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CONVEX POLYTOPES IN RIEMANNIAN MANIFOLDS

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1. INTRODUCTION.

Let M^n be a n-dimensional Riemannian manifold. By a m-dimensional convex polytope P^m embedded in M^n $(2 \leq m \leq n)$ we mean a convex Riemannian polyhedron (for definition, see [1]) embedded in M^n bounded by a finite number of totally geodesic submanifolds P_λ^{m-1} of dimension m-1 such that P_λ^{m-1} intersect at lower dimensional totally geodesic submanifolds P_μ^r $(0 \leq r \leq m-2)$.

Let various dimensional outer angles of P^m be given. One question is to find the volume $V(P^m)$ of P^m in terms of the given outer angles of P^m. When m is even $(m = 2p)$ and M^n is of constant sectional curvature K $(\neq 0)$, the Gauss-Bonnet formula of Allendoerfer, Chern, Fenchel and Weil ([1] and [2]) implies such a volume formula which might be interesting and seems not to have appeared in given classical literatures on convex polytopes.

2. GAUSS-BONNET FORMULA OF RIEMANNIAN POLYHEDRA IN RIEMANNIAN MANIFOLDS.

A Riemannian polyhedron P^m is a Riemannian manifold with a boundary consisting of polyhedra P_λ^r of lower dimensions for $0 \leq r \leq m-1$. We denote by $X'(P^m)$ the inner characteristic of P^m, that is, the Euler-Poincaré characteristic of the open complex consisting of all inner cells in an arbitrary simplicial or cellular subdivision of P^m.

From now on we shall assume $m = 2p$, that is, m is even.

Let $S(P^m)$ be the tangent sphere bundle over P^m that is the bundle of unit tangent vectors of P^m. Let $\sigma : S(P^m) \longrightarrow P^m$ be the projection. Let $\in_{i_1 \ldots i_k}$ be the Kronecker index which is equal to +1 or -1 according as $i_1 \ldots i_k$ constitute an even or odd permutation of 1,...,k. In [2], Chern constructed a (m-1)-form

$$\Phi = \frac{1}{\pi^p} \sum_{\lambda=0}^{p-1} (-1)^\lambda \frac{1}{1.3\ldots(2p-2\lambda-1)2^{p+\lambda}\lambda!} \Phi_\lambda$$

on $S(P^m)$, where for $\lambda = 0,1,\ldots,p-1$,

$$\Phi_\lambda = \sum \in_{i_1 \ldots i_{2p-1}} \Omega_{i_2}^{i_1} \wedge \Omega_{i_4}^{i_3} \wedge \ldots \wedge \Omega_{i_{2\lambda}}^{i_{2\lambda-1}} \wedge \omega_{2p}^{i_{2\lambda+1}} \wedge \ldots \wedge \omega_{2p}^{i_{2p-1}} .$$

There exists a unique closed m-form Ψ on P^m such that

$$\sigma^*(\Psi) = \frac{(-1)^p}{2^{2p}\pi^p\, p!} \sum \in_{i_1 \ldots i_{2p-1}} \Omega_{i_2}^{i_1} \wedge \ldots \wedge \Omega_{i_{2p}}^{i_{2p-1}}$$

Let $\Gamma(P^m)$ be a outer normal vector field on P^m in $S(P^m)$. Then the Gauss-Bonnet formula for Riemannian polyhedra P^m $(m = 2p)$ in Riemannian manifolds is given by ([1] and [2])

$$(1) \qquad \int_{P^m} \Psi = \int_{\gamma(P^m)} \sigma^*\Psi = \int_{\partial P^m} \int_{\Gamma(\partial P^m)} \Phi - X'(P^m)$$

where $\Gamma(\partial P^m)$ denotes the outer angle at an arbitrary point x of ∂P^m which is a spherical cell on the unit sphere S^{m-r-1} in the normal linear manifold to P_λ^r at x.

3. CONVEX POLYTOPES IN RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE K(≠0).

Let P^m be a convex polytope in a Riemannian manifold M^n of constant sectional curvature $K(\neq 0)$. We shall consider P^m as a convex polytope in a totally geodesic submanifold N^m of M^n. The curvature form $\Omega = (\Omega_j^i)$ in the principal bundle $O(N^m)$ satisfies

$$\Omega_j^i = K\, \theta^i \wedge \theta^j$$

where $\theta = (\theta^i)$ is the canonical form in $O(N^m)$.

Consequently,

$$\int_{P^m} \Psi = \frac{(-1)^p}{2^{2p}\pi^p\, p!} \int_{\gamma(P^m)} \sum \in_{i_1 \ldots i_{2p}} \Omega_{i_2}^{i_1} \wedge \ldots \wedge \Omega_{i_{2p}}^{i_{2p-1}} =$$

$$= \frac{(-1)^p}{2^{2p}\pi^p \; p!} \; [(2p)!] \; K^p \; V(P^m).$$

Since P^m is convex, $X'(P^m) = 1$. Hence (1) becomes

$$\int_{P^m} \Psi + 1 = \int_{\partial P^m} \int_{\Gamma(\partial P^m)} \Phi .$$

Let $\partial P^m = \bigcup_{r=0}^{m-1} \bigcup_{\mu} P^r_\mu$. Then we have

$$\int_{\partial P^m} \int_{\Gamma(\partial P^m)} \Phi = \sum_{\mu} \sum_{r=0}^{m-1} \int_{P^r_\mu} \int_{\Gamma(P^r_\mu)} \Phi .$$

Since P^r_μ are totally geodesic in N^m, we may choose a suitable frame $\{e_1,\ldots,e_r\}$ on a coordinate neighborhood U in P^r_μ such that $\{e_1,\ldots,e_r,e_{r+1},\ldots,e_m\}$ is a frame for N^m and the Christoffel symbol

$$\Gamma^{\delta}_{\alpha\beta} = 0 \qquad 1 \leqslant \alpha,\ \beta \leqslant r,\ r+1 \leqslant \delta \leqslant m. \qquad \text{(see [3])}.$$

We remark that under the spherical map η from $\Gamma(P^r_\mu)$ to S^{m-r-1}_μ, $\eta^*(d\sigma) = \omega^{r+1}_{2p} \wedge \ldots \wedge \omega^{2p-1}_{2p}$, where $d\sigma$ is the surface area element of S^{m-r-1}_μ.

It is not difficult to see that

$$(2) \quad \int_{P^r_\mu} \int_{\Gamma(P^r_\mu)} \Phi = \frac{(-1)^\lambda}{\pi^p} \; \frac{1}{1.3\ldots(2p-2\lambda-1)2^{p+\lambda}\lambda!} \int_{P^r_\mu} \int_{\Gamma(P^r_\mu)} \Phi_\lambda =$$

$$= \frac{(-1)^\lambda}{\pi^p} \; \frac{1}{1.3\ldots(2p-2\lambda-1)2^{p+\lambda}\lambda!} \; [(2\lambda)!(2p-2\lambda-1)!] K^\lambda V(P^{2\lambda}_\mu)\Gamma(P^{2\lambda}_\mu)$$

when $r = 2\lambda$, otherwise

$$\int_{P^r_\mu} \int_{\Gamma(P^r_\mu)} \Phi = 0.$$

Consequently, we get from (1) and (2) the following

$$\frac{(-1)^p}{2^{2p}\pi^p \; p!} \; (2p)! \; K^p \; V(P^{2p}) + 1 =$$

$$= \sum_{\lambda=0}^{p-1} \frac{(-1)^{\lambda}}{\pi^{p}} \frac{(2\lambda)!\,(2p-2\lambda-1)!\,K^{\lambda}}{1.3\ldots(2p-2\lambda-1)\,2^{p+\lambda}\,\lambda!} \sum_{\mu} V(P_{\mu}^{2\lambda})\,\Gamma(P_{\mu}^{2\lambda}).$$

Thus, we can express the volume $V(P^{2p})$ in terms of outer angles of P^{2p} and $P_{\mu}^{2\lambda}$, for $\lambda = 0,\ldots,p-1$. This will be achieved inductively. When $p = 1$, we get the usual Gauss formula for geodesic polygons.

REFERENCES

[1] C. ALLENDOERFER and A. WEIL, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc., 53, 101-129 (1943).

[2] S. CHERN, *On the Curvatura Integra in a Riemannian Manifold*, Ann. Math., 46 (4), 674-684 (1945).

[3] S. HELGASON, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, (1962).

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