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EIGENVECTORS AND CYCLIC VECTORS FOR BILATERAL WEIGHTED SHIFTS

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1. INTRODUCTION

In what follows a class of bilateral weighted shifts operators on Banach spaces is defined. Let B be one of these operators; the following results are proven to be true:

1) There exist two vectors, f,g, such that the linear span of $\{B^k f,\ B^k g:\ k = 0,1,2,\dots\}$ is dense in the whole space; moreover, if B satisfies certain additional conditions, then the intersection of the (closed) invariant subspaces generated by

$$\{B^k f:\ k = 0,1,\dots\} \text{ and } \{B^k g:\ k = 0,1,\dots\}$$

is equal to $\{0\}$, whenever f and g are suitably chosen.

2) If B has an eigenvector, then it also has a cyclic vector;

3) If the adjoint B* of B has an eigenvector, then B has no cyclic vectors;

4) If either B or B* has an eigenvector, then B has no algebraically complementary invariant subspaces, no roots and no logarithm.

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1. INTRODUCTION

Let C_0 be the Banach space of all complex (two-sided) sequences $\{c_n\}$ ($n \in Z$, the set of all integers) converging to zero in both directions, and with the sup. norm. ℓ_1 is the subset of all those sequences in C_0 such that $\sum_{n=-\infty}^{+\infty} |c_n| < \infty$ and this sum is the norm of that sequence in ℓ_1.

Throughout this paper, X will denote an intermediate space between C_0 and ℓ_1; i.e., (see [1],[7])

(1) $$\ell_1 \subset X \subset C_0$$

and, whenever T is a (bounded linear) operator on C_0 and ℓ_1, its restriction to X is also bounded there. Further, we shall assume that the sequence $\{e_m\}$ ($m \in Z$), where $e_m = \{\delta_{mn}\}$ ($n \in Z$) ($\delta_{mn} = 0$, if $m \neq n$; $\delta_{mm} = 1$) is a *Schauder basis* for X in the sense that, if $g = \{c_n\} \in X$, then

(2) $$\lim \| g - \sum_{m=-k}^{+k'} c_m e_m \| = 0 \quad (k, k' \to +\infty).$$

Using (2), g can be written as $g = \sum c_n e_n$.

It is well known ([7, *Chap.1*]) that an intermediate space X can be re-normed with an equivalent norm so that, whenever an operator T satisfies the two inequalities

(3) $$\|T\|_{(C_0)} \leqslant 1, \ \|T\|_{(\ell_1)} \leqslant 1, \text{ then } \|T\|_{(X)} \leqslant 1.$$

Assume that the X-norm satisfies (3) and let W be a *unitary* operator on C_0 and ℓ_1, simultaneously (i.e., W is an *isometric* operator mapping each of those spaces *onto* itself). Then W is also a unitary operator on X; moreover, W^* (= the adjoint of W acting on the dual space X^* of X) is a unitary operator too.

Applying these results to the operators W and V defined (on C_0 and ℓ_1) by

$$We_n = e_{\sigma(n)} \quad (n \in Z),$$

where σ is a "permutation" (i.e., σ is a bijective map of Z), and

$$Ve_n = \lambda_n e_n \quad (n \in Z),$$

where λ_n is a complex number of modulus one, for all $n \in Z$, we get

$$0 \neq \|e_n\| = \|e_m\|, \text{ for all } n,m \in Z,$$

and

$$\|\sum c_n e_n\| = \|\sum |c_n| e_n\|,$$

for all X-norm $\|\cdot\|$.

Hence, without loss of generality we can assume (and we shall!) that the X-norm satisfies (3) and, moreover, $\|e_n\| = 1$, for all $n \in Z$.

A *bilateral weighted shift* (B.W.S.) B on X is an operator defined by

$$B \sum c_n e_n = \sum c_n w_n e_{n+1}, \tag{4}$$

where w_n ($n \in Z$) is a bounded (two-sided) sequence of *non-zero* complex numbers. Since B is clearly bounded and linear on C_0 and ℓ_1 (for every bounded sequence $\{w_n\}$), then it is so on X.

2. THE SPECTRUM OF B ACTING ON X.

LEMMA 1 ([3, *thm. 6*]). *The annulus (or disc)*

$$D = \{z: R_2 \leqslant |z| \leqslant R_1\},$$

where

$$R_1 = \lim_{n\to+\infty} (\sup_m . \prod_{j=m+1}^{m+n} |w_j|)^{1/n} \quad \textit{and}$$

$$R_2 = \lim_{n\to+\infty} (\inf_m . \prod_{j=m+1}^{m+n} |w_j|)^{1/n},$$

is contained in $\sigma(B)$.

LEMMA 2 ([3, *thm.10*]). *If* $X = C_0$ *or* $X = \ell_1$, *then*

$$\sigma(B) = D.$$

COROLLARY 3. $\sigma(B) = D$, *for every intermediate subspace* X *satisfying* (2).

Proof. Let $\lambda \notin D$, then (by *Lemma 2*) $(B - \lambda)$ and $(B - \lambda)^{-1}$ are bounded on C_0 and ℓ_1; hence, they are bounded on every intermediate space X.

It follows that

$$\sigma(B) \subset D.$$

The converse inclusion is the *lemma 1*. q.e.d.

The dual space of C_0 (ℓ_1, resp.) is equal to ℓ_1 (ℓ_∞, the Banach space of all bounded sequences under the sup. norm, resp.). Thus, the dual space X^* of X (satisfying (1)) is intermediate between ℓ_1 and ℓ_∞:

(1') $$\ell_1 \subset X^* \subset \ell_\infty$$

Moreover, if $e_m^* = \{\delta_{mn}\}$ $(n \in Z)$ (though as an element of X^*), then $(e_n, e_m^*) = \delta_{nm}$, and the set of (bounded linear) functionals $\{e_m^*\}$ $(m \in Z)$ is *total* in the sense that

(5) $g \in X$, $(g, e_m^*) = 0$, for all $m \in Z$, implies $g = 0$.

It is also clear that $\|e_m^*\|^* = 1$, for all $m \in Z$.

The finite linear combinations of the e_m^*'s are not dense in X^* in general (e.g., if $X = \ell_1$, $X^* = \ell_\infty$); however, every element $f \in X^*$ can be represented as

$$f = \sum c_n e_n^* ,$$

where the sum is understood as a weak* limit.

Let B be a B.W.S. on X; as in the case when $X = \ell_2$ ([5, *prob. 75*]; see also [3], [4]), it is not hard to check that the point spectrum of B and B^* is invariant under rotations about the origin.

Since kernel $(B) = \{0\}$, every eigenvalue of B has positive modulus; thus, if $\lambda \in \sigma_p(B)$, then $\lambda \neq 0$ and $\sigma_p(B)$ contains the circle of radius $|\lambda|$. The eigenvalues of B are simple; to be more precise, the only eigenvectors of B with eigenvalue λ are the multiples of

(6) $$g_\lambda = e_0 + \sum_{n=1}^{\infty} \left(\prod_{j=0}^{n-1} w_j\right) \lambda^{-n} e_n + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} w_{-j}\right)^{-1} \lambda^n e_{-n} .$$

The analogous results are true for B^* and every eigenvector of B^* with eigenvalue λ is a multiple of

(6') $$h_\lambda = e_0^* + \sum_{n=1}^{\infty} \left(\prod_{j=0}^{n-1} w_j\right)^{-1} \lambda^n e_n^* + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} w_{-j}\right) \lambda^{-n} e_{-n}^* =$$

$$= \sum_{n=-\infty}^{+\infty} a_n \lambda^n e_n^*.$$

In the general case, $\sigma_p(B)$ ($\sigma_p(B^*)$) is a (possibly empty) annulus about the origin containing both, one or none of the two circles that form the boundary; this annulus could degenerate to a (closed or open) pointed disc, or to a circle. For example, if $w_n = [(n+1)/(n+2)]^2$ ($n = 0,1,2,\ldots$), $w_{-n} = [(n+1)/n]^2$ ($n = 1, 2,3,\ldots$), then $\sigma_p(B) = \sigma(B) = \sigma(B^*)$ is the unit circle.

Assume that $\sigma_p(B^*) \neq \phi$; without loss of generality we can assume that

(7) $$\sigma_p(B^*) \supset \Pi = \{z: |z| = 1\}.$$

Let $g = \sum c_n e_n \in X$ and define

(8) $$g \to g(\lambda) = (g,h_\lambda) = \sum c_n a_n \lambda^n$$

(the series converges absolutely; to see this, use the results of *sect.1*).

For fixed g, the function $g(\lambda)$ is continuous on $\sigma_p(B^*)$ and analytic on the interior of this set (to see this, use (2) and (6')); moreover

(9) $$|g(\lambda)| = |(g,h_\lambda)| \leqslant \|g\| . \|h_\lambda\|^*$$

and the convergence of a sequence in X implies the convergence (uniformly on compact subsets of $\sigma_p(B^*)$) of the corresponding functions (given by the map (8)).

Let $L: X \to C(\Pi)$ (the Banach algebra of all continuous functions on Π, under the sup. norm $\|.\|_\infty$) be the linear map defined by (8) when λ is restricted to the unit circle. The following facts can be easily checked:

(10) i) $L e_n = a_n^{-1}\lambda^n$; therefore $L(X)$ is dense in $C(\Pi)$;

ii) $\|L\| = \|h_1\|^* = K^*$. (use (9) and the results of *sect.1*)

iii) L is one-to-one (use (6'), (8) and the uniqueness property of the Fourier series);

iv) $(LBg)(\lambda) = \lambda g(\lambda)$, for all $g \in X$.

3. EIGENVECTORS AND CYCLIC VECTORS.

THEOREM 4. *If* $\sigma_p(B^*) \neq \phi$, *then* B *has no cyclic vectors.*

Proof. Without loss of generality we can assume that (7) holds. Assume that the statement is false; then there exists $g \in X$ such that $X = \bigvee_{k=0}^{\infty} B^k g$ (where the sign "$\bigvee$" means "the closed linear span of"). Since (by (10), i)) $L(X)$ is dense in $C(\Pi)$, it follows from ((10),ii)) that the finite linear combinations

$$\sum_{k=0}^{N} \lambda^k g(\lambda)$$

are dense in $C(\Pi)$; in other words, $g(\lambda)$ is a cyclic vector for S = "multiplication by λ" on $C(\Pi)$. To prove the theorem we only have to observe that S cannot have a cyclic vector; in fact, if $g(\lambda) \in C(\Pi)$ and $M = \bigvee_{k=0}^{\infty} S^k g(\lambda)$, then,

a) If $g(\lambda_0) = 0$, for some $\lambda_0 \in \Pi$, then M is contained in the maximal *proper* ideal $\{f \in C(\Pi): f(\lambda_0) = 0\}$; or else,

b) If $g(\lambda)$ never vanishes on Π, then $M = \{fg: f \text{ ranges over } A(\Pi)\} = gA(\Pi)$,

where $A(\Pi)$ is the closure in $C(\Pi)$ of the analytic trigonometric polynomials; then $A(\Pi) = g^{-1} M$.

In either case, $M \subsetneq C(\Pi)$.

THEOREM 5. *If* $\sigma_p(B) \neq \phi$, *then* B *has a cyclic vector.*

Proof. Without loss of generality we can assume that

$$\sigma_p(B) \supset \Pi \tag{7'}$$

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a dense subset of Π such that every λ_n has a rational argument; then $\overline{\lambda}_h \lambda_n$ is a primitive root of the identity of order k, where $k = 0$ if and only if $h = n$. Fix h and n; for large values of m, we shall certainly have

$$(1/m!) \sum_{j=1}^{m!} (\overline{\lambda}_h \lambda_n)^j = \delta_{hn}; \tag{11}$$

moreover, the absolute value of the first member of (11) is less than or equal to one for all values of m.

Let $g_n = g_{\lambda_n}$ be the eigenvector of B given by (6) and set

$$g = \sum_{n=1}^{\infty} c_n g_n,$$

where $\{c_n\}_{n=1}^{\infty}$ is any summable sequence of positive reals. By (11) we have

$$(c_h m!)^{-1} \sum_{j=1}^{m!} (\bar{\lambda}_h B)^j g = g_h + \sum_{n=M}^{\infty} \{(1/c_h m!) \sum_{j=1}^{m!} (\bar{\lambda}_h \lambda_n)^j g_n\},$$

where $h < M \to \infty$, as $m \to \infty$. This expression shows that $g_h \in \bigvee_{k=0}^{\infty} B^k g$, for all values of $h = 1,2,3,\ldots$.

Let $f \in X^*$ be an element of the annihilator of $\bigvee_{k=0}^{\infty} B^k g$; by the previous result, $(g_h, f) = 0$, for all h.

Since $\sigma_p(B) \supset \Pi$ we can repeat the above construction defining a continuous linear map L': $X^* \to C(\Pi)$ by means of

(8') $\qquad h \to h(\lambda) = (g_\lambda, h), \quad h \in X^*$

(in fact, (2) and (6) show that the map $\lambda \to g_\lambda$ is continuous). This map is well defined and it enjoys all the properties (10) (with the obvious changes); in particular, we have:

(10') iii) $\qquad L'$ is one-to-one

Applying this result to the continuous function $f(\lambda)$, which (since $(g_h, f) = 0$, $h = 1,2,\ldots$) has a zero at every point of the *dense* subset $\{\lambda_n\}_{n=1}^{\infty}$ of Π, we conclude that $f = 0$ and therefore

$X = \bigvee_{k=0}^{\infty} B^k g.$

q.e.d.

The unequal behavior of B according to $\sigma_p(B^*) \neq \phi$ or $\sigma_p(B) \neq \phi$ yields the following result:

COROLLARY 6. *Let* B *be a* B.W.S. *on* X *and let* B^* *be its adjoint (acting on* X^**); then either* $\sigma_p(B) = \phi$ *or* $\sigma_p(B^*) = \phi$.

If $X^* \subset C_0$, then the (unweighted) bilateral shift U (defined by $Ue_n = e_{n+1}$) provides an example of a B.W.S. for which *both* $\sigma_p(U)$

and $\sigma_p(U^*)$ are empty sets. On the other hand, as it is well known (see, e.g., [6]) U and U* acting on ℓ_2 have (the same!) cyclic vectors. This example suggests the following question: is it true, for an arbitrary B.W.S. B acting on a space X such that X* is *separable*, that either B has a cyclic vector, or B* has a cyclic vector?.

If we consider the same operator U* acting on ℓ_∞ (= ℓ_1^*), then $\sigma_p(U^*) = \Pi$ and the eigenvectors of U* with eigenvalue λ are the multiples of $h_\lambda = \sum \bar{\lambda}^n e_n^*$; a short analysis of these eigenvectors shows that

$$\|h_\lambda - h_{\lambda'}\| \geqslant \sqrt{3}, \text{ for all } \lambda \neq \lambda' .$$

I.e., the map $\lambda \to h_\lambda$ (defined by (5')) is continuous at no point of Π!.

4. THE MULTIPLICITY OF A B.W.S. - INVARIANT SUBSPACES.

Let P^+ (P^-, resp.) denote the projection of X onto $X^+ = \bigvee \{e_n : n \geqslant 0\}$ ($X^- = \bigvee \{e_n : n < 0\}$, resp.) defined by

$$P^+ \sum c_n e_n = \sum_{h=0}^{\infty} c_n e_n \quad (P^- = I - P^+),$$

and let T^+, T^- be the operators defined on X^+, X^-, resp., by

$$T^+ g = Bg \quad (g \in X^+), \quad T^- g = P^- Bg \quad (g \in X^-)$$

It is clear that (up to an isometric isomorphism) $(X^+)^* = P^+ X^*$ and $T^{+*} = P^+ B^*$ (on X^{+*}); similarly we can write $T^{-*} = B^*$ restricted to $X^{-*} = P - X^*$.

Following R. Gellar ([4]), we shall define

$$I^+ = \liminf_{n \to +\infty} \left(\prod_{j=0}^{n-1} w_j\right)^{1/n}, \quad Q^+ = \limsup_{n \to +\infty} \left(\prod_{j=0}^{n-1} w_j\right)^{1/n}$$

$$I^- = \limsup_{n \to +\infty} \left(\prod_{j=1}^{n} w_{-j}\right)^{1/n}, \quad Q^- = \liminf_{n \to +\infty} \left(\prod_{j=1}^{n} w_{-j}\right)^{1/n} .$$

We have $R_2 \leqslant I^+ \leqslant Q^+ \leqslant R_1$, $R_2 \leqslant Q^- \leqslant I^- \leqslant R_1$.

Even in the case when $\sigma_p(B^*) = \phi$, to every $g \in X$ we can associate a *formal* Laurent series

(8") $$g(\lambda) = \sum c_n a_n \lambda^n$$

(i.e., $g(\lambda)$ is defined in such a way that if $\lambda \in \sigma_p(B^*)$, then $g(\lambda)$ coincides with the expression (8)); the correspondence $g \to g(\lambda)$ is clearly injective and $(Bg)(\lambda) = \lambda g(\lambda)$.

THEOREM 7 ([4, *thm. 11*]). *If* $f \in X$ *then*

i) $f^+(\lambda) = (P^+f)(\lambda)$ *converges to an analytic function in the region* $|z| < I^+$.

ii) $f^-(\lambda) = (P^-f)(\lambda)$ *converges to an analytic function in the region* $|z| > I^-$.

iii) If $I^- < I^+$, *then convergence in* X *implies uniform convergence of the associated analytic functions on compact subsets of the region* $I^- < |z| < I^+$.

The region $I^- < |z| < I^+$ is precisely the *interior* of $\sigma_p(B^*)$. Similarly, the regions $|z| < I^+$ and $|z| < Q^-$ coincide with the *interior* of $\sigma_p(T^{+*})$ and $\sigma_p(T^-)$, respectively.

We recall that an operator R is called *unicellular* if its lattice of invariant subspaces is linearly ordered by inclusion ([5]). It is not hard to conclude from the above definition that a B.W.S. B on X is unicellular if and only if its lattice of invariant subspaces consists of *exactly* the following elements:

(12) $$\{0\}, X, X_m = \bigvee\{e_n : n \geqslant m\} = \bigvee_{k=0}^{\infty} B^k e_m \quad (m \in Z).$$

No example of such operator is known. Here we shall establish a *necessary* condition for the unicellularity of a B.W.S.

COROLLARY 8. *If* B *is a unicellular* B.W.S., *then*

$$I^+ = Q^- = 0.$$

In particular, an invertible B.W.S. *cannot be unicellular.*

Proof. If $\lambda_0 \in \sigma_p(T^{+*})$ and h_{λ_0} is a non-zero eigenvector of T^{+*}

with eigenvalue λ_0, then

$$M_{\lambda_0} = \{g \in X^+;\ (g,h_{\lambda_0}) = 0\} = \{g \in X^+ : g(\lambda_0) = 0\}$$

is an invariant subspace for T^+; hence M_{λ_0} is an invariant subspace for B and it is not hard to check that if $\lambda_0 \neq 0$, then M can not be of the form (12).

Similarly, if $g_0 \neq 0$ is an eigenvector of T^- with eigenvalue $\lambda_0 \neq 0$, then

$$C_{g_0} = \bigvee_{k=0}^{\infty} B^k g_0$$

is an invariant subspace of B, not of the form (12). In fact, $P^- C_{g_0} = \bigvee g_0$ is a *one-dimensional* subspace of X^-; since $\lambda_0 \neq 0$, g_0 cannot be a multiple of e_{-1}; therefore C_{g_0} is a non-trivial invariant subspace and $C_{g_0} \neq X_m$, for all $m \in Z$.

Therefore, if B is unicellular, $\sigma_p(T^{+*}) = \sigma_p(T^-) = \{0\}$; now, by the observations following *thm.7*, this is equivalent to: $I^+ = Q^- = 0$.

q.e.d.

Cor. 8 also shows that if $I^+ > 0$ or $Q^- > 0$, then B has "many" invariant subspaces (the lattice of invariant subspaces of B has the power of the continuum!). Now we want to show that some of these subspaces have an invariant *topological* complement (however, as we shall see in the next section, an invariant subspace of B has no invariant *algebraic* complement, in general). The next *lemma* has some interest in itself: it says (in the terminology of [8]) that the multiplicity of a B.W.S. cannot be greater than 2.

LEMMA 9. *If* B *is a* B.W.S. *on* X, *then there exists a vector* $g \in X^-$ *such that*

$$X = \bigvee_{k=0}^{\infty} \{B^k e_0,\ B^k g\}\ .$$

Proof. A simple modification of the argument given in [5; *prob. 126*] shows that a "backward" unilateral weighted shift operator always has a cyclic vector; since T^- (acting on X^-) belongs to this class of operators, it follows that there exists $g \in X^-$

such that

$$X^- = \bigvee_{k=0}^{\infty} (T^-)^k g.$$

Assume that g satisfies the above condition; since $X^+ = \bigvee_{k=0}^{\infty} B^k e_0$, we have that

$$(T^-)^n g = P^- B^n g = B^n g - P^+ B^n g$$

belongs to the subspace spanned by $\{B^k e_0, B^k g : k = 0,1,2,\ldots\}$ for all $n = 0,1,2,\ldots$, and therefore

$$X = X^- \oplus X^+ = X^- \vee X^+ \subset \bigvee_{k=0}^{\infty} \{B^k e_0, B^k g\} \subset X.$$

q.e.d.

THEOREM 10. *Let* B *be a* B.W.S. *such that* $\sigma_p(B^*) \supset \Pi$ *and let* $C_f = \bigvee_{k=0}^{\infty} B^k f$ *be the cyclic invariant subspace generated by an element* f *of* X.

Assume that $f, g \in X$ *satisfy the conditions:*

1) The continuous functions $f(\lambda)$, $g(\lambda)$ $(\lambda \in \Pi)$ *never vanish;*

2) $f(\lambda)\, g^{-1}(\lambda)$ *are not the radial limits. (a.e. with respect to the Lebesgue measure) of a function meromorphic on* $|z| < 1$.

Then

$$C_f \cap C_g = \{0\}. \tag{13}$$

Proof. The modulus maximum theorem, inequality (9) and 1) imply that if $h \in C_f$ then

$$h(\lambda) = f(\lambda) h_1(\lambda) \quad (\lambda \in \Pi)$$

where $h_1(\lambda) = \lim h_1(r\lambda)$ $(0 \leqslant r < 1,\ r \to 1)$, $h_1(z)$ being a function analytic on $|z| < 1$.

Let $0 \neq h \in C_f \cap C_g$; the above result shows that

$$h(\lambda) = f(\lambda) h_1(\lambda) = g(\lambda) h_2(\lambda) \not\equiv 0,$$

where $h_1(z)$ and $h_2(z)$ are two bounded and non-identically zero functions analytic on $|z| < 1$. Thus (see, e.g. [6]), the radial limits

$$h_2(\lambda) h_1^{-1}(\lambda) = f(\lambda) g^{-1}(\lambda)$$

are well-defined almost everywhere on Π; but the function $h_2(z) h_1^{-1}(z)$ is meromorphic on $|z| < 1$, contradicting 2). This contradiction proves that $h(\lambda) \equiv 0$ and therefore, by (10), iii), $h = 0$, which proves (13).

q.e.d.

COROLLARY 11. *Let* B *be a* B.W.S. *on* X *such that* $\sigma_p(B^*) \supset \Pi$ *and let* $g \in X$ *be a vector satisfying the conditions:*

1) $P^- g$ *is a cyclic vector for* T^- *(acting on* X^-*);*
2) $g(\lambda)$ *never vanishes* $(\lambda \in \Pi)$ *and*
3) $g(\lambda)$ *are not the radial limits (a.e.) of a function meromorphic on* $|z| < 1$, *or*

3') $g(\lambda)$ *is the restriction to* Π *of a function analytic on some neighborhood of the unit circle.*

Then

$$C_g \cap X_m = \{0\}, \quad C_g \vee X_m = X,$$

for every $m \in Z$.*However, the algebraic direct sum* $C_g \oplus X_m$ *is never closed.*

Proof. To see that $C_g \vee X_m = X$ we only have to repeat the *proof of lemma 9* with minor changes (use 1).

Condition 2) implies (as in the *proof* of *thm.10)*, that if $C_g \cap X_m \neq \{0\}$, then

$$g(\lambda) = \lambda^m h_1(\lambda) h_2^{-1}(\lambda),$$

where $h_1(z)$ and $h_2(z)$ are two bounded and non-identically zero functions, analytic on $|z| < 1$. If g satisfies 3'), the above expression shows that there exist a polynomial $p(z)$ such that $p(z) g(z)$ is analytic on $|z| < 1$; but this implies that the dimension of the subspace

$$P^- C_g = \bigvee_{k=0}^{\infty} (T^-)^k g^-$$

of X^- cannot be larger than degree $(p) < \infty$, contradicting 1).

We concluded that 1), 2), and 3') imply 3); now, if g satisfies 1), 2) and 3), then $C_g \cap X_m$ must be equal to $\{0\}$ by *thm. 10*.

Finally, observe that if $C_g \oplus X_m$ is closed, then

$$X = C_g \oplus X_m = C_g \oplus X_{m-1};$$

hence $e_{m-1} = f + h$, where $f \in C_g$, $h \in X_m$, and therefore e_{m-1} has *two different expressions* as an element of the direct sum

$C_g \oplus X_{m-1}$; this contradiction shows that $C_g \oplus X_m$ cannot be closed in X.

q.e.d.

The hypothesis $\sigma_p(B^*) \neq \phi$ is sufficient but not necessary for the existence of two topologically, but not algebraically, complementary invariant subspaces.

EXAMPLE. If U is the (unweighted) bilateral shift acting on ℓ_2, then U is unitarily equivalent to multiplication by λ on $L^2(\Pi, dm)$ (where dm denotes the normalized Lebesgue measure on the unit circle); if $f(\lambda)$ is a function of modulus one (a.e., dm) and $g(\lambda)$ is the characteristic function of a measurable set $E \subset \Pi$ such that $0 < m(E) < 1$, then

$$C_f \cap C_g = \{0\} \quad , \quad C_f \vee C_g = L^2,$$

but $C_f \oplus C_g$ is not closed in L^2 (the proof follows from the results contained in [6]).

CONJECTURE. If B is a B.W.S. such that $I^+ > 0$, then B has two topologically, but not algebraically, invariant subspaces.

We are going to close this section with a more precise relation between (8) and (8").

LEMMA 12. *Assume that for every* $g \in X$ *the series* (8") *is Cesàro summable to a finite limit for* $\lambda = \lambda_0$; *then* $\lambda_0 \in \sigma_p(B^*)$.

Proof. The operators C_N defined by

$$C_N\, g = g_N = \sum_{n=-N}^{+N} c_n \left(1 - \frac{|n|}{N+1}\right) e_n, \quad (N=1,2,\ldots)$$

have finite dimensional range and therefore they are bounded on X (moreover, $\|C_N\| = 1$, for all N). It follows that the linear functionals

$$j_N(g) = g_N(\lambda_0)$$

are also bounded.

By hypothesis, $j_N(g) = g_N(\lambda_0)$ converges to a finite limit; hence for every $g \in X$ there is a constant K_g such that

$$|j_N(g)| \leq K_g \quad , \text{ for all } N.$$

From this and the uniform boundedness principle, we conclude that

$$\| j_N \| \leqslant K ,$$

for some positive constant K, independent of N.

Hence

$$| j_N(g) | = | g_N(\lambda_0) | \leqslant K \| g \|$$

and therefore

$$| g(\lambda_0) | \leqslant K \| g \| , \text{ for all } g \in X.$$

Let $h^*_{\lambda_0} \in X^*$ be the bounded linear functional defined by $g \to g(\lambda_0)$; then $g(\lambda_0) = (g, h^*_{\lambda_0})$, for all $g \in X$ (it is clear that $h^*_{\lambda_0} \neq 0$), and therefore

$$(g, \lambda_0 h^*_{\lambda_0}) = \lambda_0 (g, h^*_{\lambda_0}) = \lambda_0 g(\lambda_0) = (Bg)(\lambda_0) =$$

$$= (Bg, h^*_{\lambda_0}) = (g, B^* h^*_{\lambda_0}) \ ;$$

i.e., $B^* h^*_{\lambda_0} = \lambda_0 h^*_{\lambda_0}$ and therefore $\lambda_0 \in \sigma_p(B^*)$

q.e.d.

It is not difficult to conclude (using the results of *sects.1 and 2*) that, if $g(\lambda_0)$ (given by (8")) is Cesàro summable for all $g \in X$ and for some $\lambda_0 \in \Pi$, then $\Pi \subset \sigma_p(B^*)$ and $g(\lambda) \in C(\Pi)$.

5. OPERATORS COMMUTING WITH B.

Assume that $\sigma_p(B) \supset \Pi$ and let $R \in \underline{A}'_B$, the commutant of B; then

$$0 = (BR - RB) g_\lambda = (B - \lambda) \, R g_\lambda .$$

Since every $\lambda \in \sigma_p(B)$ is a *simple* eigenvalue, the above equality implies

$$R g_\lambda = c(R, \lambda) g_\lambda , \tag{14}$$

for some complex number $c(R,\lambda)$. An elementary analysis of the results of *sects.2-3* shows that

THEOREM 13. *If* $\sigma_p(B) \supset \Pi$, *to every operator* R *commuting with* B *corresponds a function* $c(R,\lambda) \in C(\Pi)$; *the mapping* $\gamma\colon \underline{A}'_B \to C(\Pi)$ *defined by* $\gamma(R) = c(R,\lambda)$ *is an algebra isomorphism (of* $\underline{A}'_B$ *into* $C(\Pi)$) *and*

$$\| c(R,\lambda) \|_\infty \leqslant K \| R \| ,$$

where $K = \|g_1\|$.

Proof. The existence of $c(R,\lambda)$ is clear from the previous observations. The continuity of $c(R,\lambda)$ is a consequence of (2), (6) and the fact that R itself is continuous.

That γ is an algebra homomorphism is also clear. If $c(R,\lambda) \equiv 0$, then $Rg_\lambda = 0$, for all $\lambda \in \Pi$, and this implies (since the set $\{g_\lambda, \lambda \in \Pi\}$ is *complete* in X, as we saw in the proof of *thm.5*) that $R = 0$; hence, γ is one-to-one.

Finally, the inequality $\|c(R,\lambda)\|_\infty \leqslant K\|R\|$ follows immediately from (14), and the results of *sect.1*.

q.e.d.

COROLLARY 14. *If* $\sigma_p(B) \supset \Pi$, *then:*

i) B *has no non-trivial complementary invariant subspaces;*

ii) If $|z| < 1$, $(B - z)$ *is rootless;*

iii) If $|z| \leqslant 1$, $(B - z)$ *is logarithmless.*

Proof.
i) Assume that $P \in \underline{A}'_B$ satisfies the equation $P^2 = P$; then, by *thm.13*, $c^2(P,\lambda) = c(P,\lambda) \in C(\Pi)$ and therefore $c(P,\lambda) \equiv 0$ (and P=0) or $c(P,\lambda) \equiv 1$ (and $P = I$). Therefore $\underline{A}'_B$ contains no non-trivial idempotents and this condition is equivalent (see [2; *exerc.3*, p. 70]) to the non-existence of two non-trivial complementary invariant subspaces for B.

ii) If $R^k = (B - z)$, then it is clear that $R \in \underline{A}'_B$ and, by *thm.13*, $c^k(R,\lambda) = c(B - z,\lambda) = (\lambda - z)$; since $|z| < 1$, the equation $c^k(R,\lambda) = (\lambda - z)$ has no solutions in $C(\Pi)$, for any $k > 1$. Therefore $(B-z)$ is rootless.

iii) Similarly, if $e^R = \sum_{n=0}^{\infty} (1/n!)\, R^n = (B - z)$, then $R \in \underline{A}'_B$ and $c(R,\lambda) = \log(\lambda - z)$; but this equation has no solutions in $C(\Pi)$, unless $|z|$ is strictly larger than 1.

q.e.d.

COROLLARY 15. *The conclusions of thm.13 and cor.14 remain true when the hypothesis* "$\sigma_p(B) \supset \Pi$" *is replaced by* "$\sigma_p(B^*) \supset \Pi$."

Proof. Let $R \in \underline{A}'_B$; then $R^* \in \underline{A}'_{B^*}$ and we conclude as in (14) that

$$R^*h_\lambda = c(R^*, \lambda)h_\lambda, \quad (\lambda \in \Pi). \tag{14'}$$

However (as it is shown by the *example* at the end of *sect.3*), the mapping $\lambda \to h_\lambda$ is highly discontinuous in general, and we cannot apply the arguments of *thm.13*. The solution arrives by using the duality between X and X*; in fact, (8)-(10) and the equalities

$$c(R^*, \lambda) = c(R^*, \lambda)\ (e_0, h_\lambda) = (e_0, R^*h_\lambda) = (Re_0, h_\lambda),$$

show that $c(R^*, \lambda) \in C(\Pi)$.

The remaining statements are clear now.

q.e.d.

6. COMPLEMENTARY REMARKS.

The results contained in [3],[4] and those of this paper show that, if B is a B.W.S. in a Banach space X satisfying our conditions and, moreover,

1) B has two non-trivial complementary invariant subspaces, or

2) B has a k^{th} root for some integer $k > 1$, or

3) B has a logarithm, then

$\sigma(B)$ is a circle and $\sigma_p(B) = \sigma_p(B^*) = \phi$

CONJECTURE. If *both* X and X* satisfy our conditions, X is reflexive and B is a B.W.S. on X, then each of the three above conditions is equivalent to

4) B is *similar* to a positive multiple of U, i.e., there exists an invertible operator V on X and a positive r such that $B = rVUV^{-1}$.

All the results of this paper can be easily extended to a larger class of B.W.S., and even to a class of operators related with them.

For example, if Y is a Banach space satisfying (1) and (2), and the Y-norm satisfies all the conditions that we asked for the X-norm of the intermediate subspaces and if the B.W.S. B is well defined and continuous on Y, then all the results can be extended to B acting on Y. This is true, in particular, for all B.W.S. B such that

$$B(C_0) \subset \ell_1. \tag{15}$$

EXAMPLE 1. Let $\{X_k\}$ be a finite (or denumerable) family of different intermediate spaces between C_0 and ℓ_1 in the conditions of *sect.1*, and let $Z = \cup_k Z_k$ be a partition of Z (where k ranges over the same set of indices as for the X_k's). Let P_k be the projection defined by

$$P_k \sum c_n e_n = \sum_{n \in Z_k} c_n e_n$$

and let Y be the completion of $\oplus_k P_k X_k$ under some "suitable" norm (e.g., $\|g\|_Y = \sum_k \|P_k g\|_{X_k}$). Observe that, in general, the W's of *sect.1* are not unitary operators in Y, even if Y is re-normed with an equivalent norm. However, Y satisfies our requirements and, if B is a B.W.S. whose restriction to Y maps Y continuously into itself, then the results of the paper apply to B restricted to Y, except perhaps *cor.3*.

If, e.g., $Y = P^+C_0 \oplus P^-\ell_1$, then every B.W.S. defines an operator in Y. If $Y = P^e C_0 \oplus P^0\ell_1$, where $P^e \sum c_n e_n = \sum c_{2n} e_{2n}$, $P^0 = I - P^e$, then the only B.W.S. that we can "interpolate" in this Y are those satisfying the condition (15) (for each of these operators, $\sigma(B) = \{0\}$).

EXAMPLE 2. The results also apply to $Y = C(\Pi)$, though as the space of all sequences of Fourier coefficients of continuous functions on Π , even when the projections P^+ and P^- are *not* bounded here and, moreover, (2) is not satisfied in this space (see [9; *Chap.II and VIII*]).

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