\Phi) = \lambda_2 \Phi_i , \tag{5.4}$$

where $\lambda_2 = e^{-2\sigma}\lambda_1$ is an absolute scalar.

However if $u_i = g^{ab} \Phi_a T_{ib}$, it is recalled from section two that

$$\frac{\partial\sigma}{\partial x^i} = \frac{(\Phi, \frac{\partial\sigma}{\partial x})}{(\Phi,\Phi)} \Phi_i + \frac{u_i}{(\Phi,\Phi)} , \tag{5.5}$$

since the field of force $\Phi$ is near conservative and its lines of force therefore form a normal family.

Use of this result in equations (5.4), yields

(5.6)
$$\frac{\partial}{\partial x^i}(\Phi,\Phi) + 2\, u_i = \lambda_3\, \Phi_i$$

where $\lambda_3 = \lambda_2 - \dfrac{(\Phi, \frac{\partial\sigma}{\partial x})}{(\Phi,\Phi)}$, is again an absolute scalar.

THEOREM 5.1. *A near conservative field of force $\Phi$, in Riemannian space $V_n$ has as its associated family of $\infty^1$ hypersurfaces, a parallel family if and only if*

(5.7)
$$A_{ij} \log e^{-2\sigma} \Delta_1(W^*) = - T_{ij}$$

A field of force $\Phi$ in $V_n$, which obeys the conditions of Theorem 5.1, is called a *parallel near conservative field of force* $\Phi$.

Suppose that a field of force $\Phi$ is parallel near conservative. In addition let it be assumed that $\Phi$ is ultra solenoidal. Then the following result is valid.

THEOREM 5.2. *A parallel near conservative field of force $\Phi$ is ultra solenoidal if and only if it obeys*

(5.8)
$$A_{ij} \log \mathrm{div}(\Phi) + T_{ij} = 0$$

*where $T_{ij}$ is the curl of the force vector $\Phi$ .*

This follows from Theorem 5.1, and the conditions (4.8).

## 6. CARTOGRAMS AND NEAR CONSERVATIVE FIELDS OF FORCE [5] .

Suppose that a Riemannian space $V_n$ is a conformal image of a Riemannian space $\overline{V}_n$, by means of the cartogram T defined by

(6.1)
$$d\bar{s} = e^{\frac{\sigma}{2}}\, ds$$

In $V_n$, consider the near conservative field of force defined by

(6.2)
$$\Phi_i = e^{-\sigma}\, W^*_{,i}$$

where $W^*(x)$ is the associated function of the near conservative field of force $\Phi$, and $\sigma$ is an absolute scalar function.

Now under a conformal cartogram $d\bar{s} = e^{\mu}\, ds$, it is known that the fundamental metric tensor $g_{ij}$ transforms by the rules

(6.3) $$\bar{g}_{ij} = e^{2\mu} g_{ij} \quad , \quad \bar{g}^{ij} = e^{-2\mu} g^{ij}$$

Thus for the cartogram $d\bar{s} = e^{-\frac{\sigma}{2}} ds$, we have

(6.4) $$\bar{\Phi}_i = \bar{g}_{ij} \bar{\Phi}^j = e^{\sigma} g_{ij} \Phi^j = e^{\sigma} \Phi_i = \frac{\partial W^*}{\partial x^i}$$

THEOREM 6.1. *A near conservative field of force $\Phi$, in a Riemannian space $V_n$, is a conformal image of a conservative field of force in a Riemannian space $\bar{V}_n$, which is related to $V_n$ by the conformal cartogram* $d\bar{s} = e^{\sigma/2} ds$. *In $V_n$, the associated function* $W^*(x)$ *of the near conservative field of force $\Phi$, is the work function for the conservative field of force $\bar{\Phi}$. Here*

(6.5) $$\bar{\Phi}_i = \frac{\partial W}{\partial x^i} \quad .$$

Now recall that a conservative field of force, which is ultra solenoidal is termed a near Laplacean field of force.

THEOREM 6.2. *An ultra Laplacean field of force $\Phi$ is a conformal image of a near Laplacean field of force.*

For, by Theorem 6.1, a near conservative field of force $\Phi$ is a conformal image of a conservative field of force $\bar{\Phi}$ in a conformally related $\bar{V}_n$. Then since angle is preserved by a conformal map it follows that the vector

(6.6) $$\frac{\partial}{\partial x^i} \frac{\mathrm{div}(\Phi)}{(\Phi,\Phi)}$$

will have its image in $\bar{V}_n$, parallel to the force vector $\bar{\Phi}_i$ in $\bar{V}_n$. Thus from above, the ultra solenoidal property is preserved by a conformal map. The ultra Laplacean field of force is then an image of a near Laplacean field of force $\bar{\Phi}$ in a related $\bar{V}_n$ and the result is established.

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