Revista de la
Unión Matemática Argentina
Volumen 27, 1975.

PRIME IDEALS AND SYMMETRIC IDEMPOTENT KERNEL FUNCTORS

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INTRODUCTION.

Let R be a ring with unity element and τ a prime idempotent kernel functor on Mod-R, the category of unital right R-modules. We will denote by Ter(τ) the tertiary radical of a supporting module for τ. In the first part of this paper we give necessary and sufficient conditions for R/Ter(τ) to be τ-torsion free while in the second part we apply this result to study symmetric idempotent kernel functor. It is shown that, every symmetric prime idempotent kernel functor on Mod-R, where R is a right noetherian ring, is in fact the symmetric idempotent kernel functor associated with some ideal of R. This extends a result of [4].

Throughout, R will be an associative ring with unity element, "module" will mean unital right module. The category of R-module will be denoted by Mod-R.

1. TERTIARY IDEAL OF A PRIME IDEMPOTENT KERNEL FUNCTOR.

We first recall a number of definitions and results concerning idempotent kernel functors. Our main reference is Goldman [2], whose terminology we follow.

A subfunctor σ of the identity functor on Mod-R is called an *idempotent kernel functor* if σ is left exact and such thac $\sigma(M/\sigma(M)) = 0$ for all $M \in$ Mod-R. M is said to be *σ-torsion* if $\sigma(M) = M$ and *σ-torsion free* if $\sigma(M) = 0$. To each R-module S, there is an idempotent kernel functor τ_S, on Mod-R, given by

$$\tau_S(M) = \{m \in M \mid f(m) = 0 \text{ for all } f: M \longrightarrow E\}$$

where E is the injective hull of S. Note that S is τ_S-torsion free and for any idempotent kernel functor σ on Mod-R such that S is σ-torsion free, we have $\sigma \leq \tau_S$, in the sense that $\sigma(M) \subseteq \tau_S(M)$ for all $M \in$ Mod-R. In case S = R/K for some right ideal K of R, we will write μ_K for τ_S. An idempotent kernel functor σ

on Mod-R is called a *prime* if $\sigma = \tau_S$, where S is a supporting module for σ i.e., S is σ-torsion free and S/S' is σ-torsion for each non-zero submodule S' of S.

Let M be an R-module. The two-sided ideal of R consisting of all elements which annihilate a large submodule of M is called the *tertiary radical* of M and is denoted by Ter(M). It is well-known that Ter(N) = Ter(M) for any essential extension N of M. For a prime idempotent kernel functor σ on Mod-R, we define the *tertiary ideal* Ter(σ) of σ by Ter(σ) = Ter(S), where S is a supporting module for σ. Ter(σ) is well-defined because all σ-injective supporting modules for σ are isomorphic (see [2], Theorem 6.4).

EXAMPLE 1.1. Let R be a commutative or right noetherian ring and P a prime ideal of R. Then μ_P is a prime idempotent kernel functor on Mod-R and Ter(μ_P) = P.

Let M be an R-module. We will denote the two-sided ideal of R consisting of all elements which annihilate M by Ann(M). Note that if M is σ-torsion free for some idempotent kernel functor σ, then so is R/Ann(M) as we can always embed R/Ann(M) into the direct product of cyclic submodules of M.

PROPOSITION 1.2. *Let σ be a prime idempotent kernel functor on* Mod-R. *Then* $\mathrm{Ter}(\sigma) = \sum_{I \in \Phi} I$, *where Φ is the family of all two-sided ideals* I *of* R *such that* $I \neq R$ *and* R/I *is σ-torsion free.*

Proof. Let S be a supporting module for σ. Then for each $I \in \Phi$ there is a non-zero R-homomorphism $f: R/I \longrightarrow E$, where E is the injective hull of S. Let $M = f(R/I) \cap S$. Since M is also a supporting module for σ, we have $I \subseteq \mathrm{Ann}(M) \subseteq \mathrm{Ter}(M) = \mathrm{Ter}(\sigma)$. Hence $\sum_{I \in \Phi} I \subseteq \mathrm{Ter}(\sigma)$. On the other hand, if $x \in \mathrm{Ter}(\sigma)$, then $x \in \mathrm{Ann}(M)$ for some large submodule M of S. Since $\mathrm{Ann}(M) \in \Phi$, we have $x \in \sum_{I \in \Phi} I$. It follows that $\mathrm{Ter}(\sigma) = \sum_{I \in \Phi} I$.

A non-zero R-module M is called a *prime* R-module if Ann(M) = = Ann(M') for every non-zero submodule M' of M.

THEOREM 1.3. *Let σ be a prime idempotent kernel functor on* Mod-R. *Then the following are equivalent:*

(1) R/Ter(σ) *is σ-torsion free.*

(2) *Every supporting module for σ contains a prime submodule.*

(3) $\sigma = \tau_M$ *for some prime* R-*module* M.

(4) Ter(σ) = Ann(U) *for some prime supporting module* U *for* σ.

Proof. (1) ⇒ (2). Let S be a supporting module for σ. Since R/Ter(σ) is σ-torsion free the injective hull of S contains a non-zero homomorphic image of R/Ter(σ), say M. Then $M \cap S$ is a prime submodule of S because for any non-zero R-submodule U of $M \cap S$ we have $\mathrm{Ann}(U) \subseteq \mathrm{Ter}(U) = \mathrm{Ter}(\sigma) \subseteq \mathrm{Ann}(M \cap S) \subseteq \mathrm{Ann}(U)$.

(2) ⇒ (3) is clear.

(3) ⇒ (4) follows from the fact that M contains a supporting module for σ.

(4) ⇒ (1) is clear.

REMARK 1.4. Let σ be a prime idempotent kernel functor on Mod-R. If the conditions of 1.3 hold, then Ter(σ) is a prime ideal.

REMARK 1.5. Let σ be a prime idempotent kernel functor on Mod-R. If R satisfies the maximum condition on two-sided ideals I for which $I \neq R$ and R/I is σ-torsion free, then the conditions of 1.3 hold.

Indeed, let S be a supporting module for σ and U be a non-zero submodule of S such that Ann(U) is a maximal element in the set $\{\mathrm{Ann}(M) \mid M \text{ is a non-zero submodule of } S\}$. Then U is a prime submodule of S. Hence condition (2) of 1.3 holds.

The following example, which is essentially due to Fisher ([1],Example 1), shows that there exists a prime idempotent kernel functor whose tertiary ideal is not a prime. Thus the conditions of 1.3 are not always satisfied.

EXAMPLE 1.6. Let ${}_FV$ be a countably infinite dimensional vector space over a field F and let $\{e_1, e_2, e_3, \ldots\}$ be a basis for ${}_FV$. For each $i \in \mathbf{N}$, let V_i be the subspace of V generated by $\{e_i, e_{i+1}, e_{i+2}, \ldots\}$.

Consider the subset H of $\mathrm{Hom}_F(V,V)$ consisting of all linear transformations h satisfying the conditions:

(1) $(V_i)h \subseteq V_i$ for all $i \in \mathbf{N}$

(2) $(V_j)h = 0$ for some $j \in \mathbf{N}$

For each $n \in \mathbf{Z}$, let λ_n be the linear transformation given by $(e_i)\lambda_n = ne_i$ for all $i \in \mathbf{N}$. Also, let t be the linear transformation given by

$$(e_i)t = \begin{cases} e_{i+1}, & \text{if } i \text{ is odd} \\ 0, & \text{if } i \text{ is even} \end{cases}$$

for all $i \in \mathbf{N}$. Then $R = \{h + t\lambda_n + \lambda_m \mid h \in H, m,n \in \mathbf{Z}\}$ is a subring of $Hom_F(V,V)$. One can show that $\{V_i \mid i \in \mathbf{N}\}$ is the family of all non-zero R-submodules of V and then deduce that V is a supporting module for $\sigma = \tau_V$. As we have $(t\lambda_1)R(t\lambda_1) \subseteq H \subseteq Ter(V) = Ter(\sigma)$ and $t\lambda_1 \notin Ter(V) = Ter(\sigma)$, $Ter(\sigma)$ cannot be a prime ideal.

2. SYMMETRIC IDEMPOTENT KERNEL FUNCTORS.

For an idempotent kernel functor σ on Mod-R, let $F(\sigma)$ be the idempotent topologizing filter associated with σ i.e., $F(\sigma)$ is the family of all right ideals K of R such that R/K is σ-torsion.

Following [4], an idempotent kernel functor σ on Mod-R is called *symmetric* if every right ideal in $F(\sigma)$ contains a two-sided ideal in $F(\sigma)$. It is shown in [4] that if R is a right noetherian ring, then for every prime ideal P of R there is an associated symmetric idempotent kernel functor σ_{R-P}, defined on Mod-R by

$$\sigma_{R-P}(M) = \{m \in M \mid mRs = 0 \quad \text{for some } s \in R-P\}$$

PROPOSITION 2.1. *Let* R *be a right noetherian ring and* σ *a prime idempotent kernel functor with* $Ter(\sigma) = P$. *Then* $\sigma_{R-P} \leqslant \sigma$.

Proof. Since R is right noetherian, conditions of 1.3 hold and so $P = Ann(M)$ for some prime supporting module M for σ. Let $m \in \sigma_{R-P}(M)$. Then $mRs = 0$ for some $s \in R-P$ If $m \neq 0$, we would have $s \in Ann(mR) = Ann(M) = P$ which is a contradiction. Thus M is σ_{R-P} - torsion free and hence $\sigma_{R-P} \leqslant \sigma$.

PROPOSITION 2.2. *Let* R *be a right noetherian ring,* P *a prime ideal of* R *and* σ *a symmetric idempotent kernel functor on* Mod-R. *Then* $\sigma \leqslant \sigma_{R-P}$ *if and only if* R/P *is* σ*-torsion free.*

Proof. Assume that R/P is σ-torsion free. Let $K \in F(\sigma)$. Then there exists a two-sided ideal I of R such that $I \in F(\sigma)$ and $I \subseteq K$. Since R/P is σ-torsion free, $I \not\subseteq P$ and so $I \cap (R-P) \neq \emptyset$ which implies $RsR \subseteq I \subseteq K$ for some $s \in R-P$. If follows that $K \in F(\sigma_{R-P})$. Hence $\sigma \leqslant \sigma_{R-P}$. The converse is clear because R/P is σ_{R-P}-torsion free.

The following result extends Proposition 14 of [4] where much stronger condition is imposed.

THEOREM 2.3. *Let* R *be a right noetherian ring and* σ *a symmetric* *prime idempotent kernel functor on* Mod-R. *Then* $\sigma = \sigma_{R-P}$, *where* $P = Ter(\sigma)$.

Proof. 2.1 implies $\sigma_{R-P} \leqslant \sigma$ while 2.2 implies $\sigma \leqslant \sigma_{R-P}$.

Let Λ be a family of idempotent kernel functors on Mod-R. For any $M \in$ Mod-R, let $\sigma(M) = \bigcap_{\rho \in \Lambda} \rho(M)$. Then σ is also an idempotent kernel functor and we will call it the *infinimum* of Λ and denote it by Inf $\{\rho \mid \rho \in \Lambda\}$. In case $\Lambda = \emptyset$, Inf $\{\rho \mid \rho \in \Lambda\} = \infty$, the idempotent kernel functor for which every R-module is torsion.

If R is a right noetherian ring, then every idempotent kernel functor $\sigma \neq \infty$ has a supporting module. Furthermore, if S is a supporting module for σ then $Ter(S) = Ter(\tau_S)$ is a prime ideal of R.

THEOREM 2.4. *Let* R *be a right noetherian ring,* σ *an idempotent kernel functor on* Mod-R *and* π *be the family of all* Ter(S), *where* S *is a supporting module for* σ. *Then* σ *is symmetric if and only if* σ = Inf $\{\sigma_{R-P} \mid P \in \pi\}$.

Proof. In case $\sigma = \infty$ there is nothing to prove. Thus we may assume $\sigma \neq \infty$ and so $\pi \neq \emptyset$. Since R is right noetherian R/P is σ-torsion free for each $P \in \pi$.

Assume that σ is symmetric. Then, by 2.2 , we have $\sigma \leqslant$ Inf $\{\sigma_{R-P} \mid P \in \pi\}$. On the other hand, if $K \notin F(\sigma)$, then there is a right ideal L of R such that $K \subseteq L$ and R/L is a supporting module for σ. Let $P = Ter(R/L)$. Then, by (2.1), $\sigma_{R-P} \leqslant \mu_L$. As $K \notin F(\mu_L)$, K cannot be in $F(\sigma_{R-P})$. It follows that K is not a member of the idempotent topologizing filter associated with Inf $\{\sigma_{R-P} \mid P \in \pi\}$. Hence σ = Inf $\{\sigma_{R-P} \mid P \in \pi\}$.

The converse is clear.

REMARK 2.5. A different version of 2.4 can be found in ([4], Proposition 10). One can deduce from 2.4 that if R is a right noetherian ring and σ an idempotent kernel functor on Mod-R , then $\sigma = \sigma_{R-P}$ for some prime ideal P of R if and only if σ is symmetric and P is the largest member in the family of all two-sided ideals I of R such that $I \neq R$ and R/I is σ-torsion free.

We conclude this note by presenting an example which shows that for a prime ideal P of a right noetherian ring R, the symmetric idempotent kernel functor σ_{R-P} needs not be a prime.

EXAMPLE 2.6. Let F be a field of characteristic zero, and let S be the ring F[y][x], where xy-yx = 1. It is shown in ([3],Example 4.5) that R = F + xS is a right noetherian domain where xS is the only non-zero two-sided ideal and that R/xS and S/R are two non-isomorphic simple R-modules. Since $\sigma_{R-xS} = 0$, it cannot be a prime.

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Recibido en abril de 1974