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ACTIONS ON A GRAPH

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ABSTRACT. Part of the theory of flows and tensions on a graph is extended to any kind of *actions*, i.e. to any subspace of the space of real functions defined on the arcs; in particular, the theorems on the existence of flows or tensions under bilateral restraints.

1. INTRODUCTION.

Our basic setting is a graph $G = (X,S)$; here it means that $S \subset X \times X$ verifies: $(i,i) \notin S$ and $(i,j) \in S$ implies $(j,i) \in S$, $i,j \in X$. We assume G connected.

We consider functions $f,g,\ldots$ defined on S and we denote by $f.g = \sum f_{ij}\, g_{ij}$, the usual scalar product.

By E we denote the linear space of anti-symmetric functions: $f_{ij} + f_{ji} = 0$, $(i,j) \in S$.

The subspaces of flows and tensions, $\Phi,\Theta \subset E$, are defined by: $\varphi \in \Phi$, if $\sum\limits_{j} \varphi_{ij} = 0$, $i \in X$, and $\theta \in \Theta$, if $\theta_{ij} = t_j - t_i$, t defined on X ($\theta = \Delta t$). Φ and Θ are orthogonal complements in E.

If $F \subset E$ is defined as the set of solutions f of the linear system: $\lambda^{\alpha}.f = 0$, $\alpha \in M$, its orthogonal space $G = F^{\perp}$ in E, is generated by the μ^{α}, $\alpha \in M$, where $\mu^{\alpha}_{ij} = \lambda^{\alpha}_{ji} - \lambda^{\alpha}_{ij}$ When $M = X$ and the λ^{i}_{jk} vanish except for $j = i$, writing $\lambda^{i}_{ik} = \lambda_{ik}$, we have $\mu^{i}_{ij} = -\lambda_{ij}$, $\mu^{i}_{ji} = \lambda_{ij}$ and $\mu^{i}_{hk} = 0$ for the remaining $(h,k) \in S$. Then, the elements $g \in G$ are of the form $g_{ij} = t_j\lambda_{ji} - t_i\lambda_{ij}$ ($g = \Delta^{\lambda}t$), where t is a function on X. The case $\lambda_{ij} \equiv 1$ corresponds to $F = \Phi$, $G = \Theta$. Given an orientation to G - i.e. a subset U of S containing for each $(i,j) \in S$ one and only one of the pairs (i,j), (j,i)- and a positive m_i, $i \in X$, defining $\lambda_{ik} = m_i$, if $(i,k) \in U$, and

$\lambda_{ij} = 1$, if $(i,j) \notin U$, we obtain the spaces of multiplicative flows and tensions ([1] pp.225).

Actions of a certain kind can be thought as the elements f of a subspace F of E - f_{ij} representing the intensity of the action f trasmited from i to j through the link $(i,j) \in S$.

Certain notions and results of the theory of flows and tensions on a graph, can be extended to any subspace F of E. Doing that, a unified linear treatment of the outstanding cases $F = \Phi$, $F = \Theta$ - that may be useful - is obtained.

In 2 we give the notion of elementary action, corresponding to the notions of elementary cycles and cocycles, and a proposition on the decomposition of any action in elementary ones. It gives the known decomposition of a positive flow (tension) - on an oriented graph - as a positive linear combination of elementary cycles (cocycles) ([1] pp.143).

In 3 we prove the analogue of Hoffman and Roy's theorems ([2], [3]) for actions of any kind, using the appropiate geometric version of the consistence theorem of a system of linear inequalities (Farkas-Minkowsky).

2. ELEMENTARY ACTIONS.

For $f \in E$ we denote $s(f) = \{(i,j)/f_{ij} > 0\}$, the (effective) support of f.

It is seen that, for $f,g \in E$:

(A) $\phi \neq s(g) \subset s(f)$ implies $s(f-\lambda g) \subset s(f)$, properly, for the positive number $\lambda = \max\left\{\frac{f_{ij}}{g_{ij}} \;/\; g_{ij} > 0\right\}$. A function $f \in F$, $f \neq 0$, is said to be an *elementary function* of F if for any $g \in F$, $s(g) \subset s(f)$ implies $g = \lambda f$.

This means that s(f) is a minimal set of $\{s(g)/g \in F\}$. In fact, s(f) minimal implies, for each $g \in F$ with $s(g) \subset s(f)$, that $s(f-\lambda g) \subset s(f)$, properly, (A); since $f - \lambda g \in F$ it follows $s(f - \lambda g) = \emptyset$, $f = \lambda g$. The converse is clear.

Of course, if f is an elementary function of F, so is λf, $\lambda \neq 0$.

PROPOSITION. *Any* $f \in F$, $f \neq 0$, *is a sum of elementary functions*

f_α *of* F *such that* $s(f_\alpha) \subset s(f)$.

Proof. Let g_1 be an elementary function of F such that $s(g_1) \subset \subset s(f)$. From (A), for some $\lambda_1 > 0$, it is $s(f-\lambda_1 g_1) \subset s(f)$, properly. If $f - \lambda_1 g_1 \neq 0$, we apply to $f - \lambda_1 g_1 \in F$ the same argument and we get an elementary $g_2 \in F$, $\lambda_2 > 0$, such that $s(g_2) \subset \subset s(f-\lambda_1 g_1)$ and $s(f-\lambda_1 g_1 - \lambda_2 g_2) \subset s(f-\lambda_1 g_1)$, properly. After a finite number of steps we have elementary $g_1,\ldots,g_k \in F$, $\lambda_1,\lambda_2,\ldots,\lambda_k > 0$, such that $f - \lambda_1 g_1 - \ldots - \lambda_k g_k = 0$. The proposition follows with $f_\alpha = \lambda_\alpha g_\alpha$, $1 \leqslant \alpha \leqslant k$.

For a set $Z \subset S$, such that $(i,j) \in Z$ implies $(j,i) \notin Z$, we define $\zeta = \zeta(Z)$ by $\zeta_{ij} = 1, -1, 0$ according to $(i,j) \in Z$, $(j,i) \in Z$ or $(i,j),(j,i) \notin Z$, respectively. $\zeta \in E$ and $s(\zeta) = Z$.

If $Z = \{(i,j),(j,k),\ldots,(h,l),(l,i)\}$ is a cycle, ζ is a flow, if $Z = \{(i,j)/i \in A,\ j \notin A\}$ $(A,\ X - A \neq \emptyset)$ is a cocycle, ζ is the tension $\Delta(-1_A)$.

If the sequence $i,j,k,\ldots,h,l$ of the cycle Z has not repeated elements Z is an elementary cycle. If A and X - A are connected, in the graph G_Z obtained eliminating the $(i,j),(j,i)$, with $(i,j) \in Z$, the cocycle Z is said to be an elementary one.

It is clear that $\zeta = \zeta(Z)$ is an elementary flow, when Z is an elementary cycle. For an elementary cocycle Z, if $\theta = \Delta t$ is such that $s(\theta) \subset Z = s(\zeta)$, the connectedness of A and X - A in G_Z implies $t_{|A} = \alpha$, $t_{|X-A} = \beta$, then $\Delta t = \theta = (\beta-\alpha)\zeta$, with $\beta \geqslant \alpha$. Hence ζ is an elementary tension.

Conversely, if $\varphi \neq 0$ is a flow, $s(\varphi)$ verifies that $(i,j) \in s(\varphi)$ implies $(j,k) \in s(\varphi)$ for some $k \neq i$ $(\varphi_{ji} + \sum_{k \neq i} \varphi_{jk} = 0,\ \varphi_{ji} = -\varphi_{ij} < 0)$.

It follows that $s(\varphi)$ contains a cycle, and then also an elementary cycle Z. Hence if φ is an elementary flow, $\zeta = \lambda.\varphi$, $\lambda > 0$. On the other hand, if $\theta = \Delta t \neq 0$ is a tension, taking α: $\min t_i < \alpha < \max t_i$, the cocycle Z defined by means of $A = \{i/t_i \leqslant \alpha\}$ is contained in $s(\theta)$. Hence, if θ is an elementary tension we have $\zeta = \lambda\theta$, $\lambda > 0$. Z has to be an elementary cocycle, otherwise we could take a connected component A' of A (alternatively of X - A) in G_Z and define Z' in terms of A'. But this would imply $s(\zeta') \subset s(\theta)$, properly; which is a contradiction.

Resuming, the multiples $\lambda.\zeta$, $\lambda > 0$, $\zeta = \zeta(Z)$, for Z elementary cycle (cocycle), are the elementary functions of $\Phi(\Theta)$.

3. EXISTENCE THEOREM.

We consider a finite dimensional linear space with the scalar product x.y. For a cone Q - $Q+Q \subset Q$, $\lambda Q \subset Q$ for every $\lambda \geqslant 0$ - the dual cone is defined by $Q^o = \{x/x.y \geqslant 0, \text{ for any } y \in Q\}$.

We need the theorem of consistence of a system of linear inequalities under the following form:

"Given the polyhedral set C and the polyhedral cone Q, $(C, Q \neq \emptyset)$ it holds: $Q \cap C \neq \emptyset$ if and only if, for every $x \in Q^o$ there is a $c \in C$ such that $x.c \geqslant 0$".

In fact, if $Q \cap C = \emptyset$, we can separate the closed convex (polyhedral) set Q - C from 0; i.e. there is x such that $x.(c-q) < 0$, for any $c \in C$, $q \in Q$. Taking λq, $\lambda > 0$, instead of q, we conclude that $x.q \geqslant 0$, i.e. $x \in Q^o$. For $q = 0$ we have $x.c < 0$ for every $c \in C$. The converse is clear.

We will apply the theorem to a subspace Q. In this case $Q^o = Q^\perp$.

THEOREM. *Let* $c_{ij}+c_{ji} \geqslant 0$, *for any* $(i,j) \in S$, *and* F *be a linear subspace of* E. *In order that there exists* $f \leqslant c$, $f \in F$, *it is necessary and sufficient that, for each elementary* $g \in G = F^\perp$, $g^+.c \geqslant 0$.

REMARK. As it is usual: $g^+ = \max(g,0)$. The condition $c_{ij}+c_{ji} \geqslant 0$ is obviously necessary for the existence of an anti-symmetric $f \leqslant c$.

Proof. Omitting the word "elementary", the equivalence follows from the theorem of consistence applied to $Q = F$, $C = \{x/x \leqslant c\}$ in the linear space E.

In fact, $Q \cap C \neq \emptyset$, i.e. there is $f \in F$, $f \leqslant c$, is equivalent to assert that for any $g \in G = Q^o$ there is $x \in E$, $x \leqslant c$ such that $g.x \geqslant 0$. This implies $g^+.c \geqslant 0$, since from $(g^+-g^-)x \geqslant 0$, $c \geqslant x$, it follows $g^+.c \geqslant g^+.x \geqslant g^-.x$, and then $2g^+.c \geqslant (g^++g^-)x = |g|x = 0$ ($|g|$ is symmetric).

And conversely, if $g^+.c \geqslant 0$, $g \in G$, then defining, for a given $g \in G$, x by $x_{ij} = c_{ij}$, $-c_{ij}$, $-1/2(c_{ij}-c_{ji})$, according to $g_{ij} > 0$, < 0 or $= 0$, we have $g.x = 2g^+.c \geqslant 0$.

Finally, if $g^+.c \geqslant 0$ for elementary functions $g \in G$, the same holds for any $g \in G$, since from the proposition we can write $g^+ = \sum_\alpha g_\alpha^+$, for elementary functions g_α of G.

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