Revista de la
Unión Matemática Argentina
Volumen 26, 1972.

FOURIER SERIES EXPANSION FOR THE GENERAL POWER OF THE DISTANCE BETWEEN TWO POINTS

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ABSTRACT. Simply by using the property of orthogonality of cosine functions, the Fourier series for the hypergeometric function ${}_2F_1[\frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda+1; -\frac{r^2}{c^2}]$ where $r = (r_1^2 + r_2^2 - 2r_1r_2\cos\omega)^{\frac{1}{2}}$ is the distance between two points $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_2, \varphi_2)$ is given. When c = 0, it yields the expansion for $r^{-(\lambda+1)}$ valid for all values of λ, where λ can be real or complex. By specializing λ, the expansions for $(c^2 + r^2)^{-1/2}$ and $(c^2 + r^2)^{-3/2}$ are also obtained.

1. INTRODUCTION. The expansion for powers of the distance between two points is often required in various fields of Mathematical Physics. Various approaches to the expansion have been considered by several authors. Recently Sack [1] in a generalization of Laplace's expansion, presented a series expansion of the form

$$(1) \qquad r^n = \sum_{\ell=0} R_{n,\ell}(r_1, r_2)\, P_\ell(\cos\omega) ,$$

in terms of Legendre polynomials P_ℓ for arbitrary real powers of the distance $r = (r_1^2 + r_2^2 - 2r_1r_2\cos\omega)^{1/2}$ between the two points $P_1 \equiv (r_1, \theta_1, \varphi_1)$ and $P_2 \equiv (r_2, \theta_2, \varphi_2)$ referred to polar coordinates and $\cos\omega = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 . \cos(\varphi_1 - \varphi_2)$.

Ashour [2] following Sack's method obtained the Fourier series expansion:

$$(2) \qquad r^n = \sum_{\ell=0} R_{n,\ell}(r_1, r_2) \cos \ell\omega$$

Sack's approach (and in effect Ashour's as well) requires a knowl edge of the partial differential equation satisfied by r^n. Secondly, their approach assumes a preknowledge of the functional form of the desired expansion functions $R_{n,\ell}$. Thirdly, their expansion presuppose the knowledge of the expansion for a specific value of n. Furthermore, the use of series solution and the equating of coefficients which their approach entails could be cumbersome.

In this paper we show that a new Fourier expansion for r^λ (where λ can be any number real or complex satisfying $R(\lambda) > -1$) can be obtained very simply merely by using the orthogonal properties of the cosine functions. Two other new results are also given.

2. MATHEMATICAL DERIVATIONS.

Eason, Noble and Sneddon [3] have proved that

$$(3) \qquad \int_0^\infty J_\mu(at)J_\nu(bt)e^{-ct}t^\lambda dt = \frac{\Gamma(\mu-\nu+\lambda+1)}{2^{\mu-\nu}\,\pi\, c^{\mu-\nu+\lambda+1}\,\Gamma(\mu+\nu+1)} \times$$

$$\int_0^\pi (a-be^{-i\theta})^{\mu-\nu}\, e^{-i\nu\theta}\ {}_2F_1[\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2}\lambda + \tfrac{1}{2},$$

$$\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2}\lambda + 1\ ;\ \mu - \nu + 1\ ;\ -\frac{r^2}{c^2}]\ d\theta$$

where $r^2 = a^2 + b^2 - 2ab\cos\theta$, $R(\mu-\nu+\lambda+1) > 0$, and $R(c) > 0$. Using the result given by Erdelyi {4, p.373, equ.(8)} and putting $\mu = \nu$, we obtain after simplifications that

$$(4) \qquad \int_0^\infty J_m(at)J_m(bt)t^\lambda e^{-ct}dt = \frac{\Gamma(\frac{1}{2}\lambda + \frac{1}{2} + m)\ \Gamma(\frac{1}{2}\lambda + 1 + m)2^\lambda(ab)^m}{\sqrt{\pi}\ c^{2m+\lambda+1}\ [\Gamma(m+1)]^2} \times$$

$$F_4[(m + \tfrac{1}{2}\lambda + \tfrac{1}{2}),\ (m + \tfrac{1}{2}\lambda + 1);\ m+1,\ m+1;\ -\frac{a^2}{c^2},\ -\frac{b^2}{c^2}]$$

valid for $R(\lambda) > -1$. The function F_4 appearing in equation (4) is the Appell hypergeometric function of two variables defined by

$$(5) \qquad F_4[\alpha,\beta;\gamma,\delta;x,y] = \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(\alpha)_{p+q}\ (\beta)_{p+q}}{(\gamma)_p\ (\delta)_q\ p!q!}\ x^p y^q$$

With the notation,

(6) $a = r_< = \min(r_1, r_2)$, $b = r_> = \max(r_1, r_2)$, $(r_< > 0)$

we obtain from equations (3) and (4) that

$$(7) \quad \int_0^\infty \cos m\,\theta \;{}_2F_1[\tfrac{1}{2}\lambda + \tfrac{1}{2}, \tfrac{1}{2}\lambda + 1; 1; -\tfrac{r^2}{c^2}]\, d\theta =$$

$$= \frac{\sqrt{\pi}\; 2^\lambda (r_< r_>)^m\; \Gamma(\tfrac{1}{2}\lambda + m + \tfrac{1}{2})\; \Gamma(\tfrac{1}{2}\lambda + m + 1)}{c^{2m}\; \Gamma(\lambda+1)\; [m!]^2} \times$$

$$\times \; F_4[\tfrac{1}{2}(\lambda+2m+1), \tfrac{1}{2}(\lambda+2m+2); m+1, m+1; -(\tfrac{r_<}{c})^2, -(\tfrac{r_>}{c})^2]$$

valid for $R(\lambda) > -1$.
This result becomes useful in the following section.

3. THE MAIN RESULT TO BE PROVED.

We shall obtain the Fourier expansion of the hypergeometric function ${}_2F_1[\frac{1}{2}\lambda + \frac{1}{2}, \frac{1}{2}\lambda + 1; 1; -\frac{r^2}{c^2}]$.

Let

$$(8) \quad {}_2F_1[\tfrac{1}{2}\lambda + \tfrac{1}{2}, \tfrac{1}{2}\lambda + 1; 1; -\tfrac{r^2}{c^2}] = \sum_{m=0}^\infty A_{\lambda,m} \cos m\,\theta .$$

The hypergeometric function ${}_2F_1$ is continuous and of bounded variation in the interval $(0,\pi)$.
Multiplying both sides of equation (8) by $\cos n\theta$ and integrating with respect to θ from 0 to π, and using the orthogonality property of cosine functions, we obtain

$$A_{\lambda,0} = \frac{1}{\pi}\int_0^\pi {}_2F_1[\tfrac{1}{2}\lambda + \tfrac{1}{2}, \tfrac{1}{2}\lambda + 1; 1; -\tfrac{r^2}{c^2}]\, d\theta \quad , \text{ and}$$

$$(9) \quad \underset{(m\neq 0)}{A_{\lambda,m}} = \frac{2}{\pi}\int_0^\pi \cos m\theta \;{}_2F_1[\tfrac{1}{2}\lambda + \tfrac{1}{2}, \tfrac{1}{2}\lambda + 1; 1; -\tfrac{r^2}{c^2}]\, d\theta$$

With the help of equation (7), the Fourier series for ${}_2F_1$ is obtained thus:

$$(10)\quad A_{\lambda,0} = \frac{2^{\lambda}\ \Gamma(\frac{1}{2}\lambda + \frac{1}{2})\ \Gamma(\frac{1}{2}\lambda + 1)}{\sqrt{\pi}\ \Gamma(\lambda+1)}\ F_4[\tfrac{1}{2}(\lambda+1),\ \tfrac{1}{2}(\lambda+2);\ 1,\ 1;$$

$$;\ -(\frac{r_<}{c})^2,\ -(\frac{r_>}{c})^2]$$

$$A_{\lambda,m} = -\frac{2^{\lambda+1}(r_< r_>)^m\ \Gamma(\frac{1}{2}\lambda + \frac{1}{2} + m)\ \Gamma(\frac{1}{2}\lambda + m + 1)}{\sqrt{\pi}\ c^{2m}\ \Gamma(\lambda+1)\ [m!]^2} \times$$

$$\times\ F_4[\tfrac{1}{2}(\lambda+2m+1),\ \tfrac{1}{2}(\lambda+2m+2);\ m+1,\ m+1;\ -(\frac{r_<}{c})^2,\ -(\frac{r_>}{c})^2]$$

valid for $R(\lambda) > -1$.

4. PARTICULAR CASES.

On specializing the parameters in equations (9) and (10), interesting results can be obtained.

Case (1): On letting $c \to 0$, equations (8) and (10) reduce to

$$(11)\qquad r^{-(\lambda+1)} = \sum_{m=0}^{\infty} A_{\lambda,m} \cos m\theta$$

with $$A_{\lambda,0} = r_>^{-(\lambda+1)}\ {}_2F_1[\tfrac{1}{2}\lambda + \tfrac{1}{2},\ \tfrac{1}{2}\lambda + \tfrac{1}{2};\ 1;\ (\frac{r_<}{r_>})^2]$$

$$(12)\qquad \underset{(m\neq 0)}{A_{\lambda,m}} = \frac{2}{m!}\ r_<^m\ r_>^{-m-\lambda-1}(\tfrac{1}{2}\lambda + \tfrac{1}{2})_m\ \ {}_2F_1[m + \tfrac{1}{2}\lambda + \tfrac{1}{2}\ ,$$

$$,\ \tfrac{1}{2}\lambda + \tfrac{1}{2};\ m+1;\ (\frac{r_<}{r_>})^2\ ]$$

valid for $R(\lambda) > -1$. By analytical continuation, this condition can be further relaxed. For negative values of λ, the series for $A_{\lambda,m}$ is of finite terms and the expansion therefore remains valid for all values of λ. When λ is real, the expansion of this case reduces to the result of Ashour [2].

Case (ii): Let $\lambda = 0$, $c \neq 0$. Equations (8) and (10) reduce to

(13) $$(c^2 + r^2)^{-1/2} = \sum_{m=0}^{\infty} A_{\lambda,m} \cos m\theta$$

with

(14) $$A_{\lambda,0} = c\, F_4\left[\tfrac{1}{2},\ 1;\ 1,\ 1;\ -(\tfrac{r_<}{c})^2,\ -(\tfrac{r_>}{c})^2\right]$$

$$\underset{(m\neq 0)}{A_{\lambda,m}} = \frac{2(r_< r_>)^m\, \Gamma(m+1/2)}{\sqrt{\pi}\, c^{2m-1}\, m!}\, F_4[(m+1/2),\ (m+1);\ (m+1),$$

$$,\ (m+1);\ -(\tfrac{r_<}{c})^2,\ -(\tfrac{r_>}{c})^2]$$

Case (iii): Let $\lambda = 1$, $c \neq 0$. Equations (8) and (10) again yield the Fourier expansion

(15) $$(c^2 + r^2)^{-3/2} = \sum_{m=0}^{\infty} A_{\lambda,m} \cos m\theta$$

with

(16) $$A_{\lambda,0} = c^3\, F_4[1,\ 3/2;\ 1,\ 1;\ -(\tfrac{r_<}{c})^2,\ -(\tfrac{r_>}{c})^2]$$

$$\underset{(m\neq 0)}{A_{\lambda,m}} = \frac{4(r_< r_>)^m\, \Gamma(m+3/2)}{\sqrt{\pi}\, c^{2m-3}\, m!}\, F_4[m+1,\ m+3/2;\ m+1,\ m+1;$$

$$;\ -(\tfrac{r_<}{c})^2,\ -(\tfrac{r_>}{c})^2]$$

Expressions such as expanded in Cases (ii) and (iii) frequently arise in atomic physics where the constant c may be treated as an adjustable physical parameter.

By setting the parameters suitably, further results can be obtained as particular cases of expansion (8). Other interesting expansions of the function f(r) will be treated in the next communication.

ACKNOWLEDGEMENT. The author would like to thank Professor B. L. Sharma for helpful discussions.

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Recibido en octubre de 1971.