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# LOCAL SOLVABILITY AND CAUCHY PROBLEM FOR A CLASS OF DEGENERATE HYPERBOLIC OPERATORS

Jorge G. Hounie*

## INTRODUCTION.

The Cauchy problem for hyperbolic differential operators with multiple characteristics has been thoroughly studied in the constant multiplicity case ([1], [2] and [3]) but there seems to be few results when the multiplicity is not constant.

As a first step in that direction, we study here the Cauchy problem for a class of hyperbolic operators with double characteristics at t=0 and simple characteristics for $t\neq 0$. This is carried out in the abstract set up of evolution equations ([7]). This is a simplified commutative model of the microlocal behavior of pseudodifferential operators and provides a basis for the understanding of the latter.

Given an abstract Hilbert space H and an unbounded self-adjoint positive definite operator A on H, we study evolution operators of the form

$$(1) \qquad P = (\partial_t - ia(t,A)A)(\partial_t - ib(t,A)) + c(t,A)A$$

where $\partial_t$ means $\frac{\partial}{\partial_t}$ and the coefficients $a(t,A)$, $b(t,A)$ and $c(t,A)$ are power series in $A^{-1}$, with coefficients in $C^\infty(J)$; J an open subset of the real line containing the origin. These power series are assumed to converge in L(H,H) as well as each one of their t-derivatives, uniformly with respect to t on compact subsets of J.

When the leading coefficients $a_0(t,A)$, $b_0(t,A)$ *are real and vanish simultaneously at* t=0, we have (an analogue of) a *hyperbolic operator with double characteristics at the origin.*

If we further assume that

* The results of this work are part of the author's doctoral dissertation.

(2) $$a_0'(0) \neq b_0'(0)$$

P *will have simple characteristics for* $t \neq 0$.

The main result of Chapter I is theorem 2, where we prove that local Cauchy problem is well posed for an operator satisfying (1) and (2). In Chapter II we study an analogous class of operators, now in pseudo-differential form. The methods applied in Chapter I, namely, asymptotic expansions, do not lead straightforwardly to results for pseudo-differential equations and we only give sufficient conditions for local solvability.

We intend to treat the Cauchy problem for this class of pseudo-differential operators in a future work.

I am indebted to Professor Treves, who introduced me to the subject and suggested the proof of Proposition 2.1.

## CHAPTER I.

1. In the first chapter of this work, we will follow the notations of [4]. A will denote a linear, densely defined, unbounded operator which we assume self-adjoint and positive definite. Models for A are self-adjoint extensions of $(1-\Delta x)^s$ or $|D_x|$ in n space variables.

We will consider differential operators on the real line, where the variable is denoted by t, of the following kind

(1.1) $$P = \sum_{r+j \leq m} C_{r,j}(t,A)A^r \partial_t^j$$

where r,j are positive integers.

The coefficients $C_{r,j}(t,A)$ belong to the ring $Q_A(J)$ defined as follows: J is an open subset of the real line; the elements of $Q_A(J)$ are series in non-negative powers of $A^{-1}$, with coefficients in $C^\infty(J)$, which converge in L(H;H) (the B-space of bounded linear operators on H), as well as each one of their t-derivatives, uniformly with respect to t on compact subsets of J.

The operators of the kind (1.1) form an algebra which is denoted $P_A(J)$. The operator given in (1.1) is said to be of order m.

We will use the scale of "Sobolev spaces" $H^s$ $(s \in R)$ defined by A: if $s \geqslant 0$, $H^s$ is the space of elements $u \in H$ such that $A^s u \in H$, equipped with the norm $\|u\|_s = \|A^s u\|_0$, where $\| \ \|_0$ denotes the norm in $H = H^0$. If $s < 0$, $H^s$ is the completion of H for the norm $\|u\|_s = \|A^s u\|_0$. The inner product in $H^s$ will be denoted by $( \ , \ )_s$. For every $s,m \in R$ $A^m$ is an isometry of $H^s$ onto $H^{s-m}$.

We denote $H^\infty$ the intersection of the spaces $H^s$, equipped with the projective limit topology, and $H^{-\infty}$ their union with the inductive limit topology. $H^\infty$ is an F-space and $H^{-\infty}$ can be identified with the dual of $H^\infty$ by the pairing $\langle u,v \rangle = (u,v)_0$ defined on $H^\infty \times H^{-\infty}$.

Let J be an open subset of R. $C^\infty(J,H^\infty)$ is the space of $C^\infty$ functions in J valued in $H^\infty$. It has a natural F-topology. If $K \subset J$ is compact, $C_c^\infty(K,H^\infty)$ denotes the subspace of $C^\infty(J,H^\infty)$ of functions supported in K. We give $C_c^\infty(J,H^\infty)$ the inductive limit topology induced by the $C_c^\infty(K,H^\infty)$ as K ranges over all compact subsets of J.

We will denote $\mathcal{D}'(J,H^{-\infty})$ the *dual of* $C_c^\infty(J,H^\infty)$, and refer to it as the space of *distributions in* J *valued in* $H^{-\infty}$.

DEFINITION 4.1. Let $P \in P_A(J)$ be of order m, and assume that J contains the origin.

We say that the *two sided local Cauchy problem is well posed if there exist a neighborhood of the origin* $J'(0) \subset J$, such that *for every* $f \in C^\infty(J,H^\infty)$, $h_1,\dots,h_{m-1} \in H^\infty$ there exists a unique $u \in C^\infty(J',H^\infty)$ such that

$$(1.2) \qquad \begin{aligned} &P_u = f \quad \text{in } J' \\ &\partial_t^i u\big|_{t=0} = h_i \qquad 0 \leqslant i \leqslant m-1 \end{aligned}$$

If the same holds whenever f=0 we say that the *homogeneous Cauchy problem* is well posed. The *forward* Cauchy problem is defined in the same way with J' replaced by a semi-open interval $[0,\varepsilon)$.

Let $c(t,A) \in Q_A(J)$. Then $c(t,A) = \sum_{j=0}^{n} c_j(t)A^{-j}$, $c_j(t) \in C^\infty(J,C)$. If $\lambda$ is in the spectrum $Q(A)$ of A the series $c(t,A) = \sum_{j=0}^{\infty} c_j(t)\lambda^{-j}$ converges uniformly with respect to t on compact subsets of J, together with its t-derivatives.

To every differential operator

$$P = \sum_{j+k\leq m} C_{j,k}(t,A)A^j\partial_t^k \in P_A(J)$$

we associate the ordinary differential polynomial

$$P(\tfrac{d}{dt},\lambda) = \sum_{j+k\le m} C_{j,k}(t,\lambda)\lambda^j\partial_t^k \quad , \quad \lambda \in \sigma(A).$$

Here $\lambda$ plays the role of a (real) parameter.

From now on we assume that the coefficient $C_{0,m}(t,\lambda)$ of $\partial_t^m$ is identically one (hence the "t-direction" is non-characteristic).

THEOREM 1. *Let* $P \in P_A(J)$. *The following statements are equivalent*

a) *The homogeneous two-sided Cauchy problem for* P *is well posed.*

b) *There exists* $J'(0) \subset J$ *such that every solution* $u(t,\lambda)$ *of*

$$(1.3)\qquad \begin{cases} P(\tfrac{d}{dt},\lambda)u = 0 \\ \partial_t^i u\big|_{t=0} = \alpha_i \qquad \alpha_i \in C \end{cases}$$

*as well each one of its t-derivatives grows slower than a power of* $\lambda$ *when* $\lambda \in \sigma(A)$ *uniformly in* $t \in J'(0)$.

c) *The* m *solutions* $u_\alpha(t,\lambda)$ $\alpha = 1,\dots,m$ *of*

$$(1.4)\qquad \begin{cases} P(\tfrac{d}{dt},\lambda)u_\alpha = 0 \\ \partial_t^i u_\alpha\big|_{t=0} = \delta_\alpha^i \end{cases} \quad \textit{verify}$$

$$\sup_{\substack{t\in J' \\ 0\le i\le m-1}} \left|(\tfrac{d}{dt})^i u_\alpha(t,\lambda)\right| < k(1+\lambda)^p,\ \lambda \in J(A),\ \textit{for a certain}$$

*neighborhood of the origin* $J'$, *and positive constants* k, p.

REMARKS. 1) When a function $u(t,\lambda)$ verifies a growth condition as in (b) of Theorem 1, we say that $u(t,\lambda)$ is *tempered*.

The proof of Theorem 1 is rather simple and we do not include it here. It makes use of Ovsjannikov's theorem for singular operators in Banach scales (see [2] and also [3]) and the spectral resolution of the self-adjoint operator A.

2) Since Theorem 1 reduces the study of the correctness of the Cauchy problem to the study of the growth of an ordinary differential equation, it will produce immediate answers in the cases where the O.D.E. can be integrated, for instance the first

order and the constant coefficient cases.

2. We now study a class of second order evolution operators of the form

(2.1) $$P = (\partial_t - ia(t,A)A)(\partial_t - ib(t,A)A) + c(t,A)A$$

where a, b, c are elements in $Q_A(J)$ and J contains the origin. We assume that

(2.2) $a_0(t)$ *and* $b_0(t)$, *the leading coefficients of* $a(t,A)$ *and* $b(t,A)$ *are real*

(2.3) $a_0(0) = b_0(0) = 0, \qquad a'_0(0) - b'_0(0) \neq 0$

We can regard P as a hyperbolic operator with double characteristics at t=0 but simple characteristics for t≠0.

THEOREM 2. *Let* $P = (\partial_t - ia(t,A)A)(\partial_t - ib(t,A)) + c(t,A)A$ *belong to* $P_A(J)$ *and assume that* (2.2) *and* (2.3) *hold. Then, the two-sided local homogeneous Cauchy problem for* P *is well posed.*

Before embarking in the proof of Theorem 2 let us make some preliminary remarks. According to Theorem 1 we can replace the operator A by a real parameter $\lambda \in \sigma(A)$ and deal with the corresponding ordinary differential equation. Since there is no possibility of confusion *we denote the parameter with* A *instead of* $\lambda$. We use the notation

$$X = \partial_t - ia(t,A)A \quad , \quad Y = \partial_t - ib(t,A)A \quad , \quad \delta = a(t,A) - b(t,A)$$

We see that $XY + c(t,A)A = YX + c^{\#}(t,A)$ for a certain $c^{\#}(t,A) \in P_A(J)$. Therefore there is no restriction if we impose

(2.4) $$\delta'_0(0) > 0$$

To have some insight of the problem, let us look at the simplest example of operator occurring in Theorem 2, namely

$$P = (\partial_t - itA)(\partial_t + itA) + cA$$

where c is a complex constant.

It is easy to check that the change of variable $s = \sqrt{A}t$ takes $Pu = 0$ into the Weber equation $[(\partial_s - is)(\partial_s + is) + c]v = 0$ whose solutions are known to be tempered in s (for s real).

We have an estimate

$$|v(s)| + |v'(s)| \leqslant c(1+|s|^k)$$

Now the solutions of $Pu = 0$ can be written $u(t,A) = v(\sqrt{A}t)$ and we conclude that $u(t,A)$ is tempered.

In the general case, we reduce the problem introducing suitable (formal) changes of dependent and independent variables to a simple form, where standard techniques give asymptotic expansions for the solutions and provide the necessary estimates.

PROPOSITION 2.1. *Let* $P = XY_{+}c(t,A)A$ *and assume that* (2.4) *holds. Then there exist a formal series* $s(t,A) = \sum_{j=0}^{\infty} s_j(t)A^{-j}$,

$$\alpha(s,A) = \sum_{j=0}^{\infty} \alpha_j(s)A^{-j}, \quad \gamma(A) = \sum_{j=0}^{\infty} \gamma_j A^{-j}, \quad \theta(A) = \sum_{j=0}^{\infty} \theta_j A^{-j}$$

*such that*

a) $s_0(t)$, $\alpha_0(t)$ *are real*; $s'_0(0) \neq 0$

b) $s_0(0) = s_j(0) = 0, \quad j = 1,2,\ldots$

c) *The change of variable* $s = s(t,A)$ *takes* P *into*

$$s_t^2[(\partial_s - i\alpha(s,A)A)(\partial_s - i\alpha(s,A) - isA - \theta(A)) + \gamma(A)A]$$

REMARKS. 1) The functions $s_k(t)$ are defined in a certain neighborhood of the origin. The series $s(t,A)$, $\alpha(s,A)$, $\gamma(A)$, $\theta(A)$ are not convergent in general. This is not an inconvenience for we ultimately replace them by their partial sums with a large number of terms.

2) Since the $s_j(t)$ are complex valued functions for $j \geqslant 1$, $s = s(t,A)$ is not in general a real change of variable. Thus the notation

$$\int_0^{s_0} \alpha(s)ds \quad \text{will always means} \quad \int_0^{t_0} \alpha(s(t))s'(t,A)dt$$

*Proof of Prop. 2.1.* We may write

$$P = XY + cA = (\partial_t - iaA)(\partial_t - ibA) + cA =$$

$$(2.5) \qquad = s_t^2[(\partial_s - i\frac{a}{s_t}A + \frac{s_{tt}}{s_t^2})(\partial_s - i\frac{b}{s_t}A) + \frac{c}{s_t^2}A]$$

where we have used $s_t^{-1}\partial_t = \partial_s$. Since we want (2.5) to be

$(\partial_s - i\alpha A)(\partial_s - i\alpha A - isA + \theta(A)) + \gamma(A)A$ it will be enough to take

$$i\frac{a}{s_t} - \frac{s_{tt}}{s_t^2} + W = i\alpha A$$

$$i\frac{b}{s_t}A - W = i\alpha A + isA - \theta(A)$$

$$W_s - W^2 + W[i\frac{(b-a)}{s_t}A + \frac{s_{tt}}{s_t^2}) + \gamma(A)A = \frac{c}{s_t^2}A$$

Thus we need to determine W, s, γ, θ satisfying the system of equations

$$(2.6)\quad \begin{cases} \text{i)}\ W_t s_t - (s_t)^2 W^2 - W(i\delta s_t A - s_{tt}) + \gamma(s_t)^2 A = cA \\ \text{ii)}\ i\delta A s_t - s_{tt} + 2W = is(s_t)^2 A + \theta(s_t)^2 \end{cases}$$

We solve the system formally setting

$$W = \sum_{i=0}^{\infty} W_i(t)A^{-i} \qquad \gamma(A) = \sum_{i=0}^{\infty} \gamma_i A^{-i}$$

$$s(t,A) = \sum_{i=0}^{\infty} s_i(t)A^{-i} \qquad \theta(A) = \sum_{i=0}^{\infty} \theta_i A^{-i}$$

Insertion into equation (2.6) and identification of like coefficients leads to recursion formulas

$$(2.7)_0\quad \begin{cases} -i\delta_0 W_0(s_t)_0 + \gamma_0(s_t)_0^2 = c_0 \\ i\delta_0(s_t)_0 = is_0(s_t)_0^2 \end{cases}$$

$$(2.7)_k\quad \begin{cases} -iW_{k+1}\delta_0(s_t)_0 + B_k + \gamma_{k+1}(s_t)_0^2 = c_{k+1} \\ i\delta_0(s_t)_{k+1} + D_k = is_{k+1}(s_t)_0^2 + 2is_0(s_t)_0(s_t)_{k+1} + \\ \qquad + \theta_k(s_t)_0^2 \end{cases}$$

where

$$B_k = \sum_{j=0}^{k} [(s_t)_j(W_t)_{k-j} - (s_t^2)_j(W^2)_{k-j} - i(\delta s_t)_{k-j+1}W_j + (s_{tt})_{k-j}W_j + \\ + \gamma_j(s_t^2)_{k+1-j}] \qquad \text{and}$$

$$D_k = \sum_{j=0}^{k}(s_t)_k\delta_{k+1-j} + W_j(s_t^2)_{k-j} - \sum_{j=0}^{k-1}[is_{j+1}(s_t^2)_{k-j} + \theta_j(s_t^2)_{k-j}]$$

We observe that $D_k$ depends on $W_0,\dots,W_k$; $s_0,\dots,s_k$; $\theta_0,\dots,\theta_{k-1}$ and $B_k$ depends on $W_0,\dots,W_k$; $\gamma_0,\dots,\gamma_k$; $s_0,\dots,s_k,s_{k+1}$.

We proceed by induction.

The second equation in $(2.7)_0$ is $s'_0 s_0 = \delta_0$ that integrates to $s_0(t) = (2\int_0^t \delta_0(\tau)d\tau)^{1/2}$. Since $\delta_0(0) = 0$, $\delta'_0(0) > 0$, $s_0(t)$ is smooth and $s'_0(0) > 0$. Now we pick up $\gamma_0 = c_0(0)$. That makes $c_0 - \gamma_0(s_t)_0^2$ divisible by $\delta_0$ in $C^\infty$ and determines $W_0$.

Assume we have determined $W_0,\dots,W_k$, $\gamma_0,\dots,\gamma_k$, $s_0,\dots,s_k$, $\theta_0,\dots,\theta_{k-1}$.

The second equation in $(2.7)_k$ can be written after replacing $\delta_0$ by $s_0 s'_0$

$$D_k - \theta_k(s'_0)^2 = i(s_{k+1}(s'_0)^2 + s_0 s'_0 s'_{k+1}) = is'_0(s_{k+1}s_0)'$$

Hence we can define

$$s_{k+1}(t) = \frac{1}{is_0}\int_0^t \frac{D_k(\tau) - \theta_k(s'_0)^2(\tau)}{s'_0}\,d\tau$$

Moreover taking $\theta_k = \dfrac{D_k(0)}{(s'_0)^2(0)}$ we can achieve $s_{k+1}(0) = 0$.

This determines $B_k$. Now we can choose $\gamma_{k+1}$ so as to make $c_{k+1} - B_k - \gamma_{k+1}(s'_0)^2$ divisible by $\delta_0$. That determines $W_{k+1}$ and completes the induction step. Q.E.D.

Let's go back to the operator

$$P^\# = (\partial_s - i\alpha(s,A))(\partial_s - i\alpha(s,A) - isA - \theta(A)) + \gamma(A)A$$

obtained from P by the change of variable $s = s(t,A)$ and assume for simplicity that all terms are convergent series.

We observe that if

$$P = \exp-\Big[\int_0^s i\alpha(\sigma,A)Ad\sigma + i\frac{s^2}{4}A + \theta(A)\frac{s}{2}\Big]P^\#\exp\Big[\int_0^s i\alpha(\sigma,A)Ad\sigma + i\frac{s^2}{4} + \theta(A)\frac{s}{2}\Big]$$

then $P = (\partial_s + i\frac{s}{2}A + \frac{\theta(A)}{2})(\partial_s - i\frac{s}{2}A - \frac{\theta(A)}{2}) + \gamma(A)A$

Since the leading term $s_0(t)$ of $s(t,A)$ is real the change of independent variable introduced by the exponential is tempered. Thus, the Cauchy problem for $P^\#$ is well posed if and only if it is well posed for P.

We investigate how the growth of the solution u(s,A) of

$$(2.8)\qquad \begin{aligned} P\,u &= 0\\ u(0) &= \alpha\\ u'(0) &= \beta \end{aligned}$$

when $|\mathrm{Re}\ s| \leqslant M$, $|\mathrm{Im}\ s| \leqslant \frac{M}{A}$ and $A > \infty$. The change of variable $\sigma = \sqrt{A}s$ takes (2.8) into

$$(2.9)\qquad \begin{aligned} &[(\partial_\sigma + i\tfrac{\sigma}{2} + \tfrac{1}{2}\tfrac{\theta(A)}{\sqrt{A}})(\partial_\sigma - i\tfrac{\sigma}{2} - \tfrac{1}{2}\tfrac{\theta(A)}{\sqrt{A}}) + \gamma(A)]v = 0\\ &v(0) = \alpha\\ &v'(0) = \frac{\beta}{\sqrt{A}} \end{aligned}$$

That is the solution u(s,A) of (2.8) can be expressed as u(s,A) = = $v(\sqrt{A}s,A)$ where $v(\sigma,A)$ is the solution of (2.9). We must study the behavior of $v(\sigma,A)$ in the expanding sector $|\mathrm{Re}\ \sigma| \leqslant$ $\leqslant M\sqrt{A}$ $|\mathrm{Im}\sigma| \leqslant \frac{M}{\sqrt{A}}$ . Since $v(\sigma,A)$ depends analytically on $A^{-1/2}$, given $\sigma_0$, there exists $K > 0$ so that $|v(\sigma,A)| \leqslant K$ for $|\sigma| \leqslant \sigma_0$, with a similar estimate for $v'(\sigma,A)$. It is enough to consider $|\sigma| > \sigma_0$. From now on we take $\mathrm{Re}\ \sigma > 0$; the analysis for $\mathrm{Re}\ \sigma < 0$ is similar. If we set $(\partial_\sigma - i\,\frac{\sigma}{2} - \frac{1}{2}\,\frac{\theta(A)}{\sqrt{A}})v = W$ and $Y = \binom{v}{W}$ we can write (2.9) as

$$(2.10)\qquad Y' = \begin{bmatrix} i\,\frac{\sigma}{2} + \frac{\theta(A)}{2\sqrt{A}} & 1 \\ -\gamma(A) & -(i\,\frac{\sigma}{2} + \frac{\theta(A)}{2\sqrt{A}}) \end{bmatrix} Y$$

Writing $z = i\,\frac{\sigma}{2} + \frac{\theta(A)}{i2\sqrt{A}}$ for simplicity of notation (2.10) yields

$$(2.11)\qquad Y' = z\,[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \frac{1}{z}\begin{bmatrix} 0 & 1 \\ -\gamma(A) & 0 \end{bmatrix}] = z\,[A_0 + \frac{1}{z}A_1]$$

Since $A_0$ is diagonal with distinct eigenvalues one can find matrices

$$P_r = \begin{bmatrix} 0 & p_r^{12} \\ p_r^{21} & 0 \end{bmatrix}, \; r \geq 1 , \qquad B_r = \begin{bmatrix} b_r^{11} & 0 \\ 0 & b_r^{22} \end{bmatrix}, \; r \geq 0$$

such that the formal change of unknown $Y = PW$ takes (2.11) into $W' = zBW$, where $P = \sum_{r=1}^{\infty} P_r z^{-r} + I$ and $B = \sum_{r=0}^{\infty} B_r z^{-r}$. The matrices $P_r$, $B_r$ are determined by the recurrence relations

$$B_0 = A_0 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad P_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sum_{r=0}^{n} A_n P_{n-k} - P_{n-k} B_k = -(n-k) P_{n-k} \qquad n \geq 1$$

with the convention $P_{-1} = 0$.

Computation of the first terms gives

$$B_1 = 0 , \qquad B_2 = \begin{bmatrix} \frac{-\gamma(A)}{2i} & 0 \\ 0 & \frac{\gamma(A)}{2i} \end{bmatrix}$$

We conclude that for $\text{Re}\,\sigma > \sigma_0$, (2.10) has a fundamental matrix solution of the form

$$\text{(2.12)} \qquad Y(\sigma) = \exp[\, iz^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ] \; z^{\frac{-\gamma}{2i}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \; \tilde{Y}(\sigma, A)$$

with $\tilde{Y}(\sigma,A)$ bounded on $\sigma_0 < \text{Re}\,\sigma < M\sqrt{A}$; $|\text{Im}\,\sigma| \leq \frac{M}{\sqrt{A}}$.

Using the fact that $|\text{Im}\, z| \leq \frac{C_0}{\sqrt{A}}$ (for a certain constant $C_0$) and (2.12), it is easy to derive that the solutions of (2.10) are tempered and grow slower than $|\sigma|^p$ at infinity where p is any constant bigger than $|\frac{\gamma_0}{2}|$. In turn, this implies that the solutions of (2.8) are tempered. This considerations prove Theorem 2 at least when the change of variable given in Prop.2.1 gives rise to convergent series.

*Proof of Theorem 2*. Let $u(t,A)$ be the solution of

$$\text{(2.13)} \qquad \begin{aligned} P_u &= 0 \\ u(0) &= \alpha \\ u'(0) &= \beta \end{aligned}$$

We take $\frac{N}{2} > p > \frac{1}{2} C_0(0)$ and consider the partial sums of order N of the series $s(t,A)$, $\alpha(t,A)$, $\theta(A)$, $\gamma(A)$:

$$S^N(t,A) = \sum_{i=0}^{N} s_i(t)A^{-i} \qquad \alpha^N(t,A) = \sum_{i=0}^{N} \alpha_i(t)A^{-i}$$

$$\theta^N(A) = \sum_{i=0}^{N-1} \theta_i A^{-i} \qquad \gamma^N(A) = \sum_{i=0}^{\infty} \gamma_i A^{-i}$$

Then, there is a first order operator $R_N = f_N(t,A)\partial_t + g_N(t,A)$ with $f_N, g_N \in Q_A$ such that the change of variables $s = s_N(t,A)$ takes $P_N$ into $(s_t^N)^2[(\partial_s - i\alpha^N A)(\partial_s - i\alpha^N A - isA - \theta_N) + \gamma_N A]$. According to the analysis of the convergent case the solutions of $P_N v = 0$ grow slower than

$$(\sqrt{A})^p, \; p > \frac{\gamma_0^N}{2} = \frac{C_0(0)}{2}$$

Taking two linearly independent solutions $v_1$, $v_2$ of $P_N v_1 = 0$, $P_N v_2 = 0$ (say $v_1(0) = 0$, $v'_1(0) = 1$; $v_2(0) = 1$, $v'_2(0) = 0$) we can construct a Green function $G_N(t,\tau,A)$ such that if $(G_N f)(t) = \int_{-T}^{T} G_N(t,\tau,A) f(\tau)d\tau$ then

$$P_N G_N f = f$$
$$G_N f \Big|_{t=0} = 0$$
$$(G_N f)' \Big|_{t=0} = 0$$

Noticing that the Wronskian of $v_1$ and $v_2$ is bounded away from zero (uniformly in A) we get an estimate

$$|G_N(t,\tau,A)| \leq K A^p$$

Now we can write the solution $u(t,A)$ of (2.13) as

$$u(t,A) = v_N(t,A) + G_N(\frac{R_N}{A^N} u)(t,A)$$

where $v_N(t,A)$ verifies $P_N v_N = 0$, $v_N(0) = \alpha$; $v'_N(0) = \beta$.
In view of the estimates for $G_N$, $v_N$, $v'_N$ we have

$$\sup_{|t|\leq T} (|u(t,A)| + |u'(t,A)|) \leq C_1 A^p + C_2 A^{p-N} \sup_{|t|\leq T} (|u(t,A)| + |u'(t,A)|)$$

$C_1$, $C_2$ constants. When A is big enough $C_2A^{p-N} < \frac{1}{2}$ and we conclude that u(t,A) is tempered. The theorem is proved.

## CHAPTER II.

1. We consider an analogue of the operator P of Theorem 2, now in the framework of pseudo-differential operators and give sufficient conditions for local solvability.

Explicitely we assume that

(1.1) $P(x,t,D_x,D_t) \sim (D_t - ta(t,x,D_x))(D_t - tb(t,x,D_x)) + c(t,x,D_x)$

where (x,t) denotes a point in $R^n \times R$, $a(t,x,D_x)$, $b(t,x,D_x)$, $c(t,x,D_x)$ are pseudo-differential operators of degree one acting on the x-variable, depending smoothly on t such that

(1.2) $(a-b)(t,x,D_x)$ *is elliptic*

(1.3) *The principal symbols* $a_1(t,x,D_x)$, $b_1(t,x,D_x)$ *of* a *and* b *are real*

Let us write

(1.4) $P \sim D_t^2 - r_1(x,t,D_x)D_t + r_2(x,t,D_x)$

We see that the principal symbols $\sigma_1(r_1)$, $\sigma_2(r_2)$ of $r_1$, $r_2$ are

(1.5)
$$\sigma_1(r_1)(x,t,\xi) = t(a_1+b_1)(x,t,\xi)$$
$$\sigma_2(r_2)(x,t,\xi) = t^2(a_1b_1)(x,t,\xi)$$

We start getting rid of the term in $D_t$. Consider the linear operator U(t) defined by

(1.6)
$$D_tU = \frac{r_1}{2}U$$
$$U(0) = \text{Identity}$$

Since $r_1$ is essentially self-adjoint, the unique solution of this problem is a function of t with values in the group of invertible operators in $L^2x$. The inverse of U(t) is the solution V(t) of the problem

$$(1.7)\qquad \begin{aligned} D_t V &= -V\,\frac{r_1}{2} \\ V(0) &= \text{Identity} \end{aligned}$$

It has been shown in [4], that there exists a Fourier integral operator $K(t)$, depending smoothly on $t$ and acting on the x-variable, defined by an oscillatory integral

$$K(t) = \frac{1}{(2\pi)^h}\int k(x,t,\xi)e^{ih(x,t,\xi)}\hat{u}(\xi)d\xi$$

which approximates $U(t)$ in the following sense. Given a positive integer $k$ and real numbers $s$, $s'$, there exists $C = C(k,s,s') > 0$ such that

$$(1.8)\qquad \max_{0\le j\le k}\ \sup_{|t|\le T}\ \|\partial_t^j(U(t)-K(t))u\|_s \leqslant \|u\|_{s'}$$

briefly $U(t) \approx K(t)$.

Let $Q$ be pseudo-differential operator in the x-variable of degree $m$. The proof of Theorem 7.1 in [4] shows that there is a pseudo-differential operator $Q^{\#}(t)$ of degree $m$ acting on the x-variables and depending smoothly on $t$ (for $|t|$ small) such that

$$(1.9)\qquad Q^{\#}(t) \sim K^{-1}(t)QK(t)$$

where $\sim$ stands for the standard equivalence of pseudo-differential operators.

Moreover, the correspondence $Q \longmapsto Q^{\#}$ takes elliptic operators into elliptic operators.

We now eliminate the term in $D_t$ introducing the "change of unknown" $v = Uu$.

We have

$$\begin{aligned} v_t &= U_t u + Uu_t \\ v_{tt} &= U_{tt}u + 2U_t u_t + Uu_{tt} \end{aligned}$$

using (1.6) we obtain

$$PU = UD_t^2 + ir_1U_t - U_{tt} + r_2U$$

Hence

$$(1.10)\quad \begin{aligned} U^{-1}PU &= D_t^2 + U^{-1}ir_1U_t - U^{-1}U_{tt} + U^{-1}r_2U \\ &= D_t^2 - U^{-1}\left[\frac{r_1^2}{4} + i\left(\frac{r_1}{2}\right)_t - r_2\right]U = D_t^2 - U^{-1}QU \end{aligned}$$

Let $q_2$ be the principal symbol of the operator Q appearing in the right hand side of (1.10), $q_2^{\#}$ the principal symbol of $Q^{\#} \sim K^{-1}(t)QK(t)$. Then

$$(1.11)\qquad q_2(x,t,\xi) = t^2\left(\frac{a-b}{2}\right)^2(x,t,\xi) \qquad \text{and}$$

$$(1.12)\qquad q_2^{\#}(x_0,t,\xi_0) = \sigma_2(Q^{\#}) = \frac{t^2}{4}\,[(a-b)(x(x_0,t,\xi_0),t,\ \xi(x_0,t,\xi_0)]^2$$

where $x(x_0,t,\xi_0)$, $\xi(x_0,t,\xi_0)$ are the solutions of the Hamilton equations

$$\frac{dx}{dt} = -\mathrm{grad}_\xi\ \sigma_1\left(\frac{r_1}{2}\right) \qquad x\big|_{t=0} = x_0$$

$$\frac{d\xi}{dt} = \mathrm{grad}_x\ \sigma_1\left(\frac{r_1}{2}\right) \qquad \xi\big|_{t=0} = \xi_0$$

A consequence of this discussion is that it will be enough to consider operators

$$L = D_t^2 - t^2R(x,t,D_x) + s(x,t,D_x) \qquad \text{with}$$

$$(1.13)\qquad R(x,t,D_x) \text{ positive elliptic of order two}$$

$$(1.14)\qquad S(x,t,D_x) \text{ of order one}$$

THEOREM 3. *Let* $L = D_t^2 - t^2R(x,t,D_x) - S(x,t,D_x)$ *with* R, S *as in* (1.13), (1.14) *and assume that either*

$$(1.15)\qquad \mathrm{Im}\ \sigma_1(s)\ (0,0,\xi) = 0 \qquad |\xi| = 1$$

*or*

$$(1.16)\qquad \mathrm{Im}\ \sigma_1(s)\ (0,0,\xi) \neq 0 \qquad |\xi| = 1$$

*then* $\forall\, a > 0$, *there exists a neighborhood* U *of the origin so that*

$$\|u\|_0 \leqslant a\|L_u\|_0$$

REMARK. The estimate in Theorem 3 implies the local solvability of the transpose of L, ${}^tL$.

Now ${}^tL = D_t^2 - t^2\,{}^tR - {}^tS(x,t,D_x)$, so ${}^tL$ satisfies the hypothesis of Theorem 3 if and only if L does, so the theorem gives sufficient conditions for the local solvability of L.

*Proof*. Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$, H a sub space of H. An operator A with domain $\mathcal{D}$ is formally self-adjoint if $\langle Au,v\rangle = \langle u,Av\rangle$ for $u,v \in \mathcal{D}$. We recall that if A(t) is a $C^\infty$ function of $t \in R$ with compact support and values in the space of formally self-adjoint operators with domain D, and u,v are $C^\infty$ functions of t with values in $\mathcal{D}$ we have

$$2\,\mathrm{Re}\int \langle A(t)u(t),u'(t)\rangle\,dt = -\int\langle A'(t)u(t),u(t)\rangle\,dt$$

We also recall the fact that the injection $H_c^s(\Omega) \subset H^{s'}$ has arbitrarily small norm when $s' \in s$, $s \geqslant \frac{-\dim\Omega}{2}$ and the diameter of $\Omega$ tends to zero.

To take advantage of this fact we assume that the diameter of U is less than $\varepsilon$ and pick up representatives of the pseudo-differential operators occurring on L whose associated kernels have support contained in a nbdd. of the diagonal

$$\{(x,y) \in \Omega_x \times \Omega_y \mid |x-y| < \varepsilon\}$$

so if $u \in C_c^\infty(U)$, then Ru, $Su \in C_c^\infty(\Omega)$ and the diameter of their support is less than 3 $\varepsilon$. The choice of $\varepsilon$ will depend on the principal symbols of R and S and will be made in the course of the proof. It is convenient to work with $\partial_t$ rather than $D_t = \frac{1}{i}\partial_t$ so we take

$$L = \partial_t^2 + t^2R + S$$

Since R is essentially self-adjoint we may assume that it is truly self-adjoint modifying S. We notice that the principal symbol of the new S will coincide with the principal symbol of the old one at t=0.

Using the positive ellipticity of R we may assume that $R = P^2$ with $P = P^*$.

CASE I. (1.15) *holds*. We write $S = S^R + S^I$ with $S^R$ formally self-adjoint and $S^I$ formally antiself-adjoint. (1.15) implies

(1.17) $\|S^Iu\|_{L_x^2} \leqslant \delta^2(\varepsilon)\,\|u\|^2_{H_x^1}$ for $u \in C_c^\infty(U)$ and

$$\delta(\varepsilon) > 0 \quad \text{as} \quad \varepsilon > 0$$

We denote ( , ) the inner product in $L^2_{x,t}$. Consider

(1.18) $\operatorname{Re}(Lu,u+2+u_t) = -\|u_t\|^2+\|tP_u\|^2+(S^R u,u)-\|u_t\|^2-3(t^2P^2u,u)-$

$$-(t^3(P^2)_t u,u) + 2\operatorname{Re}(S^R_u,tu_t) - 2\operatorname{Re}(S^I u,tu_t)$$

We can write

(1.19) $$2\operatorname{Re}(S^R u,tu_t) = -(S^R u,u) - (tS^R_t u,u)$$

Substitution of (1.19) in (1.18) yields

(1.20) $\operatorname{Re}(Lu,u+2+u_t) = -2(\|u_t\|^2+\|tP_u\|^2)-(tS^R_t u,u) -$

$$- 2\operatorname{Re}(S^I,tu_t) - (t^3(P^2)+u,u)$$

We observe that

(1.21) $|(tS^R_t u,u)| \leqslant M\,\varepsilon\,(\|u_t\|^2+\|tP_u\|^2)$ for a certain $M > 0$

Also, using (1.17) and the ellipticity of P we get

(1.22) $2|(S^I u,tu_t)| \leqslant \|u_t\|^2+\|tS^I u\|^2 \leqslant \|u_t\|^2+M\delta^2(\varepsilon)\|tP_u\|^2$

for a certain M independent of $\varepsilon$.

Finally we also have

(1.23) $|(t^3(P^2)_t u,u)| \leqslant M\,\varepsilon\,\|tP_u\|^2$ for a certain $M > 0$

Combining (1.20), (1.21), (1.22) and (1.23) we get

(1.24) $|\operatorname{Re}(Lu,u+2+u_t)| \geqslant \|u_t\|^2+\|tP_u\|^2$ for $u \in C^\infty_c(U)$

and $\varepsilon$ small enough

On the other hand,

(1.25) $|\operatorname{Re}(Lu,u_t+u_t)| \leqslant \|Lu\|M\,\varepsilon\,(\|u_t\|^2+\|tP_u\|^2)^{1/2}$

for a certain M, so (1.24) and (1.25) give

$(\|u_t\|^2+\|tP_u\|^2)^{1/2} \leqslant M\,\varepsilon\,\|Lu\|$ which implies at once $\|u\| \leqslant M\varepsilon\|Lu\|$.

That takes care of Case I.

CASE II. (1.16) *holds*. Let $u \in C^\infty_c(U)$. We are going to denote

$$\|u\|^2 = \int \|u(\cdot,t)\|^2_{1/2}\,dt$$

where $\|u(\cdot,t)\|_{1/2}$ is the usual $\frac{1}{2}$ - Sobelev norm in the x-variable. We are assuming that the formally antiself-adjoint operator $S^I$ is elliptic, so in a small neighborhood of the origin we have

$$|(S^I u(\cdot,t),u(\cdot,t))_{L^2_x}| \geq \|u(\cdot,t)\|^2_{1/2}$$

This implies that

$$(1.26) \qquad |\mathrm{Im}(Lu,u)| = |(S^I u,u)| \geq \int \|u(\cdot,t)\|^2_{1/2} M = |||u|||^2$$

It is clear that $\|u\|_0 \leq M\,\varepsilon\, |||u|||$ for a certain M.
Since $|\mathrm{Im}\ Lu,u| \leq \|L_u\|_0\ |||u|||$ we get the desired estimate.

Q.E.D.

REMARK. We observe that at a point $(0,0,\xi)$ in the cotangent space to $\Omega_x$ at the origin, the conditions $\mathrm{Im}\ \sigma_1(S)(0,0,\xi) = 0$ and $\mathrm{Im}\ \sigma_1(S)(0,0,\xi) \neq 0$ are exhausting. That means that microlocally we always fall either on case I or case II.
The proof of Theorem 3 then shows how to obtain microlocal estimates for L, without making any assumptions on the first order term S. However it is not clear that a local estimate of the type considered in Theorem 3, can always be obtained.
If one tries to "patch up" the microlocal estimates, the commutators involved cannot be treated as perturbations, i.e. the microlocal estimates are not stable.

There is at least one case, though, where this can be done: the two variables situation.

We have

COROLLARY TO THEOREM 3. *If* $L = D_t^2 - t^2R - S$ *with* R, S *as in* (1.13), (1.14) *and* $\Omega \subset R^2$, L *is locally solvable.*

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Universidad Nacional del Sur
Bahía Blanca, Argentina

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