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AN ELEMENTARY PROOF OF THE STRUCTURE THEOREM OF FINITELY GENERATED MODULES OVER A PRINCIPAL DOMAIN

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Let A be a principal domain. It is a well known result that if M is a finitely generated A-module then M can be written as a direct sum of a free submodule and the torsion submodule. The free part is (up to isomorphism) completely determined by its rank. So, the main point is to characterize the torsion submodule. Therefore let M be a finitely generated torsion module over A. We intend to give an elementary proof of the classical result stating that M can be written as a direct sum of cyclic submodules. Let J be the ideal of A, annihilator of M. Since A is principal we have $J = \langle a \rangle$. Let

$$a = \prod_{i=1}^{n} p_i^{r_i} \quad , \quad p_i \in A \, , \; r_i \in \mathbb{N}$$

be a factorization of a in A. The elements p_i are irreducibles in A and furthermore p_i is not associated to p_j if $i \neq j$. If, for any irreducible element p in A we denote with M_p the p-primary component of M, that is

$$M_p = \{x \mid x \in M \text{ and } p^i x = 0 \text{ for some } i \in \mathbb{N}\}$$

we have that M splits in a direct sum of its p_i-primary components. We can therefore assume that $M = M_p$ for some irreducible element p in A. For any $m \in M$, $m \neq 0$, we consider the integer $o(m)$ as the maximal positive integer i satisfying:

$$p^i m = 0 \quad \text{and} \quad p^{i-1} m \neq 0$$

We call $o(m)$ the order of m.

It is clear that if m has order s then p^s generates the ideal in A, annihilator of m, so

$$\langle m \rangle \simeq A/\langle p^s \rangle$$

We have the following filtration of $\langle m \rangle$:

$$0 \subset \langle p^{s-1}m \rangle \subset \ldots \subset \langle pm \rangle \subset \langle m \rangle$$

Each $\langle p^i m \rangle$ is characterized by the property:

$$x \in \langle m \rangle \ , \quad p^i x = 0 \quad \text{iff} \quad x \in \langle p^{s-i}m \rangle$$

In fact, "if" is trivial. On the other hand if $p^i x = 0$, as $x = rm$ we have $0 = p^i x = p^i rm$. This implies $p^s \mid p^i r$, that is, $p^{s-i} \mid r$ and we get finally $x \in \langle p^{s-i}m \rangle$.

Let $m \in M$ be an element of maximal order in M. We shall prove the key result that $\langle m \rangle$ is a direct summand of M. To do this we shall define a projection of M onto $\langle m \rangle$. The next Lemma is the main device of our proof.

LEMMA. *Let* $N \subset M$ *be a submodule of* M *and let* $f\colon N \longrightarrow \langle m \rangle$ *be a morphism. Then* f *extends to a morphism of* M *into* $\langle m \rangle$.

Proof. If $N = M$ we have nothing to do. Let $N \neq M$ and take $a \in M\text{-}N$. Let I be the ideal of A defined by

$$I = \{k \mid k \in A \text{ and } ka \in N\}$$

Then $I = \langle d \rangle$ for some d in A. Since $p^s a = 0$ (by the maximality of s) we have that $p^s \mid d$, or $p^s = d.y$. By the unique factorization of A is $d = p^r.u$ with $r \leqslant s$. Without loss of generality we can assume that $u = 1$, that is $d = p^r$.

We have

$$p^{s-r} f(da) = f(p^{s-r}da) = f(p^s a) = 0$$

but by an earlier remark we can write

$$f(da) = p^r x \quad \text{for some} \quad x \in \langle m \rangle$$

We intend to extend f to $N + \langle a \rangle$ as follows:

$$\begin{aligned} f' : N + \langle a \rangle &\longrightarrow \langle m \rangle \\ n + ta &\longmapsto f(n) + tx \end{aligned}$$

Let us prove that f' is well defined. With $n, n' \in N$, $t, t' \in A$ we have

$$\begin{aligned} n + ta = n' + t'a &\Rightarrow n - n' = (t' - t)a \\ &\Rightarrow t' - t \in I \\ &\Rightarrow t' - t = zd \ , \quad z \in A \end{aligned}$$

Therefore

$$f(n - n') = f(zda) = zf(ad) = zp^r x = zdx = (t' - t)x$$

that is

$$f(n) + tx = f(n') + t'x$$

and this proves that f' is well defined. We can repeat this process, but after a finite number of steps we have to arrive to M (M is a noetherian module !). This concludes the proof of the Lemma.

Applying the Lemma to the situation: $N = \langle m \rangle$, $f = Id_{\langle m \rangle}$, we get a morphism of M into $\langle m \rangle$ which is the identity over $\langle m \rangle$. This is clearly a projection over $\langle m \rangle$. So $\langle m \rangle$ is a direct summand of M: $M = \langle m \rangle \oplus M'$. We can repeat the process with M'. But by the noetherian property of M, the process must stop after a finite number of steps. Then we get that M is a direct sum of cyclic submodules.

REMARKS. Notice the revealing property of elements of maximal order. The present proof improves the one given in our Note appeared in the American Mathematical Monthly, Vol.76, N°1, pp.60-61 (1969).

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