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# A NOTE ON THE MAXIMALITY OF THE IDEAL OF COMPACT OPERATORS

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Let A,B be rings, and $\mathcal{L}$ and A-B-bimodule, i.e., $\mathcal{L}$ is a left A-module and a right B-module and moreover $s(tu) = (st)u$ for $s \in A$, $t \in \mathcal{L}$ and $u \in B$. A subset $C \subset \mathcal{L}$ is a sub-bimodule if it is an additive subgroup and satisfies $sku \in C$ whenever $k \in C$ and $s \in A$, $u \in B$. If E,F are Banach spaces, we shall denote the space of bounded linear operators $T:E \to F$ by $\mathcal{L}(E,F)$ (and by $\mathcal{L}(E)$ when $E = F$). Consider the following situation: $A = \mathcal{L}(\ell^q)$, $B = \mathcal{L}(\ell^p)$, $\mathcal{L} = \mathcal{L}(\ell^p, \ell^q)$, where $\ell^r$, $1 \leq r < +\infty$ denotes the (real or complex) Banach space of numerical r-summable sequences.
The bimodule structure is defined by composition
$\ell^p \xrightarrow{U} \ell^p \xrightarrow{T} \ell^q \xrightarrow{S} \ell^q$ (we will use capital letters for operators).
It is clear that the set of compact operators $C = C(\ell^p, \ell^q)$ is a sub-bimodule of $\mathcal{L}$ . We aim to make a few remarks on the following results:

a) *if* $1 < q < p < +\infty$, *then* $C = \mathcal{L}$;

b) *if* $1 < p = q < +\infty$, *then* C *is a maximal sub-bimodule ( = two sided ideal) of* $\mathcal{L}$;

c) *if* $1 < p \leq q < +\infty$, *then all sub-bimodules* $S \subset \mathcal{L}$ *satisfying* $C \subset S$ *contain necessarily the identity operator* $J: \ell^p \to \ell^q$.

The statements a) and b) are known; a) goes back to Pitt [3] and is in fact a particular case of Th. A2 in [4], b) coincides with Th. 5.1 in [1] and c) seems to be new.
Our goal here is to observe that a modification of known proofs of b) actually yield c) of which b) is a particular case, and that a) is a corollary of b). This last remark would shorten the proof of Th. A2 in [4] and mildly confirms our suspicion that proving c) first has some methodological advantages. We believe (but have been unable to prove) the following:

CONJECTURE: *if* $1 < p \leq q < +\infty$, *then* C *is a maximal sub-bimodule,* from which c) follows trivially.

The proof of c) above is obtained by restating meanderingly the ingredients of the proofs of Lemma 5.1 in [1] and Lemmas 1 and 2 in [2]. We denote by $\|x\|_s$ the s-norm of $x = (x_1, x_2, \ldots)$, i.e.,

$$\|x\|_s = \left( \sum_{j=1}^{n} |x_j|^s \right)^{1/s}$$

LEMMA. *Let* $1 < s < +\infty$, $\varepsilon_n > 0$, $n = 1,2,\ldots$, $x^k \in \ell^s$, $k = 1,2,\ldots$ *and suppose that* $x^k \to 0$ *weakly and* $\inf\{\|x^k\|_s ; k = 1,2,\ldots\} = \delta > 0$. *Then there exists an increasing sequence of positive integers* $n_1 < n_2 < \ldots$ *and elements* $z^k \in \ell^s$, $k = 1,2,\ldots$ *such that:*

i) $\|x^{n_k} - z^k\|_s \leq \varepsilon_k$ *for* $k = 1,2,\ldots$;

ii) *the operator* $T_1 \in \mathcal{L}(\ell^s)$ *determined by* $T_1 e^k = \dfrac{z^k}{\|z^k\|_s}$ (*where* $e^k$ *is the* kth *unit vector* $(0,0,\ldots,1,0,\ldots)$ *in* $\ell^s$) *is an isometry and the image* $E = T_1(\ell^s)$ *of* $T_1$ *is a complemented subspace of* $\ell^s$.

*Proof*. Define $\varepsilon'_n = \min(\varepsilon_n, \frac{1}{2}\delta)$. For $x = (x_j)_{j=1}^{\infty} \in \ell^s$ and n a positive integer denote by $P_n x$ the sequence $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Let now $n_1$ be large enough for $\|x^1 - P_{n_1} x^1\|_s \leq \varepsilon'_1$ to be true and define $z^1 = P_{n_1} x^1$.

Since $x^n \longrightarrow 0$ weakly (i.e., coordinate wise) there is an integer $n_2$ such that $\|P_{n_1} x^{n_2}\|_s \leq \frac{1}{2}\varepsilon'_2$. Choose N such that $\|x^{n_2} - P_N x^{n_2}\|_s \leq \frac{1}{2}\varepsilon'_2$ and define $z^2 = P_N x^{n_2} - P_{n_1} x^{n_2}$. Clearly $\|x^{n_2} - z^2\|_s \leq \|x^{n_2} - P_N x^{n_2}\|_s + \|P_{n_1} x^{n_2}\|_s \leq \varepsilon'_2$. The procedure can be iterated in such a way that $\|x^{n_k} - z^k\|_s \leq \varepsilon'_k$ and the vectors $z^k$ have disjoint support, i.e., for each n there is at most one k with $z^k_n \neq 0$.

Since we also have $\|z^k\|_s \geq \|x^{n_k}\|_s - \|x^{n_k} - z^k\|_s \geq \delta - \frac{1}{2}\delta = \frac{1}{2}\delta > 0$, (i) and (ii) follow from Lemma 1 in [2].

*Proof of c).* Let p* be the conjugate of p defined by $p^* = p/(p-1)$. First observe that if $T \in \mathcal{L}(\ell^p, \ell^q)$, $1 < p,q < +\infty$, and $\sum \|Te_k\|_q^{p^*} < +\infty$, then T is compact.

This is obvious because if $P_n \in \mathcal{L}(\ell^p)$ is the projector on the first n coordinates defined above, then for $x \in \ell^p$ we have $\|(T - TP_n)x\|_q =$

$$= \|(T - TP_n) \sum_{k=1}^{\infty} x_k e_k \|_q = \| \sum_{k>n}^{\infty} x_k Te_k\|_q \leqslant \sum_{k>n}^{\infty} |x_k| \, \|Te_k\|_q \leqslant$$

$$\leqslant ( \sum_{k>n}^{\infty} |x_k|^p )^{1/p} ( \sum_{k>n}^{\infty} \|Te_k\|_q^{p^*} )^{1/p^*} \leqslant \|x\|_p ( \sum_{k>n}^{\infty} \|Te_k\|_q^{p^*} )^{1/p^*}$$

and therefore $TP_n \longrightarrow T$ in the operator norm.

Assume now that $S$ is a sub-bimodule of $\mathcal{L} = \mathcal{L}(\ell^p, \ell^q)$, $1 < p \leqslant q < +\infty$ such that $C \subset S$ and $C \neq S$, or equivalently, such that all compact operators belong to $S$ and there is a non-compact $T' \in S$. This means that for some sequence $x^1, x^2, \ldots$ weakly convergent to 0 in $\ell^p$, we have $\|T'x^n\|_q \geqslant \delta_1 > 0$ for some $\delta_1$ and all $n = 1,2,\ldots$; then also $\|x^n\|_p \geqslant \delta > 0$ for some $\delta$ and all $n = 1,2,\ldots$ . For $\varepsilon > 0$, choose a sequence $\varepsilon_n > 0$ such that $\sum \varepsilon_n^{p^*} = \varepsilon^{p^*}$ and let $n_1 < n_2 < \ldots$ and $z^1, z^2, \ldots$ be as in the lemma above, corresponding to these $\varepsilon_n$. It is clear that $\frac{1}{2}\delta \leqslant \|z^k\|_p \leqslant \Delta$ for some $\Delta$ and all k and therefore the operator $T_1$ in the lemma can be modified by an invertible diagonal operator $D \in \mathcal{L}(\ell^p)$ in such a way that $S_1 = T_1 D : \ell^p \longrightarrow \ell^p$ satisfies $S_1 e^k = z^k$ for all $k = 1,2,\ldots$ . Consider now, for $\lambda_1, \lambda_2, \ldots, \lambda_n$ arbitrary scalars, the estimate

$$\| \sum_{j=1}^{n} \lambda_j x^{n_j} \|_p \leqslant \| \sum_{j=1}^{n} \lambda_j (x^{n_j} - z^j)\|_p + \| \sum_{j=1}^{n} \lambda_j z^j\|_p \leqslant$$

$$\leqslant \sum_{j=1}^{n} |\lambda_j| \, \|x^{n_j} - z^j\|_p + \| \sum_{j=1}^{n} \lambda_j \, z^j\|_p \leqslant$$

$$\leqslant \sum_{j=1}^{n} |\lambda_j| \, \varepsilon_j + \|S_1( \sum_{j=1}^{n} \lambda_j e^j)\|_p \leqslant$$

$$\leqslant \| \sum_{j=1}^{n} \lambda_j e^j\|_p \, ( \sum_{j=1}^{n} \varepsilon_j^{p^*})^{1/p^*} + \|S_1( \sum_{j=1}^{n} \lambda_j e^j)\|_p \leqslant$$

$$\leqslant \varepsilon \,\| \sum_{j=1}^{n} \lambda_j e^j \|_p + \| S_1 ( \sum_{j=1}^{n} \lambda_j e^j ) \|_p .$$

This clearly shows that there is a well defined bounded operator $S: \ell^p \longrightarrow \ell^p$ satisfying $Se^k = x^{n_k}$ for $k = 1,2,\ldots$ , and in fact $\|(S - S_1)e^k\|_p = \|x^{n_k} - z^k\|_p \leqslant \varepsilon_k$. Let now $T'' = T'S \in S$. Setting $y^k = T''e^k = Tx^{n_k} \in \ell^q$ we have $\|y^k\|_q \geqslant \delta > 0$ for $k = 1,2,\ldots$ and since $e^k \longrightarrow 0$ weakly we also have $y^k \longrightarrow 0$ weakly in $\ell^q$. Hence the lemma above applies again: let $\{y^{m_k}\}$ be a sub-sequence of $\{y^k\}$ and $\{w^k\}$ satisfy $\|y^{m_k} - w^k\|_q \leqslant \varepsilon^k$ with $\{w^k\}$ equivalent to the unit basis of $\ell^q$. If $S' \in \mathcal{L}(\ell^p)$ is defined by $S'e^k = e^{m_k}$ we obviously have $T = T''S' \in S$ and $Te^k = y^{m_k}$. Let us denote by $U \in \mathcal{L}(\ell^q)$ the operator (corresponding to $T_1$ in the lemma) determined by $U e^k = w^k$ and by $J: \ell^p \longrightarrow \ell^q$ the identity map. We have $\| U Je^k - Te^k \|_q = \| w^k - Te^k \|_q \leqslant \varepsilon_k$ so that $U J - T \in \mathcal{L}$ is compact by the first part of this proof. Therefore $U J = (U J - T) + T \in S$. But the subspace generated by $\{w^k\}$ being complemented in $\ell^q$ (see lemma) and isomorphic to $\ell^q$, there is a $U' \in \mathcal{L}(\ell^q)$ such that $U' U \in \mathcal{L}(\ell^q)$ is the identity operator. Then $J = (U' U) J = U'(U J) \in S$, as claimed.

*Proof of a).* First let us observe that b) implies that every operator $W \in \mathcal{L}(\ell^q)$ of the form $W = W_1 W_2$, $W_1 \in \mathcal{L}(\ell^p, \ell^q)$, $W_2 \in \mathcal{L}(\ell^q, \ell^p)$ for some $p \neq q$, must necessarily be compact. In fact, the family $M$ of such operators is a two sided ideal in $\mathcal{L}(\ell^q)$ wich contains all operators of finite rank. Thus, the closure of $M$ contains $C(\ell^q)$. But the closure of $M$ is different from $\mathcal{L}(\ell^q)$ because the identity in $\mathcal{L}(\ell^q)$ is at distance one from any proper ideal such as $M$. But $C$ being maximal by b), it follows that $M \subset$ closure $M = C$.

Assume now that $1 < q < p < +\infty$ and $T \in \mathcal{L}(\ell^p, \ell^q)$ is not compact. Then there is a sequence $\{x^n\}$ in $\ell^p$ such that $x^n \longrightarrow 0$ weakly and $\|Tx^n\|_q \geqslant \delta > 0$ for some $\delta$. It follows that $\|x^n\|_p \geqslant \delta' > 0$ also for an appropiate $\delta'$. Now we apply the lemma again to produce a sequence $\{z^k\}$ in $\ell^p$ such that: i) there is an operator $T_1 \in \mathcal{L}(\ell^p)$ satisfying $Te^k = z^k$ and ii) $z^k$ is near $x^{n_k}$, so that also $\|Tz^k\|_q \geq$ $\geq \delta/2$ for all $k = 1,2,\dots$ . Consider now the operator $W = W_1W_2$ where $W_1 = T$ and $W_2 = T_1J$ for $J:\ell^q \longrightarrow \ell^p$ the identity. From the first remark, W must be compact, and in particular $\|We^k\|_q \longrightarrow 0$. But this contradicts $We^k = TT_1Je^k = TT_1e^k = Tz^k \not\longrightarrow 0$. Then T is compact, and the proof of a) is complete.

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