Revista de la
Unión Matemática Argentina
Volumen 26, 1972.

ON A CLASS OF POLYNOMIALS

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ABSTRACT. A class of polynomials is defined and studied. Its functional relation is given from which the properties are deduced. The polynomials are reciprocal of 1st kind, primitive, monic and positive. Their zeros are simple and real negative. The coefficients are positive integers and increasing until the center, then decreasing. They are calculated until the order 12. Their recursion and generic formulae are given.

1. INTRODUCTION. When the complex Fourier series $\sum_{-\infty}^{\infty} c_\kappa e^{i\kappa x} = f(x)$ is used for the solution of differential equations of 1st order, one has to use the result that the Fourier coefficient $c_\kappa = \frac{1}{i\kappa-\alpha}$ $(\alpha \neq i\kappa)$ corresponds to the function

$f(x) = \frac{2\pi e^{\alpha x}}{1-e^{2\pi\alpha}}$ in the interval $(0, 2\pi)$; in the case of an equation of 2nd order that $c_\kappa = \frac{1}{(i\kappa-\alpha)^2}$ belongs to

$f(x) = \frac{2\pi e^{\alpha x}}{1-e^{2\pi\alpha}} \left(x + \frac{2\pi e^{2\pi\alpha}}{1-e^{2\pi\alpha}}\right)$. The general case can be broken down to terms of the form $\frac{(i\kappa)^m}{(i\kappa-\alpha)^{n+1}}$ $(0 \leqslant m \leqslant n$, here and throughout this paper) by expansion into partial fractions. But as the factor $(i\kappa)^m$ means only the mth derivative of $f(x)$, we need to consider only $c_\kappa = \frac{1}{(i\kappa-\alpha)^{n+1}}$. By induction it can be found that this Fourier coefficient corresponds to the function

$$f(x) = \frac{2\pi e^{\alpha x}}{n!(1-e^{2\pi\alpha})} [x^n + e^{2\pi\alpha} \sum_{\nu=1}^{n} \binom{n}{\nu} \phi_{\nu-1}(e^{2\pi\alpha}) (\frac{2\pi}{1-e^{2\pi\alpha}})^\nu x^{n-\nu}]$$

where the ϕ_n are polynomials defined by formula (A).

2. DEFINING FORMULA

(A) $\phi_n(x) = (1-x)^n + x\sum_{\nu=1}^{n}\binom{n+1}{\nu}\phi_{\nu-1}(x)(1-x)^{n-\nu}$, $\phi_0(x) = 1.$

By this definition we can calculate the first polynomials as follows:

$\phi_0(x) = 1$; $\phi_1(x) = 1 + x$; $\phi_2(x) = 1 + 4x + x^2$;

$\phi_3(x) = 1 + 11x + 11x^2 + x^3$; $\phi_4(x) = 1 + 26x + 66x^2 + 26x^3 + x^4$;

$\phi_5(x) = 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5.$

For $n > 5$ see Appendix p. 124.

If we replace n in (A) by (n-1) and multiply by (1-x) we obtain

$$\phi_{n-1} = (1-x)^n + x\sum_{\nu=1}^{n}\binom{n}{\nu}\phi_{\nu-1}(x)(1-x)^{n-\nu}.$$

If we subtract this from (A) and use $\binom{n+1}{\nu}-\binom{n}{\nu} = \binom{n}{\nu-1}$ we find another form of the definition:

(B) $\phi_n(x) - \phi_{n-1}(x) = x\sum_{\nu=1}^{n}\binom{n}{\nu-1}\phi_{\nu-1}(x)(1-x)^{n-\nu}$ $(n\neq 0)$ or also

(C) $\phi_n(x) = (1+nx)\phi_{n-1}(x) + x\sum_{\nu=1}^{n-1}\binom{n}{\nu-1}\phi_{\nu-1}(x)(1-x)^{n-\nu}$ $(n\neq 0)$.

These definitions can be transformed into still other forms if one applies a priori the relation $x^n\phi_n(1/x) = \phi_n(x)$, proved on p. 118 (3.5) ff. Then the definitions (A), (B) and (C) become

(A') $\phi_n(x) = (x-1)^n + \sum_{\nu=1}^{n}\binom{n+1}{\nu}\phi_{\nu-1}(x)(x-1)^{n-\nu}$

(B') $\phi_n(x) - \phi_{n-1}(x) = \sum_{\nu=1}^{n}\binom{n}{\nu-1}\phi_{\nu-1}(x)(x-1)^{n-\nu}$ $(n\neq 0)$

(C') $\phi_n(x) = (x+n)\phi_{n-1}(x) + \sum_{\nu=1}^{n-1}\binom{n}{\nu-1}\phi_{\nu-1}(x)(x-1)^{n-\nu}$ $(n\neq 0)$

Definition (A) can be rewritten in shorter form as

(D) $\phi_n(x) = x\sum_{\nu=0}^{n}\binom{n+1}{\nu}\phi_{\nu-1}(x)(1-x)^{n-\nu}$, but then we have to define $\phi_{-1}(x)$ to be $1/x$. Also, (D) cannot be transformed by the relation (3.5) because the latter is not valid for $n = -1$.

All the above defining formulae are used in this paper only with $\phi_0(x) = 1$. It should be pointed out that (A), (B), and (C) can be generalized either by another choice of $\phi_0(x)$ or by a [a general] $\phi_0(x)$. (A'), (B'), (C') and (D) then become useless.

From (A) or any of the other definitions it follows that the $\phi_n(x)$ are polynomials of degree n. Their absolute term is always 1, since (2.1) $\phi_n(0) = 1$.

If we set x = 1 in (A) we obtain $\phi_n(1) = (n+1)\,\phi_{n-1}(1)$ and by iteration (2.2) $\phi_n(1) = (n+1)!$

As to other properties of the $\phi_n(x)$ it is advisable to establish first a functional relation for $\phi_n(x)$ and then find the properties from it.

3. FUNCTIONAL RELATIONS.

A functional relation for the $\phi_n(x)$ of fundamental importance is

$$(3.1)\quad \phi_n(x) = (1+nx)\,\phi_{n-1}(x) + x(1-x)\,\phi'_{n-1}(x).$$

Proof. The derivative of (D) is $\phi'_n(x) = \sum_{\nu=0}^{n} \binom{n+1}{\nu}\,\phi_{\nu-1}(x)(1-x)^{n-\nu} +$

$$+\, x\sum_{\nu=0}^{n} \binom{n+1}{\nu}\,\phi'_{\nu-1}(x)(1-x)^{n-\nu} - x\sum_{\nu=0}^{n} \binom{n+1}{\nu}\phi_{\nu-1}(x)(n-\nu)(1-x)^{n-\nu-1}\ .$$

In the second sum we insert $\phi_{\nu-1}(x)$ from formula (3.1) which is supposed to be valid for n as stated. Then the second addend becomes $\sum_{\nu=0}^{n} \binom{n+1}{\nu}\,\phi_\nu(x)(1-x)^{n-\nu-1} - \sum_{\nu=0}^{n} \binom{n+1}{\nu}(1+\nu x)\phi_{\nu-1}(x)(1-x)^{n-\nu-1}$

from which we obtain that

$$\phi'_n(x) = \sum_{\nu=0}^{n} \binom{n+1}{\nu}\,\phi_\nu(x)(1-x)^{n-\nu-1} - (n+1)x\sum_{\nu=0}^{n} \binom{n+1}{\nu}(1-x)^{n-\nu-1}\ .$$

Using (D) on the second sum we find that

$$\phi'_n(x) = \sum_{\nu=1}^{n} \binom{n+1}{\nu-1}\,\phi_{\nu-1}(x)(1-x)^{n-\nu} + \binom{n+1}{n}\,\phi_n(x)(1-x)^{-1} -$$

$$-\,(n+1)\,\phi_n(x)(1-x)^{-1}\ .$$

On the right side, only the first term remains. Applying to it definition (C) we find finally

$x(1-x)\ \phi_n'(x) = \phi_{n+1}(x) - [1+(n+1)x]\ \phi_n(x)$, and this is (3.1) for (n+1) instead of n.

Formula (3.1) is better suited for the numerical calculation of the $\phi_n(x)$ that the defining formulae of section 2. The table of coefficients on page 124 has been calculated by the use of (3.1) through n = 9.

An immediate consequence of (3.1) is obtained by use of (C),

$$(3.2)\qquad \phi_n'(x) = \sum_{\nu=1}^{n} \binom{n+1}{\nu-1}\ \phi_{\nu-1}(x)(1-x)^{n-\nu}\ .$$

From (3.2) we can find

$$(3.3)\quad \phi_n'(0) = 2(2^n-1) - n \qquad\qquad (3.4)\quad \phi_n'(1) = \binom{n+1}{2}\ .\ n!$$

An important consequence of (3.1) is the relation

$$(3.5)\quad x^n\ \phi_n(1/x) = \phi_n(x).$$

Proof. Let (3.5) be valid for n as stated. Its derivative is $nx^{n-1}\ \phi_n(1/x) - \frac{x^n}{x^2}\ \phi_n'(1/x) = \phi_n'(x)$. We eliminate the two derivatives by use of (3.1) and have

$$nx^{n-1}\phi_n(1/x) - x^{n-2}\ \frac{\phi_{n+1}(1/x) - [1+(n+1)/x]\ \phi_n(1/x)}{(1-\frac{1}{x})\ \frac{1}{x}} =$$

$$= \frac{\phi_{n+1}(x) - [1+(n+1)x]\ \phi_n(x)}{(1-x)x}$$. Adding the identity

$$\frac{1+(n+1)x}{(1-x)x}\ x^n\ \phi_n(1/x) = \frac{1+(n+1)x}{(1-x)x}\ \phi_n(x)$$ we obtain

$$\frac{x^{n+1}}{(1-x)x}\ \phi_{n+1}(1/x) = \frac{\phi_{n+1}(x)}{(1-x)x}$$, or finally $x^{n+1}\ \phi_{n+1}(1/x) = \phi_{n+1}(x)$,

which is (3.5) for (n+1) instead of n.

The relation (3.5) characterizes reciprocal polynomials.

4. PROPERTIES OF THE COEFFICIENTS.

Let $\phi_n(x) = \sum_{\nu=0}^{n} a_{n\nu}x^\nu$ and insert it into (3.1), the latter written for (n+1) instead of n. Comparison of the coefficients of x^ν

$(1 \leqslant \nu \leqslant n)$ yields

$$a_{n+1,\nu} = a_{n,\nu} + (n+1)a_{n,\nu-1} + \nu a_{n,\nu} - (\nu-1)a_{n,\nu-1} \quad \text{or shorter}$$

$$a_{n+1,\nu} = (\nu+1)a_{n,\nu} + (n+2-\nu)a_{n,\nu-1} \ . \tag{4.1}$$

This recursion formula holds also when $\nu = n+1$ and $\nu = 0$ if $a_{n,\nu} = 0$ for negative indices.

Consequences of (4.1)

(4.2) *All coefficients* $a_{n,\nu}$ *are positive integers.*

Proof. If $\phi_n(x)$ has only positive integer coefficients, the right side of (4.1) is a positive integer, therefore also $a_{n+1,\nu}$. The coefficients through $n = 5$ are positive integers, see page 116.

(4.3) *The coefficients are symmetrical*: $a_{n,\nu} = a_{n,n-\nu}$.

Proof. It is well known that from (3.5) follows $|a_{n,\nu}| = |a_{n,n-\nu}|$ (reciprocal polynomials). By (4.2) the coefficients are all positive, thus $a_{n,\nu} = a_{n,n-\nu}$, i.e. the $\phi_n(x)$ are symmetrical polynomials, also called reciprocal of 1st kind.

(4.4) $a_{n,0} = a_{n,n} = 1$. This follows from (4.3) for $\nu = 0$ and from (2.1). Thus, the $\phi_n(x)$ are monic and primitive.

$$a_{n,\nu} < a_{n,\nu+1} \qquad (0 \leqslant \nu \leqslant \tfrac{n}{2}-1). \tag{4.5}$$

Proof.

$$a_{n+1,\nu+1} - a_{n+1,\nu} = (\nu+2)a_{n,\nu+1} + (n-2\nu)a_{n,\nu} - (n+2-\nu)a_{n,\nu-1} >$$

$$> (\nu+2)a_{n,\nu+1} + (n-2\nu)a_{n,\nu} - (n+2-\nu)a_{n,\nu} = (\nu+2)(a_{n,\nu+1} - a_{n,\nu}) > 0.$$

Thus, if (4.5) holds for n, it holds for (n+1). That it holds for $n = 2,3,4,5$ is seen on page (p.116).

Because of the symmetry (4.3) the coefficients must decrease monotonically in the second half of the polynomials.

Two particular cases of (4.1) are useful.

(4.6) $a_{n+1,1} = 2a_{n,1} + n + 1$ (Set $\nu = 1$ in (4.1) and use (4.4))

(4.7) $a_{2n,n} = 2(n+1)a_{2n-1,n}$ (Set $n = 2\kappa-1$, $\nu=\kappa$ in (4.1) and use (4.3)).

The recursion formula (4.1), together with (4.3), (4.6) and (4.7) as well as (2.2), seems to be the most convenient way to calculate the coefficients. The table on page 124 has been extended until $n = 12$ by this method.

$$(4.8) \qquad a_{n,\nu} = \sum_{\kappa=0}^{\nu} (-1)^{\kappa} \binom{n+2}{\kappa} (\nu+1-\kappa)^{n+1}.$$

Proof. Substitute (4.8) into the right side of (4.1)

$$(\nu+1)a_{n,\nu} + (n+2-\nu)a_{n,\nu-1} = (\nu+1) \sum_{\kappa=0}^{\nu} (-1)^{\kappa} \binom{n+2}{\kappa} (\nu+1-\kappa)^{n+1} +$$

$$+ (n+2-\nu) \sum_{\kappa=0}^{\nu} (-1)^{\kappa} \binom{n+2}{\kappa} (\nu-\kappa)^{n+1} = (\nu+1) \sum_{\kappa=1}^{\nu} (-1)^{\kappa} \binom{n+2}{\kappa} (\nu+1-\kappa)^{n+1} +$$

$$+ (\nu+1)(\nu+1)^{n+1} + (n+2-\nu) \sum_{\kappa=1}^{\nu} (-1)^{\kappa} \binom{n+2}{\kappa-1} (\nu+1-\kappa)^{n+1} =$$

$$= \sum_{\kappa=1}^{\nu} (-1)^{\kappa} (\nu+1-\kappa)^{n+1} \left[(\nu+1)\binom{n+2}{\kappa} - (n+2-\nu)\binom{n+2}{\kappa-1} \right] + (\nu+1)^{n+2} =$$

$$= \sum_{\kappa=1}^{\nu} (-1)^{\kappa} (\nu+1-\kappa)^{n+1} \left[(\nu+1-\kappa)\binom{n+3}{\kappa} \right] + (\nu+1)^{n+2} =$$

$$= \sum_{\kappa=0}^{\nu} (-1)^{\kappa} \binom{n+3}{\kappa} (\nu+1-\kappa)^{n+2} = a_{n+1,\nu} . \qquad \text{q.e.d.}$$

5. THE ZEROES OF THE $\phi_n(x)$.

It is desirable to know where the zeros of the $\phi_n(x)$ are situated, not only to help their numerical computation but also for theoretical reasons.

The zeros of the first five polynomials (p.116) can be found by elementary methods. They are

$n = 1$, -1 ; $n = 2$, $-2 \pm \sqrt{3} \approx \begin{cases} -0.268 \\ -3.73 \end{cases}$

$n = 3$ $\quad -1$

$\qquad\quad -5 \pm 2\sqrt{6} \approx \begin{cases} -0.101 \\ -9.90 \end{cases}$

$$n = 4 \qquad -\frac{13+\sqrt{105}}{2} \pm \frac{1}{2}\sqrt{270+26\sqrt{105}} \approx \begin{cases} -0.0431 \\ -23.20 \end{cases}$$

$$-\frac{13-\sqrt{105}}{2} \pm \frac{1}{2}\sqrt{270-26\sqrt{105}} \approx \begin{cases} -0.431 \\ -2.32 \end{cases}$$

$$n = 5 \qquad -1$$

$$-(14+3\sqrt{15}) \pm \sqrt{330+84\sqrt{15}} \approx \begin{cases} -0.0195 \\ -51.22 \end{cases}$$

$$-(14-3\sqrt{15}) \pm \sqrt{330-84\sqrt{15}} \approx \begin{cases} -0.220 \\ -4.54 \end{cases}$$

The zeros for $n = 6$ and $n = 7$ are (by approximation)

$n = 6$: -0.00915 ; -0.123 ; -0.535 ; -1.87 ; -8.16 ; -109.3 .

$n = 7$: -0.00438 ; -0.0717 ; -0.319 ; -1.000 ; -3.14 ; -13.96 ; -228.5.

The zeros are all simple, real negative and are separated by the zeros of the next lower polynomial.

(5.1) *These are general properties of the* $\phi_n(x)$.

Proof. Let $x_{n\nu}$ $(1 \leqslant \nu \leqslant n)$ be the (simple) zeros of $\phi_n(x)$ such that $x_{nn} < x_{n,n-1} < \ldots < x_{n2} < x_{n1} < 0$. Then, the derivative $\phi_n'(x)$ has different signs at two consecutive zeros, and equation (3.1) becomes, for $(n+1)$ instead of n and for $x = x_{n\nu}$,

$\phi_{n+1}(x_{n\nu}) = (1-x_{n\nu})x_{n\nu}\phi_n'(x_{n\nu})$ where the factor of $\phi_n'(x_{n\nu})$ is negative for all ν. Therefore also $\phi_{n+1}(x)$ has different signs at two consecutive zeros of $\phi_n(x)$ so that between them must lie at least one zero of $\phi_{n+1}(x)$. Thus we have at least $(n-1)$ different zeros of $\phi_{n+1}(x)$. Another zero must lie between x_{n1} and 0 $(x_{n1} < x_{n+1,1} < 0)$ because of $\phi_{n+1}(0) = 1$ and $\phi_{n+1}(x_{n1}) < 0$. Corresponding to this zero $x_{n+1,1}$, another zero must exist, found beyond x_{nn}, as the polynomials are reciprocal and $x_{nn} = \frac{1}{x_{n1}}$, $x_{n+1,n+1} = \frac{1}{x_{n+1,1}}$. Since we have now $(n+1)$ different zeros of $\phi_{n+1}(x)$, they must be simple. They lie all on the real negative axis and are separated by those of $\phi_n(x)$. Thus, the statement (5.1) has been proved.

The fact that the zeros of the polynomials $\phi_n(x)$ lie on the negative real axis makes the $\phi_n(x)$ a subclass of the class of Hurwitz polynomials, i.e. of those polynomials whose zeros lie all in the left half plane. Hurwitz polynomials play a role in considerations of stability.

As the Hurwitz polynomials are positive, i.e. assume for values of the right half plane only values of the same, the $\phi_n(x)$ are also positive polynomials.

Some more properties of the zeros of $\phi_n(x)$ are easily obtained.

If we set $x = -1$ in (3.5) we have $(-1)^n\phi_n(-1) = \phi_n(-1)$ which implies $\phi_n(-1) = 0$ for odd n. For even n, $x = -1$ cannot be a zero of a reciprocal polynomial .

The zero $x = -1$ is the only rational one; all others are irrational because the real zeros of a polynomial with integer coefficients are either integer or irrational.
Besides, our polynomials are reciprocal so that if p (integer) is a zero, also $\frac{1}{p}$ would be a zero, which is only possible if $p = \pm 1$.
But $p = 1$ cannot be a zero because of the positivity of the coefficients (4.2) so that $x = -1$ is the only rational one.

A look at the cases $1 \leqslant n \leqslant 7$ shows that the zeros of $\phi_n(x)$ lie between $-a_{n,n-1}$ and $-\frac{1}{a_{n1}}$ .It can be proved, that the relation

(5.2) $\quad -a_{n,n-1} \leqslant x_{n\nu} \leqslant -\frac{1}{a_{n1}} \quad (1 \leqslant \nu \leqslant n)$ holds for all n.

Proof. For $1 \leqslant n \leqslant 5$ is $\max \frac{a_{n,\nu+1}}{a_{n\nu}} = a_{n1}$. (p.116).

For arbitrary n we have, using (4.1),

$$\frac{a_{n+1,\nu+1}}{a_{n+1,\nu}} = \frac{(\nu+2)a_{n,\nu+1} + (n+1-\nu)a_{n\nu}}{(\nu+1)a_{n\nu} + (n+2-\nu)a_{n,\nu-1}} = \frac{\frac{a_{n,\nu+1}}{a_{n\nu}} + \frac{n+1-\nu}{\nu+2}}{\frac{\nu+1}{\nu+2} + \frac{n+2-\nu}{\nu+2}\,\frac{a_{n,\nu-1}}{a_{n\nu}}}\,.$$

This fraction assumes a maximum when $\nu = 0$:

$$\frac{a_{n+1,\nu+1}}{a_{n+1,\nu}} = 2a_{n1} + n + 1 = a_{n+1,1} \text{ (cfr. 4.6) } = a_{n+1,n}\,.$$

Therefore, we find by Kakeya's theorem $\left(|x_{n+1,\nu}| \leqslant \max \frac{a_{n+1,\nu+1}}{a_{n+1,\nu}}\right)$

that $-a_{n+1,n} \leqslant x_{n+1,\nu}$ for all $\nu \leqslant n+1$ and , as the polynomials are reciprocal, also $x_{n+1,\nu} \leqslant -\frac{1}{a_{n+1,1}}$. Thus, we have (5.2) for (n+1) instead of n.

The zeros x_{n1} *accumulate toward the origin if* $n \to \infty$.

Proof. If $x_{n-1,1} > -\frac{1}{n}$ we have, on ground of (5.1), $-\frac{1}{n} < x_{n-1,1} < x_{n1}$. If $x_{n-1,1} < -\frac{1}{n}$ we have $\phi'_{n-1}(-\frac{1}{n}) > 0$ and, because of (3.1),

$\phi_n(-\frac{1}{n}) = \frac{n+1}{n^2}\,\phi'_{n-1}\,(-\frac{1}{n}) < 0$, which implies, together with $\phi_n(0)=1$,

that $-\frac{1}{n} < x_{n1} < 0$. Thus we have $\lim_{n\to\infty} x_{n1} = -0$, and as the $\phi_n(x)$ are reciprocal also $\lim_{n\to\infty} x_{nn} = -\infty$.

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Recibido en mayo de 1972.

6. Appendix. Coefficients of the $\phi_n(x)$ $(0 \leqslant n \leqslant 12)$.

n													
0	1												
1	1	1											
2	1	4	1										
3	1	11	11	1									
4	1	26	66	26	1								
5	1	57	302	302	57	1							
6	1	120	1191	2416	1191	120	1						
7	1	247	4293	15619	15619	4293	247	1					
8	1	502	14608	88234	156190	88234	14608	502	1				
9	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1			
10	1	2036	152637	2203488	9738114	15724248	9738114	2203488	152637	2036	1		
11	1	4083	478271	10187685	56580360	172250400	172250400	56580360	10187685	478271	4083	1	
12	1	8178	1479726	45533450	374590965	1486145280	2411505600	1486145280	374590965	45533450	1479726	8178	1