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# MEAN CONVERGENCE OF SERIES OF BESSEL FUNCTIONS

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1. INTRODUCTION. Let be $\nu > -1$ , $\psi_n(x) = \sqrt{2}\, J_\nu(xs_n)/\,|\,J_{\nu+1}(s_n)\,| = J_\nu(xs_n)/\,\|J_\nu(xs_n)\,\sqrt{x}\|_2$ , $0 < x < 1$ , and $\{s_n: n = 1,2,\ldots\}$ the set of positive zeros of $J_\nu$. Let us denote with $L^p(\gamma;a,b)$ the $L^p$-space of functions defined on (a,b) with respect to the measure $\mu$ such that $d\mu = x^\gamma dx$. When $(a,b) = (0,1)$ we shall write $L^p(\gamma)$ for this space. When $\gamma = 0$ it will be denoted with $L^p(a,b)$, or simply with $L^p$ if $(a,b) = (0,1)$. If $c_n(f)$ denotes the nth Fourier coefficient of f with respect to the orthonormal system $\{\psi_n\}$ , $c_n = \int_0^1 f(x)\,\psi_n(x)\,x\,dx$ , and $\beta = 1/2$ or $1/p$, then according to [12] and [2], it holds:

$$(1)\qquad \int_0^1 |\sum_1^N c_n\,\psi_n(x) - f(x)|^p \,.\, x^{\beta p}\,dx \underset{N\to\infty}{\to} 0 \quad ,$$

whenever

$$(2)\quad 1<p<\infty,\ \{(\nu+1/2)\wedge 0\} + 3/2 - \beta > 1/p > 1/2 - \beta - \{(\nu+1/2)\wedge 0\}.$$

From (1), it follows that

$$(3)\qquad \int_0^1 |\sum_1^N c_n\,\psi_n|^p\, x^{\beta p}\,dx \leqslant K \int_0^1 |f|^p\, x^{\beta p}\,dx \quad ,$$

with $K = K(\beta,p,\nu)$ independent of N.

From [9], theorem 2, it follows that inequality (3) must also hold for other weight functions $x^{\beta p}$ obtained by interpolation between x

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and $x^{p/2}$. This suggests that (1) could also be true for those weight functions. More than that can be said. Precisely, we shall prove the following result:

THEOREM 1. *If* $\nu > -1$ *and* p *and* $\beta$ *verify* (2) *then for any* $f \in L^p(\beta p)$, (1) *holds.*

A particular case of this theorem is obtained when

(4) $\beta p = (p/2)\,[(1-2\alpha)/2(1-\alpha)] + \alpha/(1-\alpha)\ ,\ \nu = \pm|(1-2\alpha)/2(1-\alpha)| > -1,$

where $\alpha \in (-\infty,1)$. It is not difficult to see that the results contained in [3], theorems 2 and 3, could be obtained from theorem 1 with $\beta$ and $\nu$ defined by (4). Besides in this case $x^{\beta p}$ is in general not obtainable by interpolation, (take, for example, $\alpha = 1/4$). Dini series are analogous to Bessel series but with the $s_n$'s replaced by the set $\{\lambda_n\}$ of positive solutions of the equations:

(5) $$z\,J'_\nu(z) + H\,J_\nu(z) = 0\ .$$

If $\nu + H \leqslant 0$ it is necessary to add to the system $\{\varphi_n(x)\ ;\ n = 1, 2,\ldots\}$, $\varphi_n(x) = J_\nu(x\lambda_n)/\|J_\nu(x\lambda_n)\sqrt{x}\|_2$, a function $\varphi_o(x)$, (cf. [11], ch. XVIII). Assuming that $\varphi_o \equiv 0$ when $\nu + H > 0$, we have:

THEOREM 2. *If* $\nu > -1$ *and* p *and* $\beta$ *verify* (2) *then for the system* $\{\varphi_n(x) : n = 0,1,2,\ldots\}$ *and any* $f \in L^p(\beta p)$, (1) *holds.*

This result has been obtained by Wing for $\beta = 1/2$ and $\beta = 1/p$ when $\nu \geqslant -1/2$, (cf. [12]).

Askey in [1] has given a general account on theorems of this type for several orthonormal systems. Theorems 1 and 2 are the Bessel analogues to results due to B. Muckenhoupt for Jacobi series (cf. [14]). Moreover, this paper is of a technical nature, since the main ideas used in the proof of th. 1, are already present in the literature.

2. Since there exists a constant $M = M(n)$ such that

$|\psi_n(x)| \leqslant M\,x^\nu\ ,\quad 0 < x < 1\ ,$ we have:

$$|c_n(f)| \leqslant M\left(\int_0^1 x^{(\nu+1-\beta)q}dx\right)^{1/q}\left(\int_0^1 |fx^\beta|^p dx\right)^{1/p} \quad , \quad q = p/(p-1).$$

Because of (2), $(\nu+1-\beta)q > -1$, and it follows that $c_n(f)$ *is a continuous linear functional in* $L^p(\beta p)$.

To prove theorem 1 it is sufficient to show that:

i) (3) *holds for a set of functions dense in* $L^p(\beta p)$,

ii) (1) *holds for a set of functions dense in* $L^p(\beta p)$.

In fact, i) together with the preceding consideration on Fourier coefficients imply (3) for any function in $L^p(\beta p)$, and in consequence from ii), it is also possible to conclude that (1) holds for any function in $L^p(\beta p)$. The dense sets in i) and ii) *do not* necessarily coincide.

The proof of theorem 2 will follow the same lines as that of theorem 1. To prove i) we shall use the following result proved in [5] for $\nu \geqslant -1/2$ and in [2], § 4, for $-1 < \nu < -1/2$.

THEOREM 3. *Let* $d_n$ *be the Dirichlet kernel of the system* $\{\psi_j\}$, $d_n(x,y) = \sum_1^n \psi_j(x)\,\psi_j(y)$ *and assume* $\nu > -1$. *Then, if* $M_n = (s_n + s_{n+1})/2$ *it holds :*

$$d_n(x,y) = \sum_{j=1}^{6} A_j(x,y) \quad , \quad 0 < x,y < 1 \quad , \tag{6}$$

*where:*

$$\begin{cases} A_1(x,y) = O(1)\, x^\nu y^\nu \quad ; \quad A_2(x,y) = O(1)/(xy)^{1/2}(2-x-y) \quad ; \\ A_3(x,y) = J_\nu(xM_n)\, J_{\nu+1}(yM_n)\, M_n/2(y-x) \quad ; \quad A_4(x,y) = A_3(y,x) \; ; \\ A_5(x,y) = J_\nu(xM_n)\, J_{\nu+1}(yM_n)\, M_n/2(x+y) \quad ; \quad A_6(x,y) = A_5(y,x) \, . \end{cases} \tag{7}$$

*The* O'*s that appear in* (7) *are uniformly bounded with respect to* x,y *and* n.

To prove i) for theorem 1 it is sufficient to see that for

$i = 1,\dots,6$ ,

$$T_i f(x) = \text{P.V.} \int_0^1 A_i(x,y)\, f(y)\, y\, dy \quad ,$$

is uniformly continuous (with respect to n) in the space $L^p(\beta p)$ whenever p verifies (2). For this we shall make use of the following results:

THEOREM 4. *Assume* $-\infty < s,t < +\infty$ , $0 < x,y < 1$. *The operator*

$$\text{(8)} \qquad Uf(x) = \int_0^1 x^s y^t\, (2-x-y)^{-1}\, f(y)\, dy \quad ,$$

*is continuous in* $L^p$ *if*

$$\text{(9)} \qquad (-s) \vee 0 < 1/p < (1+t) \wedge 1 \quad .$$

This proposition can be proved in the same way as Lemma 3 of §6 in [2].

THEOREM 5. j) *The operator*

$$\text{(10)} \qquad K_{a,b;A,B} f(x) = \text{P.V.} \int_{-\infty}^{+\infty} \frac{f(t)}{x-t} \frac{|x|^b (1+|x|)^{a-b}}{|t|^b (1+|t|)^{A-B}}\, dt \quad ,$$

*is continuous in* $L^p(-\infty,+\infty)$ *if*

$$\text{(11)} \qquad 1 < p < \infty \ ; \ b \geqslant B \ ; \ A \geqslant a \ ; \ -1/p < b \wedge a \ ; \ a \vee B < 1/p'$$

*where* $1/p' = 1 - 1/p$.

jj) *If* $1 \leqslant p \leqslant \infty$ *and the operator* (10) *is continuous in* $L^p(-\infty,+\infty)$ *then* (11) *holds.*

j) for $b = B = a = A$ is due to Hardy and Littlewood, [4]. As it is stated here, this result can be seen in Muckenhoupt's paper [6]. The last section of the present paper contains a proof of theorem 5 in which the proof of j) follows the line of argumentation given in [4] and [8]. jj) is well-known when $A = a = B = b = 0$ and suggests that theorems 1 and 2 are not true for the limiting values of p in the second inequality in (2). In fact, the following result will be proved:

THEOREM 6. *Assume* $1 \leqslant p \leqslant \infty$, $-\infty < \beta < +\infty$. *Inequality* (3) *holds for Bessel and Dini series only if* (2) *holds.*

REMARK 1. The 0's that appear in (7), theorem 3, are not only uniformly bounded with respect to x, y and n but also with respect to $\nu$ if this variable is restricted to a closed interval contained in $(-1,\infty)$. This can be seen by a careful examination of the constants that appear in the proof given in [2], §4.

3. AUXILIARY LEMMAS. The first of the following two lemmas is proved in Tolstov's book [10], ch. 8, sections 14, 19, 20 and 23. The second one is analogous to Dirichlet-Jordan test for trigonometric series and is proved in Watson's book, [11], chapter XVIII.

LEMMA 1. i) *There exist two positive constants* A *and* B *depending only on* $\nu$ $(> -1)$ *such that for* $\lambda$ *great enough it holds:*

$$A/\lambda \leqslant \int_0^1 J_\nu^2(\lambda t)\, t\, dt \leqslant B/\lambda \quad . \tag{12}$$

ii) *If* f *is* 2k *times continuously differentiable* $(k \geqslant 1)$ *and equal to zero in neighbourhoods of* 0 *and* 1 *then for* $\nu > -1$:

$$b_n = b_n(f) = \int_0^1 J_\nu(xs_n) f(x) x dx \Big/ \int_0^1 J_\nu^2(xs_n) x dx = 0(1)/s_n^{2k-1/2} \quad . \tag{13}$$

iii) (13) *holds also when instead of* $\{s_n\}$ *the set* $\{\lambda_n\}$ *of positive zeros of* (5) *are used.*

COROLLARY 1. *If* f *verifies the hypothesis* ii) *in lemma 1 and* p *satisfies* (2) *then its Bessel and Dini series converge in* $L^p(\beta p)$.

*Proof.* Let $\sum_{n=1}^{\infty} b_n J_\nu(xs_n)$ be the Bessel series of f. Because of the asymptotic expansions of $J_\nu(x)$, $x > 1$, we have for a certain constant $C = C(\nu)$:

$$|\sqrt{x}\, J_\nu(x)| \leqslant C(x^{(\nu+1/2)\wedge 0} \vee 1) \ , \quad \nu > -1 \ , \quad x > 0 \ . \tag{14}$$

Therefore:

$$|J_\nu(xs_n)| \leqslant \frac{C}{\sqrt{xs_n}}\,(1 \vee (xs_n)^{(\nu+1/2)\wedge 0}) \leqslant \frac{C}{\sqrt{xs_n}}\, x^{(\nu+1/2)\wedge 0} \quad .$$

Then, because of (2), we get:

$$(14') \quad (\int_o^1 J_\nu(xs_n)^p x^{\beta p} dx)^{1/p} \dot{\leqslant} C\, s_n^{-1/2} (\int_o^1 x^{[(\nu+1/2)\wedge 0+\beta-1/2]\,p} dx)^{1/p} =$$

$$= O(s_n^{-1/2}) \ .$$

From (14') and lemma 1, it follows that

$$\| b_n J_\nu(xs_n) \|_{p,\beta p} = O(s_n^{-2k}) \quad ,$$

and therefore, the Bessel series of f converges in $L^p(\beta p)$. The same argument holds for the case of Dini series. QED.

If one uses now the $L^2(1)$-completeness of the Bessel system, (cf. [13], or [11] and [15]), one could assert that the series associated to f converges to this function in $L^p(\beta p)$. Thus proving ii), §2. We shall not assume that result, which will follow as a corollary. At this point this paper tries to be self contained, using when possible, results of elementary nature.

LEMMA 2. *If* f *is of bounded variation and continuous in* (0,1) *then* $f(x) = \sum_{n=1}^{\infty} b_n J_\nu(xs_n)$ , $0 < x < 1$ , *whenever* $\nu \geqslant -1/2$.

(The coefficient $b_n$ is defined by formula (13)).

*Let* D *be the set of functions twice continuously differentiable which are null in neighbourhoods of* 0 *and* 1. From the preceding results it follows immediately next corollary, (cf. remark 2, §5).

COROLLARY 2. *If* $f \in D$, $\nu \geqslant -1/2$ *and* p *verifies* (2) *then* $\sum_1^{\infty} b_n(f)\, J_\nu(xs_n)$ *converges to* f *in* $L^p(\beta p)$.

LEMMA 3. *Let* $F(x) \in D$ *and define* $f(x) = x^{-(\nu+1)} \int_o^x F(t)\, t^{\nu+1}\, dt$ , *then, if* $\nu > -1$:

$$(15)\qquad \frac{\int_0^1 F(x)J_\nu(xs_n)\ x\ dx}{\int_0^1 J_\nu^2(xs_n)\ x\ dx} = \frac{s_n\int_0^1 f(x)J_{\nu+1}(xs_n)\ x\ dx}{\int_0^1 J_{\nu+1}^2(xs_n)\ x\ dx}$$

*Proof.* From the formulae

$$(16)\qquad \frac{d}{dz}(z^{-\nu}J_\nu(z)) = -z^{-\nu}J_{\nu+1}(z)\ ;\ \frac{d}{dz}(z^{\nu+1}J_{\nu+1}(z)) = z^{\nu+1}J_\nu(z)\ .$$

it follows

$$(17)\quad \frac{d}{dz}(z^{-\nu}J_\nu(\lambda z)) = -\lambda z^{-\nu}J_{\nu+1}(\lambda z)\ ;\ \frac{d}{dz}(z^{\nu+1}J_{\nu+1}(\lambda z)) = \lambda z^{\nu+1}J_\nu(\lambda z).$$

Therefore

$$(18)\qquad \frac{d}{dx}(xJ_\nu(xs_n)\ J_{\nu+1}(xs_n)) = xs_n(J_\nu^2(xs_n) - J_{\nu+1}^2(xs_n))\ .$$

The equality of the denominators in (15) follows after integrating (18) between zero and one. The equality of the numerators follows after integrating by parts the first of them and using the first of formulae (18). QED.

4. RELATIONS BETWEEN BESSEL AND DINI SERIES. Let $D_n$ be a number in the interval $(\lambda_n + \pi/8\ ,\ \lambda_{n+1} - \pi/8)$ whose distance to any $s_j$ is at least $\pi/8$. Such a number exists if n is great enough, (cf. [11], p. 598). Let $N = N(n)$ be the greatest integer j such that $s_j < D_n$. Obviously, $n_2 > n_1$ implies $N(n_2) \geqslant N(n_1)$ and if $n \to \infty$ then $N(n) \to \infty$. The difference between the Dirichlet kernels of Bessel and Dini series is:

$$R_n(x,y,H)(xy)^{-1/2} = \sum_{j=1}^{N} \psi_j(x)\ \psi_j(y) - \sum_{i=o}^{n} \varphi_i(x)\ \varphi_i(y)\quad ,$$

where H is the constant that appears in (5).

*Let us call* S *the set of simple functions defined on* (0,1), i.e., the family of linear combinations of characteristic functions of intervals.

Then we have:

LEMMA 4. *Let* $f_o$ *be a function in* S *equal to zero in a neighbourhood of* $x = 1$ *and assume* $\nu > -1$. *If* $f(x) = x^{\nu} f_o(x)$ *and* p *verifies* (2), *then it holds in* $L^p(\beta p)$ *that*

$$\text{(19)} \qquad \int_o^1 R_n(x,y,H)(xy)^{-1/2} f(y)\, y\, dy \underset{n \to \infty}{\longrightarrow} 0$$

*Proof.* It is sufficient to prove the lemma for $f_o$ the characteristic function of $(0,T)$, $T < 1$. In this case using the following equality which appears in [11], §18.31, (19) becomes:

$$\text{(20)} \qquad \int_o^T y^{\nu+1/2} x^{-1/2} R_n(x,y,H)\, dy = \frac{T^{\nu+1}}{2\pi i} \int_{D_n - i\infty}^{D_n + i\infty} \frac{2J_\nu(xw) J_{\nu+1}(Tw)\, dw}{J_\nu(w)(wJ_\nu'(w) + HJ_\nu(w))}$$

The asymptotic expansion of $J_\nu$ and its behaviour around the origin yields that there exists a constant M verifying:

$$\text{(21)} \qquad \begin{cases} |\sqrt{tw}\, J_{\nu+1}(tw)| \leqslant M\, e^{t|\eta|} \ , \ \eta = \operatorname{Im} w \ , \quad w \in (D_n - i\infty, D_n + i\infty) \ , \\ \\ |\sqrt{xw}\, J_\nu(xw)| \leqslant M\, e^{x|\eta|} (1 + (xD_n)^{(\nu+1/2)\wedge 0}) \ . \end{cases}$$

Then, from (20) and (21) it follows that with a certain constant C it holds:

$$\left|\int_o^1 f(y)\, R_n(x,y,H)\, x^{-1/2}\, y^{1/2}\, dy\right| \leqslant \frac{CT^{\nu+1/2}(1 + (xD_n)^{(\nu+1/2)\wedge 0})}{D_n(2-x-T)\sqrt{x}} \leqslant$$

$$\leqslant \frac{CT^{\nu+1/2}}{(1-T)D_n}(1 + x^{(\nu+1/2)\wedge 0}) x^{-1/2} \leqslant \frac{2CT^{\nu+1/2}}{(1-T)D_n}\, x^{(\nu+1/2)\wedge 0 - 1/2} \ .$$

In consequence, for a certain constant $K = K(T)$ we have:

$$\text{(22)} \qquad \int_o^1 \left|\int_o^1 f(y)\, R_n(x,y,H)\, x^{-1/2}\, y^{1/2}\, dy\right|^p x^{\beta p}\, dx \leqslant$$

$$\leqslant \frac{K}{D_n} \int_0^1 x^{[(\nu+1/2)\wedge 0+\beta-1/2]p}\, dx .$$

From (2) and (22) the thesis follows immediately. QED.

LEMMA 5. *Let be* $\nu > -1$. *Then,*

$$(23) \qquad R_n(x,y,H) = O(1)\, \frac{x^{(\nu+1/2)\wedge 0}\; y^{(\nu+1/2)\wedge 0}}{2-x-y}$$

*and the operator defined by*

$$(24) \qquad \int_0^1 R_n(x,y,H)\, (xy)^{-1/2}\, f(y)\, y\, dy \quad ,$$

*is uniformly continuous with respect to* n *in* $L^p(\beta p)$ *whenever* p *verifies* (2).

*Proof*. From [11], §18.31, we get

$$(25) \qquad \frac{R_n(x,y,H)}{\sqrt{xy}} = \frac{1}{\pi i} \int_{D_n-i\infty}^{D_n+i\infty} \frac{w\, J_\nu(xw)\, J_{\nu+1}(yw)}{J_\nu(w)(wJ_\nu'(w)+HJ_\nu(w))}\, dw \quad ,$$

Since $w \in (D_n-i\infty, D_n+i\infty)$, from the asymptotic expansion of $J_\nu$ it follows for a certain positive constant M' and n great enough, that

$$(26) \qquad |\sqrt{w}\, J_\nu(w)| \geqslant M'\, e^{|\eta|} \quad , \quad \eta = \operatorname{Im} w \quad .$$

Taking into account the relation $2J_\nu'(w) = -J_{\nu+1}(w)+J_{\nu-1}(w)$, it also follows from the asymptotic expansions that

$$(27) \qquad |\sqrt{w}\, J_\nu'(w) + H\, J_\nu(w)/\sqrt{w}| \geqslant M''\, e^{|\eta|} \quad ,$$

for n great enough. Replacing (27), (26) and (21) in (25) we finally obtain for certain constants C and C' and n great enough that:

$$R_n(x,y,H) \leqslant C\int_{-\infty}^{+\infty} e^{(x+y-2)|\eta|}\{1+(xD_n)^{(\nu+1/2)\wedge 0}\}\{1+(yD_n)^{(\nu+1/2)\wedge 0}\}d\eta \leqslant$$

$$\leqslant C'(xy)^{(\nu+1/2)\wedge 0}(2-x-y)^{-1} .$$

This proves the first part of the lemma. From Theorem (4) it follows that the operator

$$(28) \qquad \int_0^1 F(y)\ (2-x-y)^{-1}\ x^{(\beta+\nu)\wedge(\beta-1/2)}\ y^{(\nu+1-\beta)\wedge(1/2-\beta)}\ dy$$

is continuous in $L^p$ if p verifies (2). Then, using (23), we obtain that (24) is uniformly continuous in $L^p(\beta p)$. QED.

From lemmas 4 and 5 it follows immediately next result.

COROLLARY 3. *If* $\nu > -1$, p *verifies* (2) *and* $f \in L^p(\beta p)$, *then in* $L^p(\beta p)$.

$$\int_0^1 R_n(x,y,H)\ x^{-1/2}\ y^{1/2}\ f(y)\ dy \underset{n\to\infty}{\to} 0 .$$

That is, the Bessel series of f(x) converges to f in $L^p(\beta p)$ if and only if the Dini series of f converges to f in the same space.

LEMMA 6. *Assume that theorem 2 is true for* $\nu \in (0,1/2)$. *If* $F(x)\in D$, p *verifies* (2) *and* $-1 < \nu < -1/2$, *then the Bessel series of* F, $\sum_{n=1}^{\infty} b_n J_\nu(xs_n)$, *converges to* F *in* $L^p(\beta p)$.

*Proof.* Because of Corollary 1 in the preceding section the series under consideration converges to a certain function G(x) in the space $L^p(\beta p)$. Therefore, in view of (2), the series

$$(29) \qquad \sum_{n=1}^{\infty} b_n\ x^{1+\nu}\ J_\nu(xs_n) ,$$

converges in $L^1(0,1)$ to $x^{1+\nu}.G(x)$. Integrating (29) term by term and using the second formula in (17) we obtain,

$$(30) \qquad \sum_{n=1}^{\infty} s_n^{-1}\ b_n\ x^{1+\nu}\ J_{\nu+1}(xs_n) = \int_0^x t^{1+\nu}G(t)dt ,$$

the convergence being uniform. Then, for any $x \in (0,1)$ ,

(31) $$\sum_{n=1}^{\infty} s_n^{-1} b_n J_{\nu+1}(xs_n) = x^{-(1+\nu)} \int_o^x t^{1+\nu} G(t)\, dt \,.$$

On the other hand we have, again from second formula in (17), that $s_n$ is the nth positive zero of $(\nu+1)J_{\nu+1}(z) + z J'_{\nu+1}(z)$, i.e., if in formula (5) we replace H and $\nu$ by $\nu+1$, its nth zero $\lambda_n$ coincides with $s_n$. Since $(\nu+1) + H = 2(\nu+1) > 0$ , we have in this case $\varphi_o \equiv 0$ . It follows from lemma 3 that the series in (30) is the Dini series of

(32) $$f(x) = x^{-(\nu+1)} \int_o^x F(t)\, t^{1+\nu}\, dt \quad ,$$

with respect to the preceding system. Because of $\nu < -1/2$ the right hand side of the second inequality in (2) implies that $p(\beta - 1 - \nu) > -1$ and therefore $f(x) \in L^p(\beta p)$. Besides, if (2) holds with $\nu,\beta,p$, it also clearly holds with $\nu+1,\beta,p$. Hence, from the hypothesis, the series in (31) converges to f(x) in $L^p(\beta p)$. Comparing (31) and (32) we obtain F(x) = G(x) a.e.. QED.

5. PROOF OF THEOREMS 1 AND 2. I) The first part of the proof will consist in proving i) for Bessel series (cf. section 2). The analogous result for Dini series will follow then from lemma 5. Assume $1 < p < \infty$.

1) $T_1$ *is continuous in* $L^p(\beta p)$ *if* p *verifies* (2). In fact, the continuity of this operator in $L^p(\beta p)$ is implied by the continuity in $L^p$ of the operator:

$$\int_o^1 y^{\nu+1-\beta}\, x^{\beta+\nu}\, g(y)\, dy \quad .$$

2) $T_2$ *is continuous in* $L^p(\beta p)$ *if* p *verifies*

(33) $$3/2 - \beta > 1/p > 1/2 - \beta$$

In fact, it is implied by the continuity in $L^p(0,1)$ of the operator:

$$\int_o^1 g(y)\, (2-x-y)^{-1}\, (x/y)^{\beta-1/2}\, dy \ ,$$

and this follows from theorem 4.

3) $T_3$ *is uniformly continuous in* $L^p(\beta p)$ *if*

(34) $$3/2 - \beta > 1/p > -\nu - \beta \ , \quad -1 < \nu < -1/2 \ ,$$

*and if* $\nu \geqslant -1/2$ *and* p *verifies* (33). *(Idem for* $T_5$*).*

It is sufficient to see that

(35) $$\sup_f \int_0^\infty |P.V. \int_0^\infty A_3(x,y) f(y) y dy|^p \, x^{\beta p} dx / \int_0^\infty |f(x)|^p x^{\beta p} dx$$

has a finite upper bound independent of $M_n$ ($\geqslant 1$). Here f may be supposed to run on the set of functions null outside an interval. It follows easily after a change of variables (as in [2], (29)) that (35) is independent of $M_n$ which can be thus replaced by one. Then it only remains to show that

(36) $$P.V. \int_0^\infty \frac{J_\nu(x) \; J_{\nu+1}(y)}{y - x} f(y) \; y \; dy$$

defines a continuous operator in $L^p(\beta p;(0,\infty))$.

From the following estimations for $x > 0$ and $-1 < \nu < -1/2$:

(37) $$\sqrt{x}\, J_\nu(x) = O(1)(1+x^{\nu+1/2}), \quad \sqrt{x}\, J_{\nu+1}(x) = O(1)/(1+x^{\nu+1/2}).$$

(38) $$\sqrt{x}\, J_\nu(x) = O(1) \quad , \quad x > 0 \quad , \quad \nu \geqslant -1/2 \ ,$$

we get for $-1 < \nu < -1/2$ , $x > 0$ :

(39) $$\sqrt{x}\, J_\nu(x) = O(1) \; x^{\nu+1/2}(1+x)^{-(\nu+1/2)},$$

(40) $$\sqrt{x}\, J_{\nu+1}(x) = O(1) \; x^{-(\nu+1/2)}(1+x)^{\nu+1/2}.$$

Using (38),(39) and (40) one sees readily that the continuity of (36) in $L^p(\beta p;(0,\infty))$ when $-1 < \nu < -1/2$ and $-1/2 \leqslant \nu$, respectively, is implied by the continuity of the following operators in $L^p(0,\infty)$:

$$(41) \qquad \text{P.V.} \int_0^\infty \left(\frac{1+x}{1+y}\right)^{-(\nu+1/2)} \left(\frac{x}{y}\right)^{\nu+\beta} \frac{F(y)}{x-y}\, dy \ , \ -1 < \nu < -1/2 \ ,$$

$$(42) \qquad \text{P.V.} \int_0^\infty \left(\frac{x}{y}\right)^{\beta-1/2} \frac{F(y)}{x-y}\, dy \ , \ -1/2 \leqslant \nu \ .$$

An application of theorem 5 proves the continuity of these operators, therefore the uniform continuity of $T_3$.

4) $T_4$ *is uniformly continuous in* $L^p(\beta p;(0,1))$ *if*

$$(43) \qquad \nu + 2 - \beta > 1/p > 1/2 - \beta \ , \ -1 < \nu < -1/2 \ ,$$

*and if* $\nu \geqslant -1/2$ *and* p *verifies* (33). *(Idem for* $T_6$*).*

It is seen as before that it is sufficient to prove the continuity of $T_4$ in $L^p(\beta p;(0,\infty))$ when $M_n = 1$. Call $a(x,y)$ the kernel obtained from $A_3$ (cf. (7)) when $M_n$ is equal to one and observe that $a(y,x) = A_4(x,y)$ for this value of $M_n$. Then for $f \in L^p(\beta p;(0,\infty))$ the continuity of $T_4$, $M_n = 1$ :

$$(44) \qquad \int_0^\infty \left|\int_0^\infty a(x,y)\ f(x)\ x\ dx\right|^p y^{\beta p}\ dy \leqslant C \int_0^\infty |f(x)|^p\ x^{\beta p}\ dx$$

holds if and only if for $g \in L^q((1-\beta)q;(0,\infty))$ , $1/p + 1/q = 1$ ,

$$(45) \qquad \int_0^\infty \left|\int_0^\infty a(x,y)\ g(y)\ y\ dy\right|^q x^{(1-\beta)q}\ dx \leqslant C \int_0^\infty |g(x)|^q\ x^{(1-\beta)q}\ dx$$

holds, where C is independent of f and g. According to the point 3) proved above, (45) is verified when $-1 < \nu < -1/2$ if $-\nu - (1-\beta) < 1/q < 3/2 - (1-\beta)$, i.e., if p verifies (43). On the other hand, if $\nu \geqslant -1/2$, (45) holds if $1/2-(1-\beta) < 1/q < 3/2-(1-\beta)$, i.e., if p verifies (33). 1) - 4) imply i).

II) Assume $\nu \geqslant -1/2$. Corollary 2 of section 3 implies ii) for Bessel series and therefore for those $\nu$'s theorem 1 is proved. From Corollary 3, section 4, theorem 2 for $\nu \geqslant -1/2$ follows. This result and lemma 6 imply ii) for Bessel series if $\nu \in (-1,-1/2)$ . This completes the proof of theorem 1. This theorem together with

Corollary 3, section 4, imply theorem 2. QED.

REMARK 2. E. Murphy communicated to us that a closer observation of [11], pp. 580-592, shows that lemma 2, and therefore Corollay 2, still hold for $\nu > -1$. This makes superfluous Lemmas 3 and 6, and reduces part II of the preceding proof to a line. Expository reasons make us to follow the present procedure.

6. PROOF OF THEOREM 6. First, we show that given $p \in (1,\infty)$, $\nu > -1$, and $\beta$, if p is a limiting point of the second inequality in (2), then (3) does not hold. Since the proof is the same for Bessel and Dini series, we shall restrict ourselves to Bessel's system. In this case, (3) implies that

$$\left|\int_0^1 \psi_n \, f \, y \, dy\right| . \|\psi_n x^\beta\|_p \leq K \|f \, x^\beta\|_p \quad , \forall \, f \in L^p(\beta p) \ ,$$

that is

$$\left|\int_0^1 \psi_n \, F \, y^{1-\beta} \, dy\right| . \|\psi_n \, x^\beta\|_p \leq K\|F\|_p \ , \ \forall \, F \in L^p \ .$$

From this inequality and therefore from (3) it follows that:

$$\|\psi_n \, y^{1-\beta}\|_q . \|\psi_n \, x^\beta\|_p \leq K \ . \tag{46}$$

It is sufficient to show that if p is a limiting point, (46) cannot hold. Because of lemma 1, $|\psi_n(y)| \geq C\sqrt{s_n}|J_\nu(ys_n)|$ for a certain constant C independent of n. Then,

$$\|\psi_n(y) \, y^{1-\beta}\|_q \geq C \, s_n^{-1/2-1/q+\beta} \left(\int_0^{s_n} |y^{1-\beta} \, J_\nu(y)|^q dy\right)^{1/q} \ . \tag{47}$$

Analogously,

$$\|\psi_n(x) \, x^\beta\|_p \geq C \, s_n^{1/2-1/p-\beta} \left(\int_0^{s_n} |y^\beta \, J_\nu(y)|^p dy\right)^{1/p} \ . \tag{48}$$

If (46) would hold, from (47) and (48) we had for a certain constant M,

$$(49)\quad M \geqslant s_n^{-1}\Big(\int_o^1 {}^n y^{(1+\nu-\beta)q}dy + \int_1^{s_n} y^{(1/2-\beta)q}dy\Big)^{1/q}\Big(\int_o^1 y^{(\beta+\nu)p}dy + \int_1^{s_n} y^{(\beta-1/2)p}dy\Big)^{1/p}.$$

Assume now that $1/p = -[\beta+(\nu\wedge -1/2)]$ . In this case, if $-1<\nu<-1/2$, $p(\beta+\nu) = -1$ and the right-hand side in (49) equals $\infty$, contradiction. If $\nu \geqslant -1/2$ , $1/p = 1/2-\beta$ , and we have: $1/q = 1/2+\beta$ . Then, from (49) we obtain for a certain positive constant C:

$$M \geqslant s_n^{-1}\Big(\int_1^{s_n} y^{q-1}dy\Big)^{1/q} \cdot \Big(\int_1^{s_n} x^{-1}dx\Big)^{1/p} \geqslant C(\lg s_n)^{1/p}.$$

This is again a contradiction if n is great enough.

The left-hand side of (46) does not change if we replace p by q, q by p and $\beta$ by $1-\beta$. Therefore (46) does not hold when $1-1/p = -[(1-\beta)+(\nu\wedge -1/2)]$, i.e. when $1/p = 2-\beta+(\nu\wedge -1/2) = 3/2-\beta+(\nu+1/2)\wedge 0$.

It also easily follows for $p = 1$ or $\infty$ that if $1/p$ belongs to the closed interval defined by the second inequality in (2), then (46) *cannot* be satisfied. From these observations and an interpolation theorem due to E. Stein ([9], th. 2), the desired result for $1 \leqslant p \leqslant \infty$ follows. QED.

7. PROOF OF THEOREM 5. j) It is sufficient to prove the following lemma:

LEMMA 7. *The operator* K *defined by* $K = K_{\alpha,\beta,\alpha,\beta}$ *is continuous in* $L^p(-\infty,\infty)$ *if*

$$(50)\qquad -1/p < \alpha,\beta < 1/q \quad , \quad 1/p + 1/q = 1 \quad .$$

In fact, because of the hypothesis of theorem 5 there exists $\alpha$ and $\beta$ such that $B \leqslant \beta \leqslant b$ , $a \leqslant \alpha \leqslant A$ , $-1/p < \alpha,\beta < 1/q$ and therefore the following functions are bounded on all the real line:

$$P(x) = |x|^{b-\beta}(1+|x|)^{a-\alpha-b+\beta} \ ; \ Q(x) = |x|^{\beta-B}(1+|x|)^{B-\beta-A+\alpha} \ .$$

Since $K_{a,b;A,B}f = P(x)\ K(Qf)$, theorem 5 follows from lemma 7. To prove this lemma we shall use the next general lemma.

LEMMA 8. *Let* $w(t)$ *be a measurable function defined on the real line and suppose that for some* $p$, $1 < p < \infty$, *there exist a positive* (a.e.) *measurable function* $r(x,t)$ *and a constant* $M$ *such that:*

$$\text{I)} \qquad \int_{-\infty}^{+\infty} \left| \frac{w(t)}{w(x)} - 1 \right| \frac{r^q(x,t)}{|x-t|}\, dt \leq M \quad , \quad \text{a.e.} \quad x \in (-\infty,\infty) \quad ,$$

$$\text{II)} \qquad \int_{-\infty}^{+\infty} \left| \frac{w(t)}{w(x)} - 1 \right| \frac{r^{-p}(x,t)}{|x-t|}\, dx \leq M \quad , \quad \text{a.e.} \quad t \in (-\infty,\infty) \ .$$

*Then the operator*

$$H_w f(x) = \text{P.V.} \int_{-\infty}^{+\infty} \frac{w(t)}{w(x)}\ \frac{f(t)}{x-t}\, dt \quad ,$$

*is well-defined and continuous in* $L^p(-\infty,\infty)$.

This result follows easily from an application of Hölder's inequality to the difference of $H_w$ and the Hilbert transform $H_1$. The proof is left to the reader, (cf. [2], th. 3).

*Proof of lemma 7.* Calling $w(t) = |t|^{-b}(1+|t|)^{b-a}$ and $r(x,t) = |x/t|^{1/pq}$, I) and II) of lemma 8 are reduced to:

$$(51) \qquad \int_{-\infty}^{+\infty} \left| \left(\frac{1+|x|}{1+|yx|}\right)^{a-b} |y|^{-b} - 1 \right| |y|^{-1/p}\ |1-y|^{-1}\, dy \leq M \quad , \quad \text{a.e. } x,$$

$$(52) \qquad \int_{-\infty}^{+\infty} \left| \left(\frac{1+|xy|}{1+|x|}\right)^{a-b} |y|^{b} - 1 \right| |y|^{-1/q}\ |1-y|^{-1}\, dy \leq M \quad , \quad \text{a.e. } x.$$

Assume $u$ and $v$ positive, then,

$$(53) \qquad \frac{1}{v \vee 1} \leq \frac{1+u}{1+uv} \leq \frac{1}{v \wedge 1}$$

From this follows:

$$(54)\qquad \frac{1}{v^a \vee v^b} \leqslant \left(\frac{1+u}{1+uv}\right)^{a-b} v^{-b} \leqslant \frac{1}{v^a \wedge v^b} ,$$

and therefore:

$$(55)\qquad \left|\left(\frac{1+u}{1+uv}\right)^{a-b} v^{-b} - 1\right| \leqslant \left|\frac{1}{v^a \wedge v^b} - 1\right| \vee \left|\frac{1}{v^a \vee v^b} - 1\right| ,$$

$$(56)\qquad \left|\left(\frac{1+u}{1+uv}\right)^{a-b} v^{-b} - 1\right| / |v-1| \leqslant (|v^{-a} \vee v^{-b} - 1| \vee$$

$$\vee\ |v^{-a} \wedge v^{-b} - 1|)/|v-1| .$$

Since the right-hand side of (56) is bounded in a neighbourhood W of 1, the left-hand side in (56) is uniformly bounded if v belongs to W. Therefore, instead of (51) and (52) it suffices to prove (57) and (58):

$$(57)\qquad \int_{-\infty}^{+\infty} \{ \left(\frac{1+|x|}{1+|xy|}\right)^{a-b} |y|^{-b} + 1\} |y|^{-1/p} (1+|y|)^{-1} dy \leqslant M , \text{ a.e. } x.$$

$$(58)\qquad \int_{-\infty}^{+\infty} \{ \left(\frac{1+|xy|}{1+|x|}\right)^{a-b} |y|^{b} + 1\} |y|^{-1/q} (1+|y|)^{-1} dy \leqslant M , \text{ a.e. } x.$$

Again from (54) it follows that instead of (57) and (58) it is sufficient to show that the following integrals are convergent:

$$(59)\qquad \int_{o}^{\infty} \frac{y^{-a} \vee y^{-b} + 1}{y+1}\, y^{-1/p}\, dy ,$$

$$(60)\qquad \int_{o}^{\infty} \frac{y^{a} \vee y^{b} + 1}{y+1}\, y^{-1/q}\, dy .$$

This last integral converges at $\infty$ if $1/q > a \vee b$ and converges at the origin if $a > -1/p$ , $b > -1/p$; i.e. it converges if (50) is satisfied. Since (59) is obtained from (60) changing p with q and the signs of a and b, (50) is reduced to $1/p > -a$ , $-b > -1/q$ ; this is again inequality (50). If f and g are bounded functions with bounded supports away from zero, then the following equality between scalar products holds:

$$(61) \qquad (K_{a,b;A,B}f\ ,\ g) = (f\ ,\ K_{-A,-B;-a,-b}g)\ .$$

Therefore, if $1 \leqslant p \leqslant \infty$ *and one of the operators in* (61) *is continuous in* $L^p$ *then the other one is continuous in* $L^q$.

jj) Given an interval $[c,d]$ , the Hilbert transform of its characteristic function is $\lg|(x-d)/(x-c)|$. From this easily follows that $K_{a,b;A,B}$ does not transform bounded functions into bounded functions, and in consequence $p \leqslant \infty$ *is a necessary condition* for the operator (10) to be continuous. Therefore, by the previous statement $1 < p$ *is also a necessary condition* for the operator (10) to be continuous.

Assume now $1 < p < \infty$ and call $\rho(x) = |x|^b(1+|x|)^{a-b}$ , $\sigma(x) = |x|^B(1+|x|)^{A-B}$. If $u > 0$ and (10) is $L^p$-continuous, we have:

$$(62) \qquad \int_{-\infty}^{+\infty} \rho^p |H(I)|^p\, dx \leqslant C \int_u^{ku} \sigma^p\, dx \quad , \quad k > 1 \quad ,$$

where $H = H_1$ and I is the characteristic function of (u,ku).

Suppose $Ap \leqslant -1$ , then from (62) we get for u great enough:

$$(63) \qquad \int_{-\infty}^{+\infty} \rho^p |H(I)|^p\, dx \leqslant 2C \lg k \quad .$$

Since $H(I)(x) = \lg|(x-ku)/(x-u)|$ , if $u \to \infty$ we have:

$$(64) \qquad \lg|k|^p \int_{-\infty}^{+\infty} \rho^p\, dx \leqslant 2C \lg k \quad .$$

Since the integral in the last inequality is not zero and $p > 1$ , (64) does not hold for k large enough. We have proved so that for the continuity of (10) in $L^p$ *it is necessary that* $Ap > -1$. If $bp \leqslant -1$, then $\rho^p$ is not integrable in any neighbourhood of the origin and therefore (62) cannot hold. In consequence $bp > -1$ *is nec* *essary for* (10) to be continuous. If $K_{a,b;A,B}$ is continuous in $L^p$, we already know that $K_{-A,-B;-a,-b}$ is continuous in $L^q$. Because of the necessary conditions shown to hold, we must have: $-1/q <$ $< -B,-a$. That is, $B,a < 1/q$ *are necessary conditions* for (10) to be $L^p$-continuous.

It only remains to show that if $b < B$ or $A < a$, (10) is not continuous. Because of what we have already shown we may assume that the remaining conditions in (11) are satisfied. Suppose that f is a bounded function with bounded support away from zero. Using the well-known fact that $H^{-1} = -H/\pi^2$, it easily follows that:

$$K_{A,B;a,b}\,(K_{a,b;A,B}f) = -\pi^2 f\ . \tag{65}$$

Then, if $a = A$, $b = B$, (10) is not only continuous but also invertible and except for a constant factor it is its own inverse. In consequence, (10) maps $L^p$ onto $L^p$. But

$$K_{a,b;A,B}f = (K_{A,B;A,B}f)\cdot g\ ,\quad g = \rho/\sigma\ . \tag{66}$$

If the operator on the left-hand side is $L^p$-continuous, then the right-hand side must be in $L^p$ for every f in $L^p$. Since the function between parentheses may be any function of $L^p$, g must be bounded, a contradiction. QED.

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