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SEMITENSIONS IN THE SOLVING OF THE "POTENTIAL'S RESTRICTED PROBLEM" *

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1. INTRODUCTION.

A) We shall be concerned with the linear problem:

"to maximize $L(t) = \sum_{i=1}^{n} c_i\ t_i$

subject to the restraints:

a) $t_j - t_i \leqslant q_{ij}$, $i, j = 0,1, \ldots , n.$

b) $t_0 = 0$ ".

This problem is seen to be equivalent to the "potential's restricted problem" (see [1], p.205).

The polyhedron Z_q of solutions of a) is assumed non void; and we will denote by Z_q^0 the set of solutions of a) and b). Note that Z_q is a cilinder, parallel to the vector $e = (1,1,\ldots,1)$.

Henceforth, we set $X = \{0,1,\ldots,n\}$; and a "potential of X" will be any real function t, eventually extended either by $+\infty$ or by $-\infty$ values.

B) In the inequalities a), their bounds q_{ij} can be replaced by their "strictest bounds" ε_{ij}, obtained either by

$$\varepsilon_{ij} = \inf_{j_1\ldots j_n} (q_{ij_1} + q_{j_1j_2} + \ldots + q_{j_nj}) , \qquad (1)$$

([2]), or by some alternative way (see [3]). Hence, $Z_q = Z_\varepsilon , Z_q^0 = Z_\varepsilon^0$.
As in [3], we shall use the term *semitensions* for these strictest bounds ε_{ij}. They have the basic properties:

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i) $\varepsilon_{ii} = 0$

ii) $\varepsilon_{ij} + \varepsilon_{jk} \geqslant \varepsilon_{ik}$

and we summarize below the few other properties that will henceforth be needed.

For a fixed potential t of X, the highest potential in Z_ε that precedes t is E(t), given by

$$E(t)_j = \inf_i (t_i + \varepsilon_{ij}) . \qquad (2)$$

Thus, $E(t) \leqslant t$, and $E(t) = t$ is equivalent to $t \in Z_\varepsilon$.

Alternatively, the lowest potential in Z_ε that follows t is

$$F(t)_j = \sup_i (t_i - \varepsilon_{ji}) . \qquad (2')$$

Thus, $F(t) \geqslant t$, and $F(t) = t$ iff $t \in Z_\varepsilon$.

Now, if δ^j is the potential that has the value 0 at j and $+\infty$ at each $i \neq j$:

$$E(\delta^j)_i = \varepsilon_{ji} \quad , \quad F(-\delta^j)_i = -\varepsilon_{ij}$$

2. THE (A,B)-MINIMAX POINTS OF Z^0_ε .

A) In this section, we will show that: if t is a potential defined on a subset A of X, where it satisfies the inequalities a), then t has both a maximum and a minimum extension on X, where they also satisfy a).

Let E_A denote the operator defined as in (2), but "on A"; that is, relative to the restriction $\varepsilon_{/A}$. For any potential t we set

$$t^A_j = \begin{cases} t_j \ , \text{ when } j \in A \ , \\ +\infty \ , \text{ otherwise} \ . \end{cases}$$

Clearly:

$$E(t^A)_{/A} = E_A(t) \ , \qquad (3)$$

since for $j \in A$, $E(t^A)_j = \inf_{i \in X} (t^A_i + \varepsilon_{ij}) = \inf_{i \in A} (t_i + \varepsilon_{ij}) = E_A(t)_j$.

From (3) it follows that if $t \in Z_{\varepsilon/A}$, then $E(t^A)$ is the maximum extension of t on X. In particular, for any $t \in Z_\varepsilon$, $E(t^A) \geqslant t$, $E(t^A)_{/A} = t_{/A}$.

In analogous manner, we define t_A as:

$$(t_A)_j = \begin{cases} t \ , \text{ if } j \in A \ , \\ -\infty \ , \text{ otherwise} \ . \end{cases}$$

Thus, $F(t_A)_{/A} = F_A(t)$, and therefore, if $t \in Z_{\varepsilon/A}$, then $F(t_A) \in Z_\varepsilon$ is the minimun extension of t on X. Moreover, for any $t \in Z_\varepsilon$, $F(t_A) \leqslant t$, $F(t_A)_{/A} = t_A$.

We add a remark on the monotony of extensions: $t \geqslant t'$ implies both $E(t^A) \geqslant E(t'^A)$ and $F(t_A) \geqslant F(t'_A)$.

B) With every $t \in Z_\varepsilon$, we now associate a relation $R_t \subset X \times X$, by means of:

$$(i,j) \in R_t \quad \text{if} \quad t_j = t_i + \varepsilon_{ij} \ .$$

Now, it is clear that, for $t, t' \in Z_\varepsilon$, we shall have $R_t = R_{t'}$ if and only if both t and t' belong to the same open face of Z_ε; by reason of that, and since $c = \{x/x_j = x_i + \varepsilon_{ij}, (i,j) \in R_t\} \cap Z_\varepsilon$ is a closed face, the $R = R_t$ is seen to be associated to the closed face c of Z_ε.

Moreover, notice that if $A_1, A_2, \ldots, A_s$ are the connected components of R - i.e., of its representative graph-the dimension of c is precisely s. This conclusion relies on the fact that the solutions of the linear system $x_j = x_i + \varepsilon_{ij}$, $(i,j) \in R$, can be given by: $x = \lambda_1(t\ 1_{A_1}) + \lambda_2(t\ 1_{A_2}) + \ldots + \lambda_s(t\ 1_{A_s})$, where 1_{A_j} is the characteristic function of the set A_j.

Also, it is clear that the dimension of a face c^0 of Z^0_ε is one unity less than the dimension of the corresponding face c of Z_ε, since Z^0_ε is the intersection of Z_ε with the hyperplane $x_0 = 0$.

In particular, a vertex $t^0 \in Z^0_\varepsilon$ has a connected associated relation $R^0 = R_{t0}$; an edge of Z^0_ε, a relation with two connected components.We note that R^0 determines uniquely t^0.

In this way, the relations R associated to edges meeting a given vertex $t^0 \in Z^0_\varepsilon$, can be characterized as follows:

"$R \subset R^0$ *has two connected components* $C \ni 0$ *and* $D = X - C$, *with* $R_{/C} = R^0{}_{/C}$ *and* $R_{/D} = R^0{}_{/D}$; *and moreover satisfies*

either I) $(i,j) \in R^0 - R$ *iff* $i \in C$, $j \in D$, (α)

or II) $(i,j) \in R^0 - R$ *iff* $i \in D$, $j \in C$. "

For brevity, we shall speak of the "blocks" C and D, "fixed" and "loose", respectively.

The edge associated to this R has the supporting line $t = t^0 + \lambda 1_D$. Thus, t is a point in this edge iff either

a) case I: $\lambda_1 \leqslant \lambda \leqslant 0$, where $\lambda_1 = -\inf_{\substack{i\varepsilon D \\ j\varepsilon C}} (\varepsilon_{ij} - (t^0_j - t^0_i))$, or

b) case II: $0 \leqslant \lambda \leqslant \lambda_2$, where $\lambda_2 = \inf_{\substack{i\varepsilon C \\ j\varepsilon D}} (\varepsilon_{ij} - (t^0_j - t^0_i))$. (β)

Joining to R the pairs (i,j) where (β) is realized, we obtain the relation R_{t^1} associated to the opposed vertex t^1 in this edge.

We see that case I (resp. II) corresponds to a "descent" ("ascent") of t^0 on the loose block D.

For a geometric view, it could be useful to consider the line $t = t^0 + \lambda 1_D$ even when R is neither in case I nor in case II. This situation may be named case III, and corresponds to:

"There are some $(i,j) \in R^0$, $i \in C$, $j \in D$, and some $(k,l) \in R^0$, $k \in D$, $l \in C$".

In this case, the line has in common with Z^0_ε the only point t^0, and thus can be most conveniently considered as supporting a "virtual edge" - i.e., reduced to t^0 .

C. The relation $R = R_t$ associated to a point $t \in Z_\varepsilon$, has the following properties of interest:

1. R *is a preorder in* X.

Clearly, reflexivity is immediate. For transitivity, we have:

a) for (i,j), $(j,k) \in R$, it must be $t_j = t_i + \varepsilon_{ij}$, $t_k = t_j + \varepsilon_{jk}$,

and thus $t_k = t_i + \varepsilon_{ij} + \varepsilon_{jk} \geqslant t_i + \varepsilon_{ik}$ from property (ii) of ε.
b) $t_k \leqslant t_i + \varepsilon_{ik}$. From a) and b), it follows that $(i,k) \in R$.

2. If $\varepsilon_{ij} + \varepsilon_{jk} = \varepsilon_{ik}$, then: $(i,k) \in R$ iff $(i,j),(j,k) \in R$.

D. Let A,B,be sets of X such that $A \cup B = X$, $A \cap B = \{0\}$. A point $t \in Z^0_\varepsilon$ will be called an "(A,B)-minimax"-in notation, $t \in M_{AB} = M$- if for any $t' \in Z^0_\varepsilon$ such that $t'_{/A} \geqslant t_{/A}$ and $t'_{/B} \leqslant t_{/B'}$, necessarily $t = t'$.

Observe that this can be easily expressed in terms of the extensions above considered. In fact, $t \in Z^0_\varepsilon$ is an (A,B)-minimax iff $t = E(t^B) = F(t_A)$. From that, $t \in M$ implies: 1) For any $j \in A$, $t_j = \inf\limits_{i\varepsilon B} (t_i + \varepsilon_{ij})$, and 2) for any $j \in B$, $t_j = \sup\limits_{i\varepsilon A} (t_i - \varepsilon_{ji})$.

Thus, when $t \in M$, its associated relation $R = R_t$ verifies: 1) for any $j \in A$, there is some $i \in B$ such that $(i,j) \in R$, and 2) for any $j \in B$, there is some $i \in A$ such that $(j,i) \in R$. A relation R with these properties will be conveniently named "(A,B)-total", and we can summarize our conclusions as follows:

"$t \in Z^0_\varepsilon$ is an (A,B)-minimax point iff its associated relation R is (A,B)-total".

Incidentally, M is an union of open faces of the polyhedron Z^0_ε; and even more, of closed faces, since $R_{t'} \supset R_t = R$ implies that $R_{t'}$ is also (A,B)-total.

Moreover, let R^0 denote the connected and (A,B)-total relation associated to a vertex $t^0 \in M$. Then, to any edge in M and meeting t^0, corresponds a relation $R \subset R^0$ that is (A,B)-total and verifies condition (α).

E. For our purposes, it will be useful to consider not only preorder relations R that are (A,B)-total, but also their "traces", $S = R \cap (B\times A)$ on $B\times A$. It is seen than R and S have the same connected components.

Furthermore, if R and R^0 are two preorder and (A,B)-total relations that both satisfy condition (α), then their respective traces S and S^0 are also (A,B)-total, satisfy (α), and moreover to cases I, II for R corresponds respectively cases I, II for S. By reason of that, the λ_1, λ_2, values can be obtained by

$$\lambda_1 = - \inf_{\substack{i\in D\cap B \\ j\in C\cap A}} (\varepsilon_{ij} - (t_j^0 - t_i^0)) ,$$

$$\lambda_2 = \inf_{\substack{i\in C\cap B \\ j\in D\cap A}} (\varepsilon_{ij} - (t_j^0 - t_i^0)). \qquad (\beta')$$

For brevity, we confine ourselves to the verification in the case $\lambda_2 = \varepsilon_{ij} - (t_j^0 - t_i^0)$, $i \in C \cap B$, $j \in D \cap B$. Since R is (A,B)-total and C, D, are their connected components, there is some $k \in D \cap A$ such that $(j,k) \in R$, i.e. $\varepsilon_{jk} = t_k^0 - t_j^0$. Then, $\lambda_2 = \varepsilon_{ij} - (t_j^0 - t_i^0) = \varepsilon_{ij} - (t_j^0 - t_k^0 + t_k^0 - t_i^0) = \varepsilon_{ij} + \varepsilon_{jk} - (t_k^0 - t_i^0) \geqslant \varepsilon_{ik} - (t_k^0 - t_i^0)$, hence $\lambda_2 = \varepsilon_{ik} - (t_k^0 - t_i^0)$, $i \in C \cap B$, $k \in D \cap A$.

F. We can summarize these results as follows:
To each $t^0 \in M = M_{AB} \subset Z_\varepsilon^0$ we assign its "characteristic relation" $S_{t^0} = S^0 \subset B\times A$, defined as $S^0 = \{(i,j)/t_j^0 - t_i^0 = \varepsilon_{ij}, (i,j) \in B\times A\}$ and which is (A,B)-total: $\mathrm{Proj}_A(S^0) = A$, $\mathrm{Proj}_B(S^0) = B$. The dimension of the open face of Z_ε^0 that contains t^0 is one unity less than the number of connected components of S^0.

In particular, if S^0 is connected - and therefore t^0 is a vertex- any (A,B)-total subrelation S of S^0 which has two connected components determines a partition $C \ni 0$, D, of X.
According to the above considerations, S corresponds to an edge in M whenever we have: either I): $(i,j) \in S^0 - S$ iff $i \in C$, $j \in D$, or II): $(i,j) \in S^0 - S$ iff $j \in C$, $i \in D$. The opposed vertex in this edge will be obtained by replacing λ in $t^0 + \lambda 1_D$ by the corresponding λ_1 or λ_2, evaluated from (β'); and its characteristic relation S' will result by joining to S all the pairs (i,j) that verify either $\lambda_1 = -(\varepsilon_{ij} - (t_j^0 - t_i^0))$ or $\lambda_2 = \varepsilon_{ij} - (t_j^0 - t_i^0)$,

according to the case in consideration.

G. We confine ourselves to a way of associating to a point $t \in Z^0_\varepsilon$ another point $t^* \in M$, namely by means of:

$$t^* = E((F(t_A))^B)$$

By doing that, we will have:

PROPOSITION 1.

1) $t \in Z^0_\varepsilon$ *implies* $t^* \in M$,

2) $t^*_{/A} \geqslant t_{/A}$, $t^*_{/B} \leqslant t_{/B}$,

3) $t = t^*$ *iff* $t \in M$,

4) *The mapping* $t \longrightarrow t^*$ *is continuous.*

Proof. First, it is a clear consequence of the properties of the extensions that for any $t \in Z^0_\varepsilon$ we have:

$$F(t_A)_{/A} = t_{/A} \leqslant E(t^B)_{/A} \quad , \quad F(t_A)_{/B} \leqslant t_{/B} = E(t^B)_{/B} \qquad (\gamma)$$

Now, is self-evident that $t^* \in Z^0_\varepsilon$. Therefore, by use of (γ):

a) $t^*_{/B} = E((F(t_A))^B)_{/B} = F(t_A)_{/B}$, and thus $t^{*B} = (F(t_A))^B$ concludes that $t^* = E(t^{*B})$.

b) From (γ) and a) we have:

$$F(t^*_A)_{/A} = t^*_{/A} \quad , \quad F(t^*_A)_{/B} \leqslant t^*_{/B} = F(t_A)_{/B} \ .$$

But now $t^*_{/A} = E((F(t_A))^B_{/A} \geqslant F(t_A)_{/A} = t_{/A}$ ensures $t^*_A \geqslant t_A$ and by monotony $F(t^*_A) \geqslant F(t_A)$, whence $F(t^*_A)_{/B} \geqslant F(t_A)_{/B} = t^*_{/B}$. Then $t^* = F(t^*_A)$.

This proves that $t^* \in M$, the rest is a straightforward consequence from the definitions for E, F and minimax.

H. We remark that $(E(\delta^0))^*$ and $(F(-\delta^0))^*$ are both vertices of Z^0_ε and, of course, (A,B)-minimax.

In fact, $\rho = E\delta^0$ has coordinates $\rho_j = \varepsilon_{0j}$, so that from $\rho_j - \rho_0 = \varepsilon_{0j}$ it follows that $(0,j) \in R_\rho$ for any $j \in X$. Because R_ρ is

connected, ρ is a vertex of Z^0_ε. Moreover, as $\rho^*_{/A} \geqslant \rho_{/A}$ and ρ is the highest potential in Z^0_ε, we have $\rho^*_{/A} = \rho_{/A}$, and thus $(0,j) \in R_{\rho*}$ for any $j \in A$.
Furthermore, the fact that $\rho*$ is (A,B)-minimax ensures that for any $i \in B$ there is some $j \in A$ such that $(i,j) \in R_{\rho*}$, and thus $R_{\rho*}$ is connected and $\rho*$ is proved to be a vertex of Z^0_ε.
The proof for $\sigma* = (F(-\delta^0))*$ proceeds by arguments completely anal ogous.

3. THE LINEAR FORM L AND THEIR RELATED MINIMAX.

We start by considering the linear form $L(t) = \sum_{i=1}^{n} c_i\, t_i$, and by setting $A = \{i/c_i > 0\} \cup \{0\}$, $B = \{i/c_i \leqslant 0\} \cup \{0\}$.
From the definition of t^*, it follows that $t \in Z^0_\varepsilon$ implies $L(t) \leqslant L(t^*)$. In particular, the maximum of L is attained at a point $t \in M$. The converse is ensured whenever $c_i \neq 0$ for any i.

PROPOSITION 2. *If* $t^0 \in M$ *is a local maximum of* L *in* M, *then it is a maximum of* L *in* Z^0_ε.

Proof. Let us consider any $t^1 \in Z^0_\varepsilon$, and form $t^\lambda = t^0 + \lambda(t^1 - t^0)$, $0 \leqslant \lambda \leqslant 1$. Then, $\lambda \to 0$ determines $t^\lambda \to t^0$, and from Prop. 1, $(t^\lambda)^* \to (t^0)^* = t^0$. From the assumed conditions, $L((t^\lambda)^*) \leqslant L(t^0)$ for suitably small values of λ, and the above considerations imply that $L(t^1) \leqslant L(t^0)$.

B. For a given vertex $t^0 \in M$ whose characteristic relation is S^0, let $S \subset S^0$ be the characteristic relation of an edge in M meeting t^0 and having C,D, as, respectively, fixed and loose blocks.
From $t = t^0 + \lambda\, 1_D (\lambda_1 \leqslant \lambda \leqslant 0$ when in case I, $0 \leqslant \lambda \leqslant \lambda_2$ when in case II), we see that L increases on the edge iff: either $L(1_D) < 0$ in case I, or $L(1_D) > 0$ in case II. $(L(1_D) = \sum_{i \in D} c_i)$

In the simplex method, the consideration of all the edges meeting a vertex is generally out of reach. Very much the same happens with the determination of all relations $S \subset S^0$ corresponding to edges in M and meeting t^0.

In the following section, we present an algorithm that obviates this difficulty.

4. ALGORITHM.

A) The algorithm proposed for solving the problem posed in § 1, can be summarized in the following steps:

0) The starting point is a vertex $t^0 \in M$, of characteristic relation S^0. We construct on S^0 a tree T^0 which is maximal and without any path of length two. Such a tree will be called an "(A,B)-tree" - for short, a TAB of t^0.

1) For a given $(h,k) \in T^0$, we consider $T^0-(h,k)$ and the corresponding connected components $C \ni 0$ and D that it determines. After doing that, we evaluate $L(1_D) = \sum_{i\varepsilon D} c_i$. We have then the alternative:

a) either (I) $L(1_D) < 0$ with $h \in C$, $k \in D$, or (II) $L(1_D) > 0$ with $h \in D$, $k \in C$.

b) The situation a) does not happen .

Clearly, a) expresses the possibility of an increase of L by displacing the loose block D.

2) When (h,k) is such that a) holds, we evaluate:

either (case I): $\lambda_1 = -\inf_{\substack{i\varepsilon B\cap D \\ j\varepsilon A\cap C}} (\varepsilon_{ij} - (t_j^0 - t_i^0))$,

or (case II) $\lambda_2 = \inf_{\substack{i\varepsilon B\cap C \\ j\varepsilon A\cap D}} (\varepsilon_{ij} - (t_j^0 - t_i^0))$.

For each case, there will be always at least a pair such that the inf. is realized. From among them, we choose if possible a pair (i,j) that excludes the point 0.

After this has been carried out, we construct the TAB $T^1 = T^0 -$

$-(h,k)+(i,j)$ and obtain its corresponding vertex $t^1=t^0+\lambda 1_D \in M$, where of course λ is either λ_1 or λ_2.
We return to 1) and reinitiate the procedure, considering now T^1 instead of T^0.

3) If (h,k) corresponds to b) we start again with another pair $(1,m)$ of T^0.

4) The algorithm is concluded when every (h,k) of T^0 appears in the situation b). When that happens, t^0 maximizes L.

REMARKS.

1) For the starting point in 0), we may take either ρ^* or σ^* (see 2.H).

2) The construction of the TAB T^0 is always possible not only if

a) for any i,j, $(i,0)$ and $(0,j)$ are not both in S^0, but also when

b) there are $(i,0)$ and $(0,j)$ in S^0, in which situation it suffices to suppress every $(i,0) \in S^0$ and to take a maximal tree in the residual connected relation.

3) When a) holds, it is clear that if λ_1 (resp.λ_2) $\neq 0$, then $t^1 \neq t^0$ and $L(t^1) > L(t^0)$. But when $\lambda_1(\lambda_2) = 0$, although $t^1 = t^0$ we nevertheless have $T^1 \neq T^0$.

4) The instructions in step 3 preclude the possibility of T^1 having a path of length 2, so that T^1 is truly a TAB. In fact, assume case I and let $(i,0)$ be a pair that realizes the infimum: if moreover there is some $(0,j) \in T^0$, we single out (i,j) that also realizes the inf. (see 2.C).
Case II is similar.

To justify the algorithm, it remains to prove: first, the second assertion of step 4); and next, that the algorithm can conclude in a finite number of steps.

B. We begin by considering a relation U in X, which is connected and without paths of length 2 - i.e., the ocurrence of both (i,j) and (j,k) in U is excluded. Then $A' = \{i/(i,j) \notin U$, for any $j\}$, $B' = \{j/(i,j) \notin U$, for any $i\}$ determine a partition of X.

We now define the mapping η: $X \times X \to R$, by setting: $\eta_{ij} = 0$ whenever either $(i,j) \in U$ or $i = j$; and for the rest: $\eta_{ij} = 1$ when $i \in B'$ and $j \in A'$, $\eta_{ij} = 2$ when i,j are either both in A' or both in B', and finally $\eta_{ij} = 3$ when $i \in A'$ and $j \in B'$.

By a rather tedious discussion of cases, we can verify the following:

LEMMA 1. *η is a semitension on X, such that:* $\eta_{ij} = 0$, $i \neq j$, *iff* $(i,j) \in U$; $\eta_{ij} \geqslant 1$ *when* $(i,j) \notin U$, *and* $\eta_{ij} + \eta_{jk} - \eta_{ik} \geqslant 1$ *when* $i \neq j$ *and* $j \neq k$.

As the foregoing conditions hold in particular for any arbitrary TAB, this Lemma permits us to assert:

PROPOSITION 3. *Let* t^0 *be a vertex in* M. *If* T^0 *is a* TAB *of* t^0, *and* b) *holds for it, then* t^0 *is optimal for* L.

Proof. The semitension $\varepsilon' = \varepsilon + \eta$ coincides with ε on T^0 and is strictly greater than ε in the remnant. First, we note that $Z^0_\varepsilon \subset Z^0_{\varepsilon'}$. Next, $t^0 \in Z^0_{\varepsilon'}$, and its associated relation is precisely T^0, since $t^0_j - t^0_i = \varepsilon_{ij} = \varepsilon'_{ij}$ whenever $(i,j) \in T^0$ and furthermore $t^0_j - t^0_i \leqslant \varepsilon_{ij} < \varepsilon'_{ij}$ for any $(i,j) \notin T^0$.

From the above it follows that, as T^0 is connected, t^0 is a vertex of $Z^0_{\varepsilon'}$ (and even more, is a regular vertex). By reason of that, b) implies (3.B) that t^0 is optimal of L on $Z^0_{\varepsilon'}$, and thus also on Z^0_ε.

C. Let t^0 be a vertex in M, and T^0 a TAB of t^0. As seen before, if T^0 is in b) of the alternative, then t^0 is optimal for L. But whenever T^0 is in a) of the alternative, one of two possibilities happens: Whether

a_1) It is possible to determine an "algorithmic sequence" T^0, T^1, T^2, ..., of TABs-i.e., each one is obtained from the precedent by means of the algorithm - for which there is some T^m corresponding to a vertex $t^m \neq t^0$; and then, $L(t^m) < L(t^0)$. Or

a_2) The possibility a_1 is excluded.

When a_1) holds, then T^m is discussed as T^0 - that is, determining for it the validity of b), a_1) or a_2).

When a_2) appears, we will show below that is possible to determine a (finite) algorithmic sequence $T^0, T^1, T^2, \ldots, T^N$, such that T^N is in b) and thus $t^0 = t^N$ is optimal for L.
For proving that, we proceed by "perturbating" ε so as to split t^0 into a finite number of regular vertices. For doing that, let U be the relation obtained from the characteristic relation S^0 of t^0 by taking out, whenever there is some $(0,j) \in S^0$, the pairs $(i,0) \in S^0$.

Now, let us assume that we can determine - as it will be carried out later - a semitension ε' such that:

A1) t^0 is a vertex of $Z^0_{\varepsilon'}$ whose associated relation is T^0.

A2) If t^r is any vertex of $Z^0_{\varepsilon'}$ whose associated relation is a TAB $T^r \subset U$, and t^{r+1} is obtained from t^r by means of the algorithm as applied in $Z^0_{\varepsilon'}$, then the relation associated to t^{r+1} in $Z^0_{\varepsilon'}$ is also a TAB $T^{r+1} \subset U$, and moreover T^{r+1} proceeds from T^r by means of the algorithm in Z^0_{ε}.

By assuming this for the moment, the desired conclusion follows. In fact, note first that A2) implies $L(t^{r+1}) > L(t^r)$, hence it precludes repeated elements in the sequence $t^0, t^1, t^2, \ldots$ and therefore any two of the corresponding TABs $T^0, T^1, T^2, \ldots$ are different.
From that, and since U contains only a finite number of TABs, the sequence must end in some TAB T^N, so that the passing of t^N to another vertex t^{N+1} of $Z^0_{\varepsilon'}$ is not possible by means of the algorithm in $Z^0_{\varepsilon'}$. The conclusion follows that T^N must be, and is, in b) of the alternative.

Now for the proof that we can determine ε' complying with the aforesaid requirements. First, and as the starting assumption is that we cannot by means of the algorithm in Z^0_{ε} pass from a vertex

t^0 to another one t, we will modify ε with a view to maintain this impossibility on the future "perturbated" t^r. This may, and shall, be done by adding to ε the semitension η defined as in Lemma 1. Finally, we "perturbate" $\varepsilon + \eta$ by means of a suitably small function, so as to split t^0 into regular vertices of $Z^0_{\varepsilon'}$.

For this we state:

LEMMA 2. (Variation of η). *In the conditions of Lemma 1, and if* x *satisfies*: $x_{ii} = 0$, $0 \leqslant x_{ij} < 1/3$ *for* $i \neq j$, *then* $\eta + x$ *is a positive semitension.*

With that, we have the starting point for the proof; however, we introduce previously the following notation.

Let $\gamma = [\beta_0\, \alpha_0\, \beta_1\, \alpha_1\, \dots\, \beta_n\, \alpha_n\, \beta_{n+1}]$, with $\beta_0 = \beta_{n+1}$, be a cycle in U such that $\beta_i \in B$, $\alpha_i \in A$. We denote by x_γ the sum of the $x_{\beta_i\alpha_i}$ and the $-x_{\alpha_i\beta_{i+1}}$. In particular, we note that $\varepsilon_\gamma = 0$, and also $(\varepsilon + \eta)_\gamma = 0$, since $\varepsilon_{\beta_i\alpha_i} = t^0_{\alpha_i} - t^0_{\beta_i}$, $-\varepsilon_{\alpha_i\beta_{i+1}} = t^0_{\alpha_i} - t^0_{\beta_{i+1}}$ and $\eta_{/U} = 0$.

Let us prove that we can determine x complying with the requirements:

$$x_{ij} = 0 \text{ for } (i,j) \in T^0 ,$$

$$0 < x_{ij} < 1/3 \text{ for } (i,j) \in U - T^0 ,$$

$$x_\gamma \neq 0 \text{ for any cycle } \gamma \text{ of } U.$$

In fact, $x_\gamma = 0$ determines an hyperplane in R^{U-T^0}, since $x_\gamma \equiv 0$ would imply $\gamma \subset T^0$ and that is impossible. As there is a finite number of cycles γ in U, we cannot cover with the corresponding hyperplanes the open set $\{x / 0 < x_{ij} < 1/3,\ (i,j) \in U-T^0\}$.

Once selected one such x, and for $0 < \delta < 1$, we set $\varepsilon' = \varepsilon + \eta + \delta x$. Since δx remains in the conditions of Lemma 2, ε' is a semitension.

Now, it only remains to prove that ε' satisfies the above requirements A1 and A2.

A1) $\eta + \delta x = 0$ on T^0, $\eta + \delta x > 0$ elsewhere. Therefore, T^0 is the associated relation of t^0 in $Z^0_{\varepsilon'}$, and t^0 is a vertex.

A2) For this property - more lengthy to prove - we suppose inductively that there is $\delta_r > 0$ such that, for any $\delta \in [0,\delta_r)$, $t^r = t^r(\delta)$ is a vertex in $Z^0_{\varepsilon'}$, whose associated relation T^r is a TAB in U which does not depend on δ; and, moreover, such that $t^r(\delta)$ is continuous, with $t^r(0) = t^0$.

Now, let t^{r+1} be a vertex in $Z^0_{\varepsilon'}$, obtained from t^r by means of the algorithm.
We assert that t^{r+1} satisfies the same conditions that t^r, and furthermore than its TAB T^{r+1} is obtained from T^r through the algorithm in Z^0_{ε}.
For that, note first that from $t^{r+1} = t^r + \lambda 1_D$ if follows that t^{r+1} is a continuous function of δ in $[0,\delta_r)$, with $t^{r+1}(0) = t^0$.
In fact, λ is an infimum of continuous functions of δ, each having the expression $\varepsilon'_{ij} - (t^r_j - t^r_i)$; and for $\delta = 0$ and $(i,j) \in U$, we have $\varepsilon'_{ij} - (t^r_j - t^r_i) = \varepsilon_{ij} - (t^0_j - t^0_i) = 0$, thus ensuring $\lambda(0) = 0$.
Next, suppose that the algorithmic passing from t^r to t^{r+1} corresponds to case II - the case I is clearly analogous. Then, we have:

$$\lambda = \lambda_2 = \inf_{\substack{i \varepsilon B \cap C \\ j \varepsilon A \cap D}} (\varepsilon'_{ij} - (t^r_j - t^r_i))$$

We once more consider the pairs (i,j), $i \in B \cap C$, $j \in A \cap D$.

1) For $(i,j) \notin U$, $\varepsilon'_{ij} - (t^r_j - t^r_i) = \varepsilon_{ij} + \eta_{ij} + \delta x_{ij} - (t^0_j - t^0_i) +$ $+ (t^0_j - t^r_j - (t^0_i - t^r_i)) \geqslant 1 + (t^0_j - t^r_j) - (t^0_i - t^r_i)$ since $\eta_{ij} \geqslant 1$, $x_{ij} > 0$, $\varepsilon_{ij} - (t^0_j - t^0_i) > 0$, and thus, for suitably small δ we have: $\varepsilon'_{ij} - (t^r_j - t^r_i) \geqslant 1/2$.

2) For $(i,j) \in U$, $\varepsilon'_{ij} - (t^r_j - t^r_i)$ converges to $0 = \varepsilon_{ij} - (t^0_j - t^0_i)$.

Then, for suitably small δ_{r+1} and $\delta \in [0,\delta_{r+1})$, $\lambda(\delta) = \varepsilon'_{ij} -$

$-(t_j^r - t_i^r)$, with $(i,j) \in U$, $i \in B \cap C$, $j \in A \cap D$. Assuredly that, for any $\delta \in [0,\delta_{r+1})$, there is an unique pair (i,j) with these requirements. For, if there were two of them, a cycle would appear in the associated relation T^{r+1} of t^{r+1}, and from that, $\varepsilon'_\gamma = 0 = \varepsilon_\gamma + \eta_\gamma + \delta x_\gamma = \delta x_\gamma$ gives a contradiction.

Finally, let $\Delta_{ij}(\delta) = \varepsilon'_{ij} - (t_j^r - t_i^r)$, $(i,j) \in U$, $i \in B \cap C$, $j \in A \cap D$.

From the results above, it is not possible for any two of such functions to have the same value at any $\delta \in [0,\delta_{r+1})$, and therefore one of those functions is strictly lesser than the others. This shows that we have $\lambda(\delta) = \varepsilon'_{ij} - (t_j^r - t_i^r)$ for the same unique pair (i,j) and any $\delta \in [0,\delta_{r+1})$; in particular, it enables us to conclude not only that T^{r+1} is a TAB in U which does not depend on δ, but furthermore that we can pass from T^r to T^{r+1} by means of the algorithm in Z_ε^0.

The existence of δ_0 for the starting T^0 is easily verifiable.

CONCLUDING REMARK. In the proposed algorithm, the eventuality of a "circling" is not precluded; but we rely on the fact of its highly improbable ocurrence in practice. That is why no special corrective devices were considered nor deemed necessary.

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