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NORM LIMITS OF NILPOTENT OPERATORS AND WEIGHTED SPECTRA IN NON-SEPARABLE HILBERT SPACES

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In his survey article "Ten Problems in Hilbert Space", P.R.Halmos has raised the following questions: Is every quasinilpotent operator in an infinite dimensional separable Hilbert space the norm limit of nilpotent ones?. What is the closure (in the norm) of the set of nilpotent operators in a separable Hilbert space? ([11], Problem 7, p.915). The first question has been affirmatively answered independently by C.Apostol and D.Voiculescu ([3]) and by the author ([15]). The complete characterization of the set of norm limits of nilpotent operators in a *separable* Hilbert space was finally obtained by C.Apostol, C.Foias and D.Voiculescu in [2], in terms of the different parts of the spectra of the approximable operators.

The purpose of this paper is to give a complete characterization of the norm closure of the set of all nilpotent operators in a Hilbert space of arbitrary dimension, in terms of the different parts of the spectra and the *weighted spectra* of the approximable operators. The literature about this problem contains several *necessary* (and easy to verify!) conditions for an operator to be a norm limit of nilpotent ones (see [2];[12];[13];[15];[20]). Theorem 1 below says that those conditions are also *sufficient* (Roughly speaking: The set of norm limits of nilpotents is as large as one could expect).

The first part of the paper is devoted to an analysis of the weighted spectra of an operator A acting on a non-separable Hilbert space; it has some interest in itself. This analysis may be considered as a continuation of the article [6] by G.Edgar, J. Ernest and S.G.Lee. A decomposition of A related to its weighted spectra is given.

The second part is devoted to the proof of the characterization theorem (Theorem 1) and several related results on approximation of operators. In particular, it follows from these results that the first question of Halmos has an affirmative answer in any

Hilbert space.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT.

Throughout this paper $\mathcal{H}$ will denote a non-separable Hilbert space of (topological) dimension h. The closed bilateral ideals of the algebra $L(\mathcal{H})$ of all (bounded linear) operators acting on $\mathcal{H}$ have been completely characterized by several authors (see [5];[7]; [17]): to each cardinal α, $\aleph_0 \leqslant \alpha \leqslant h$, there corresponds a unique ideal $J_\alpha = \{T \in L(\mathcal{H}): \dim (\text{Ran } T)^- < \alpha\}^-$ (the upper bar denotes norm closure in both cases) and these are the only non-trivial (i.e., different from $\{0\}$ and $L(\mathcal{H})$) closed bilateral ideals of $L(\mathcal{H})$. Clearly, these ideals are well-ordered by inclusion.

Let $\pi_\alpha: L(\mathcal{H}) \to L(\mathcal{H})/J_\alpha$ denote the canonical projection onto the quotient algebra. The *spectrum of weight* α, $\Lambda_\alpha(T) =$ = spectrum $[\pi_\alpha(T)]$ in $L(\mathcal{H})/J_\alpha$, as well as each of its parts has been analyzed in [6] and we shall use those results without further quotation. Given $T \in L(\mathcal{H})$ and $\varepsilon > 0$, there exists a (closed) subspace $\mathcal{H}_\varepsilon$ of $\mathcal{H}$, containing the kernel of T, such that $\|Tx\| < \varepsilon\|x\|$ for all $x \in \mathcal{H}_\varepsilon$, $x \neq 0$, and $\|Tx\| \geqslant \varepsilon\|x\|$ for all $x \perp \mathcal{H}_\varepsilon$. Let $\delta_\varepsilon(T) = \dim \mathcal{H}_\varepsilon$; then the approximate nullity $\delta(T)$ of T is defined by $\delta(T) = \min(\varepsilon > 0)\ \delta_\varepsilon(T)$. $\Pi_\alpha(T) = \{\lambda : \delta(\lambda - T) \geqslant \alpha\}$ is the *approximate point spectrum of* T, *of weight* α, and it coincides with the *left* spectrum of $\pi_\alpha(T)$. Let T^* be the adjoint of T; then $\Lambda_\alpha(T^*) = \{\bar{\lambda}: \lambda \in \Lambda_\alpha(T)\}$ ($= \Lambda_\alpha(T)^*$) and $\Lambda_\alpha(T) = \Pi_\alpha(T) \cup \Pi_\alpha(T^*)^*$. Hence, to every $\lambda \in \mathbf{C}$ (the complex plane) we can associate a positive real number $\varepsilon(\lambda)$ and a subspace $\mathcal{H}_\lambda$ such that: (1) $\|(\lambda - T)x\| < \varepsilon(\lambda)\|x\|$ for all $x \in \mathcal{H}_\lambda$ and $\|(\lambda - T)x\| \leqslant \varepsilon(\lambda)\|x\|$ for all $x \perp \mathcal{H}_\lambda$; (2) $\dim \mathcal{H}_\lambda = \delta(\lambda - T)$; (3) If $\delta(\lambda - T) < h$, then $\varepsilon(\lambda)$ is the largest possible number such that either (1) or (2) is false whenever $\varepsilon(\lambda)$ is replaced by $2\varepsilon(\lambda)$; (4) If $\delta(\lambda - T) = h$ then $\varepsilon(\lambda) = 1$. It readily follows that, if $\delta(\lambda - T) < \alpha$, then $\varepsilon(\lambda) <$ $< 1/2 \text{ dist } [\lambda, \Lambda_\alpha(T)]$. Finally, $\varepsilon^*(\lambda)$ and $\mathcal{H}_\lambda{}^*$ are defined by: $\varepsilon^*(\lambda)[T] = \varepsilon(\bar{\lambda})[T^*]$ and $\mathcal{H}_\lambda{}^*[T] = \mathcal{H}_{\bar{\lambda}}[T^*]$.

If $\delta(T)$ and $\delta(T^*)$ are finite, then $\text{ind}(T) = \delta(T) - \delta(T^*)$ is precisely the Fredholm index of T ([16]). For arbitrary values of the approximate nullities, we shall define ind(T) as follows: (1) If $\delta(T) = \delta(T^*)$, then $\text{ind}(T) = 0$; (2) If $\delta(T) > \delta(T^*)$, then $\text{ind}(T) = \delta(T) - \delta(T^*)$ ($= \delta(T)$, if $\delta(T)$ is an infinite cardinal); (3) If $\delta(T) < \delta(T^*)$, then $\text{ind}(T) = -\text{ind}(T^*)$. The "extended index" ind(T) is not invariant under small (norm) changes and not invariant under h-compact perturbations. However, if $\delta(T) = \alpha$ and

$\delta(T^*) = \beta$, then $\delta(T+K) = \alpha$ and $\delta(T^*+K^*) = \beta$ for all $K \in J_\eta$, provided $\eta < \min[\alpha,\beta]$ $(\alpha,\beta \geq \aleph_o)$.

The spectrum and the essential spectrum ($\Lambda_{\aleph_o}(T)$) play a special role here. They will be denoted by $\Lambda(T)$ and $E(T)$, respectively, to simplify the notation. Similarly, we shall denote $J_{\aleph_o}$ (compact operators) by $H(\mathcal{H})$.

The main result of this paper is the following

THEOREM 1. *Let* $\underset{\sim}{N}(\mathcal{H})$ *and* $\underset{\sim}{A}(\mathcal{H})$ *be the subsets of* $L(\mathcal{H})$ *consisting of all nilpotent and all algebraic operators, respectively.*
Then: (i) $T \in \underset{\sim}{A}(\mathcal{H})^-$ *if and only if for every* λ *in* $\mathbf{C}$, $\operatorname{ind}(\lambda-T)=0$.

(ii) $T \in \underset{\sim}{N}(\mathcal{H})^-$ *if and only if* $T \in \underset{\sim}{A}(\mathcal{H})$ *and* $\Lambda(T)$ *and* $\Lambda_\alpha(T)$, $\aleph_o \leq \alpha \leq h$, *are connected sets containing the origin.*

2. THE ANALYSIS AND THE STRUCTURE OF THE WEIGHTED SPECTRA. THE DECOMPOSITION THEOREM.

Our first result shows that the family of weighted spectra of a given operator cannot be "too arbitrary".

THEOREM 2. *With the above notation, let* $T \in L(\mathcal{H})$ *and let* $\Lambda(T)$, $\Lambda_\alpha(T)$, $\aleph_o \leq \alpha \leq h$, *be the spectrum and the weighted spectra of* T. *Then:* (i) $\{\Lambda_\alpha(T): \aleph_o \leq \alpha \leq h\}$ *is a well-ordered (by inclusion) decreasing family of compact subsets of* $\Lambda(T)$ *with a last member* $\Lambda_h(T) \neq \emptyset$ *(the heavy spectrum of* T).

(ii) *There are only countably many different weighted spectra.*

(iii) *Let* $F = \{\Lambda_o=\Lambda(T), \Lambda_1=\Lambda_{\alpha_1}(T),\ldots,\Lambda_\omega(T)=\Lambda_{\alpha_\omega}(T),\ldots\}$ *be the family of all different spectra of* T *well-ordered by inclusion. If* μ *is a limit ordinal (We shall necessarily have* $\mu \leq \gamma < \Omega$, *where* $\Lambda_\gamma = \Lambda_h(T)$ *and* Ω *is the first uncountable ordinal), then* $\Lambda_\mu = \cap\{\Lambda_\nu: \nu < \mu\}$.

(iv) *Given* $\Lambda_\nu \in F$, *there exists a unique cardinal* β_ν *such that* $\Lambda_{\beta_\nu}(T) = \Lambda_\nu$ *and* $\Lambda_\alpha(T)$ *is strictly contained in* Λ_ν *for every* $\alpha > \beta_\nu$.

Proof. (i) and (ii) follow immediately from the fact that the cardinals are well-ordered and the definition of $\Lambda_\alpha(T)$. The details are left to the reader.

(iii) Let μ be a limit ordinal corresponding to the segment of the ordinals in the index set of F. For each $\nu < \mu$, let α_ν be a cardinal such that $\Lambda_{\alpha_\nu}(T) = \Lambda_\nu$ and define $\alpha_\mu = \sup (\nu < \mu)\, \alpha_\nu = \sum_{\nu<\mu} \alpha_\nu$.

CLAIM: $\bigcap_{\nu<\mu} \Lambda_\nu = \Lambda_\mu = \Lambda_{\alpha_\mu}(T)$

In fact, we clearly have $\Lambda_{\alpha_\mu}(T) \subset \bigcap_{\nu<\mu} \Lambda_\nu$. Assume that λ belongs to the latter intersection;this means that,for every $\nu < \mu$, either $\delta(\lambda - T) \geqslant \alpha_\nu$, or $\delta(\bar{\lambda} - T^*) \geqslant \alpha_\nu$. Since $\alpha_\mu = \sup (\nu < \mu)\, \alpha_\nu$, it readily follows that either $\delta(\lambda - T) \geqslant \alpha_\mu$, or $\delta(\bar{\lambda} - T^*) \geqslant \alpha_\mu$, and therefore $\lambda \in \Lambda_{\alpha_\mu}(T) = \Lambda_\chi$, for some $\Lambda_\chi \in$ F. On the other hand, since $\Lambda_\chi = \bigcap_{\nu<\mu} \Lambda_\nu$, and F is well-ordered by inclusion, it is obvious that $\chi = \mu$.

(iv) Let $J_\nu = \{\alpha : \Lambda_\alpha(T) = \Lambda_\nu\}$ and let $\beta_\nu = \sup\{\alpha \in J_\nu\} = \sum \{\alpha \in J_\nu\}$. Obviously, $\Lambda_{\beta_\nu}(T) \subset \bigcap \{\Lambda_\alpha(T) : \alpha \in J_\nu\} = \Lambda_\nu$. The inverse inclusion follows exactly as in (iii).

The cardinals α_μ of the proof of (iii) are $\aleph_0$-*irregular* (as defined in [7]) because F is countable. This is not necessarily the case of the β_ν's of (iv).

It is convenient to observe that the same results hold for the family of all approximate point spectra of T, considered as subsets of the approximate point spectrum $\Pi(T)$. In fact, the proof of the analogues of (i) and (ii) follows by using exactly the same arguments; the proof of the analogues of (iii) and (iv) is even easier, since we only have to consider $\delta(\lambda - T)$. The details are left to the reader.

Our next step will be the proof of the *decomposition theorem*. To this end, we shall need some auxiliary results.

LEMMA 1. *Let* $\{T_n\}_{n=1}^\infty$ *be a denumerable family of operators in* $L(\mathcal{H})$. *Then there exists a separable (closed) subspace* $\mathcal{H}_0$ *of* $\mathcal{H}$ *such that* $\mathcal{H}_0$ *reduces all the operators* T_n *and* $\Lambda(T_n|\mathcal{H}_0) = \Lambda(T_n)$, $\Lambda(T_n|\mathcal{H}_0^\perp) \subset E(T_n)$, *for all* n, *where* $T|M$ *denotes the restriction of the operator* T *to the subspace* M.

Proof. Let $\{\lambda_{nm}\}_{m=1}^\infty$ be a sequence with values in $\Lambda(T_n)$ and having $\Lambda(T_n)$ as its cluster set. Since $\Lambda(T_n) = \Pi(T_n) \cup \Pi(T_n^*)^*$, for every pair (n,m) there exists a sequence $\{x_k(n,m)\}_{k=1}^\infty$ of unitary vectors such that $\lim(k\to\infty) \min\{\|(\lambda_{nm} - T_n)x_k(n,m)\|, \|(\bar{\lambda}_{nm} - T_n^*)x_k(n,m)\|\} = 0$.

Each of the sets $\Lambda(T_n)\setminus E(T_n)$ consists of countably many components which are bounded open subsets of $\mathbf{C}$. Let $\{\mu_{nk}\}$ be a subset of $\Lambda(T_n)\setminus E(T_n)$ having exactly one point in common with every component. It is well known (see [16]) that for every positive j, $\ker(\mu_{nk}-T_n)^j$ and $\ker(\overline{\mu}_{nk}-T^*)$ are finite dimensional subspaces. Define $\mathcal{H}_o$ to be the minimal subspace of $\mathcal{H}$ containing all the vectors $x_k(n,m)$ $(n,m,k=1,2,\dots)$ and all the subspaces $\ker(\mu_{nk}-T_n)^j$, $\ker(\overline{\mu}_{nk}-T^*)^j$ $(n,k,j=1,2,\dots)$ which reduces every T_n. A straightforward verification shows that $\mathcal{H}_o$ has all the desired properties.

COROLLARY 1. *Let* $T \in L(\mathcal{H})$, $\dim \mathcal{H} = h > \aleph_o$ *and assume that* $\Lambda(T) = \Lambda_h(T)$. *Then* $T = \bigoplus_{k\in\Gamma} T_k$, *where* $\not c(\Gamma) = h$, T_k *acts on a separable infinite dimensional subspace* $\mathcal{H}_k$ *reducing* T *and* $\Lambda(T_k) = E(T_k) = \Lambda(T)$ *for all* $k \in \Gamma$. *Furthermore, the result remains valid if the operator* T *is replaced by a denumerable family in* $L(\mathcal{H})$.

Proof. The proof will be given for the case of a single operator. The general case follows by a formal modification of the same arguments.

It is immediate that $E(T) = \Lambda(T) = \Lambda_h(T)$. By Lemma 1, the family $\mathcal{R}$ of all separable reducing subspaces $\mathcal{H}_o$ such that $\Lambda(T|\mathcal{H}_o) = \Lambda(T)$ is nonempty. Let $\{\mathcal{H}_j'\}$ be a maximal orthogonal family of subspaces in $\mathcal{R}$ and let $\mathcal{H}' = \bigoplus_{j\in\Gamma'}\mathcal{H}_j'$, $T' = T|\mathcal{H}' = \bigoplus_{j\in\Gamma'}T_j'$ and $T'' = T|(\mathcal{H}')^\perp$; then $\Lambda(T_j') = \Lambda(T') = \Lambda(T)$ and $\Lambda(T'') \subset \Lambda(T)$.

Suppose that $\Lambda(T'') = \Lambda(T)$; then Lemma 1 can be used to obtain a new separable reducing subspace $\mathcal{H}_o \perp \mathcal{H}'$, such that $\Lambda(T|\mathcal{H}_o) = \Lambda(T)$, contradicting the maximality of the family $\{\mathcal{H}_j'\}$. Hence, $\Lambda(T'')$ is a *proper* subset of $\Lambda(T)$.

Let $\lambda_o \in \Lambda(T)\setminus\Lambda(T'')$. Since $\lambda_o \in \Lambda_h(T)$, either $\delta(\lambda_o-T) = h$ or $\delta(\overline{\lambda}_o-T^*) = h$. On the other hand, (λ_o-T'') is invertible and therefore there exists an $\eta > 0$ such that $\min\{\|(\lambda_o-T)x\|, \|(\overline{\lambda}_o-T^*)x\|\} \geqslant \eta\|x\|$ for all $x \perp \mathcal{H}'$. Let $0 < \varepsilon < \eta/2$ and assume that $\delta(\lambda_o-T)=h$. Then there exists a subspace $\mathcal{H}_\varepsilon$ of dimension h such that $\|(\lambda_o-T)x\| < \varepsilon\|x\|$ for all $x \in \mathcal{H}_\varepsilon$, $x\neq 0$, and $\|(\lambda_o-T)x\| \geqslant \varepsilon\|x\|$ for all $x \perp \mathcal{H}_\varepsilon$. It is completely apparent that $\mathcal{H}_\varepsilon \cap (\mathcal{H}')^\perp = \{0\}$, therefore $\mathcal{H}_\varepsilon$ must be a subspace of $\mathcal{H}'$, whence we conclude that $\delta(\lambda_o-T') = h$. Hence, $\lambda_o \in \Pi_h(T')$. The case when $\delta(\overline{\lambda}_o-T^*) = h$ can be similarly analyzed in order to obtain that $\delta(\overline{\lambda}_o-T'^*) = h$ and therefore $\lambda_o \in \Pi_h(T'^*)^*$.

Since $\lambda_o \in \Lambda(T_j')$ and therefore $\delta(\lambda_o-T_j') > 0$ or $\delta(\overline{\lambda}_o-T_j'^*) > 0$ (for each j), it follows that either $\delta(\lambda_o-T_j') > 0$ for h different indices j or $\delta(\bar{\lambda}_o - T_j'^*) > 0$ for h different indices j. We conclude that $T' = \bigoplus_{j \in \Gamma'} T_j'$, where $\not c(\Gamma') = h$. Moreover, since $h > \aleph_o$, we can write Γ' as a union of pairwise disjoint denumerable subsets: $\Gamma' = \bigcup_{k \in \Gamma} J_k$. Let $B_k = \oplus\{T_j' : j \in J_k\}$. It is immediate that $\Lambda(B_k) = E(B_k) = \Lambda(T)$ for all $k \in \Gamma$.

Let $T'' = \bigoplus_{\ell \in \Gamma''} T_\ell''$ be any decomposition of T'' corresponding to an orthogonal direct sum $\mathcal{H}'' = \bigoplus_{\ell \in \Gamma''} \mathcal{H}_\ell''$ into separable reducing subspaces. Clearly, $\not c(\Gamma'') \leq h$ and therefore there exists an injective map $\varphi : \Gamma'' \to \Gamma$. Define $T_k = B_k$, if $k \in \Gamma \setminus \varphi(\Gamma'')$, and $T_k = B_k \oplus T_\ell''$, if $k = \varphi(\ell)$ for some ℓ in Γ''.

It readily follows that $T = \bigoplus_{k \in \Gamma} T_k$, where $\Lambda(T_k) = E(T_k) = \Lambda(T) = \Lambda_h(T)$, for all $k \in \Gamma$.

With minor modifications of the same proof, we can obtain the following results

COROLLARY 2. *Let* $T \in L(\mathcal{H})$, $\dim \mathcal{H} = h > \aleph_o$ *and assume that* $\Lambda(T) = \Pi_h(T)$ ($\Lambda(T) = \Pi_h(T^*)^*$; $\Lambda(T) = \Pi_h(T) = \Pi_h(T^*)^*$). *Then* $T = \bigoplus_{k \in \Gamma} T_k$, *where* $\not c(\Gamma) = h$, T_k *acts on a separable reducing subspace and* $\Lambda(T_k) = E(T_k) = \Pi(T_k) = \Pi_{\aleph_o}(T_k) = \Lambda(T)$ ($\Lambda(T_k) = E(T_k) = \Pi(T_k^*)^* = \Pi_{\aleph_o}(T_k^*)^* = \Lambda(T)$; $\Lambda(T_k) = E(T_k) = \Pi(T_k) = \Pi(T_k^*)^* = \Pi_{\aleph_o}(T_k) = \Pi_{\aleph_o}(T_k^*)^* = \Lambda(T)$, *respectively.*). *Furthermore, the same results remain valid if the operator* T *is replaced by a denumerable family in* $L(\mathcal{H})$.

LEMMA 2. *Let* $T \in L(\mathcal{H})$ *and let* $\Lambda_\nu = \Lambda_{\beta_\nu}(T)$, $\Lambda_{\nu+1} = \Lambda_{\beta_{\nu+1}}(T) \in F$, *where* F, β_ν, $\beta_{\nu+1}$ *have the meaning of* Theorem 2. *Then there exists a reducing subspace* $\mathcal{H}_\nu$ *of dimension* β_ν *such that, if* $T = B_\nu \oplus C_\nu$, $B_\nu = T|\mathcal{H}_\nu$ *and* $C_\nu = T|(\mathcal{H}_\nu)^\perp$, *then* $\Lambda(C_\nu) = \Lambda_{\nu+1} = \Lambda_{\beta_{\nu+1}}(C_\nu)$, $\Lambda(B_\nu) = \Lambda(T)$ *and* $\Lambda_{\beta_\nu}(B_\nu) = \Lambda_\nu$.

Proof. Let $\lambda \in \Lambda_\nu \setminus \Lambda_{\nu+1}$, then $\delta(\lambda - T) \leq \beta_\nu$, $\delta(\bar{\lambda} - T^*) \leq \beta_\nu$ and at least one of these two cardinals must be *equal* to β_ν. Hence, if $\varepsilon(\lambda)$, $\mathcal{H}_\lambda$, $\varepsilon^*(\lambda)$ and $\mathcal{H}_\lambda^*$ are defined as in the Introduction, then $\dim[\mathcal{H}_\lambda + \mathcal{H}_\lambda^*]^- = \beta_\nu$.

Let $D(\lambda) = \{z : |z-\lambda| < \min[\varepsilon(\lambda), \varepsilon^*(\lambda)]\}$; then $\Lambda_\nu \setminus \Lambda_{\nu+1} \subset \cup\{D(\lambda) : \lambda \in \Lambda_\nu \setminus \Lambda_{\nu+1}\}$ and this covering admits a denumerable subcovering $\bigcup_{m=1}^{\infty} D(\lambda_m)$. Define $\mathcal{H}_\nu'$ to be the minimal reducing subspace of T containing $\{\mathcal{H}_{\lambda_m} + \mathcal{H}_{\lambda_m}^*\}_{m=1}^{\infty}$. It is clear from the above construction that $\dim \mathcal{H}_\nu' = \beta_\nu$.

Let $x \perp \mathcal{H}_\nu'$ and let $\lambda \in \Lambda_\nu \setminus \Lambda_{\nu+1}$. Assume that $\lambda \in D(\lambda_m)$; then it is easy to see that $\|(\lambda-T)x\| \geq \|(\lambda_m-T)x\| - |\lambda-\lambda_m|.\|x\| \geq \eta(\lambda).\|x\|$ and, similarly, $\|(\bar{\lambda}-T^*)x\| \geq \eta(\lambda).\|x\|$, for some $\eta(\lambda) > 0$, whence we conclude that $\lambda \notin \Lambda(C_\nu')$, where $C_\nu' = T|(\mathcal{H}_\nu')^\perp$. It is completely apparent that $\Lambda_{\beta_\nu}(C_\nu') = \Lambda_{\beta_{\nu+1}}(C_\nu') = \Lambda_{\nu+1}$. On the other hand, if $B_\nu' = T|\mathcal{H}_\nu'$, then $\Lambda(B_\nu') \supset \Lambda_{\beta_\nu}(B_\nu') \supset \Lambda_\nu \setminus \Lambda_{\nu+1}$.

By considering the complex numbers $\lambda \in \Lambda(T)\setminus\Lambda_\nu$ and by a formal repetition of the above arguments, we can find a second subspace $\mathcal{H}_\nu'' \subset (\mathcal{H}_\nu')^\perp$ such that $\dim \mathcal{H}_\nu'' \leq \beta_\nu$ and the decomposition $T = B_\nu''\oplus C_\nu''$ with respect to the orthogonal direct sum $\mathcal{H} = (\mathcal{H}_\nu'\oplus\mathcal{H}_\nu'')\oplus(\mathcal{H}_\nu'\oplus\mathcal{H}_\nu'')^\perp$ satisfies the properties: $\Lambda(B_\nu'') \supset$ $\supset \Lambda(T)\setminus\Lambda_{\nu+1}$, $\Lambda(C_\nu'') = \Lambda_{\beta_{\nu+1}}(C_\nu'') = \Lambda_{\nu+1}$.

Now, by an easy adaptation of the arguments of the *proof* of Corollary 1, it is not difficult to see that C_ν'' can be written as $C_\nu'' = \bigoplus_{k\in\Gamma} C_{\nu k}''$, where $\not{c}(\Gamma) = h$, $C_{\nu k}''$ acts on a separable reducing subspace for every $k \in \Gamma$ and $\Lambda(C_{\nu k}'') = \Lambda_{\nu+1}$ for exactly $\beta_{\nu+1}$ different indices k. Let Γ_ν be a subset of Γ such that $\not{c}(\Gamma_\nu) = \beta_\nu$ and $\Lambda(C_{\nu k}'') = \Lambda_{\nu+1}$ for all $k \in \Gamma_\nu$; let $\mathcal{H}_{\nu k}$ be the separable reducing subspace on which $C_{\nu k}''$ acts and define $B_\nu = B_\nu''\oplus[\bigoplus_{k\in\Gamma_\nu} C_{\nu k}''] \in$ $\in L(\mathcal{H}_\nu)$, where $\mathcal{H}_\nu = \mathcal{H}_\nu''\oplus[\bigoplus_{k\in\Gamma_\nu}\mathcal{H}_{\nu k}]$, and $C_\nu = C_\nu''|\bigoplus_{k\in(\Gamma\setminus\Gamma_\nu)} C_{\nu k}''$.

It readily follows that the decomposition $T = B_\nu\oplus C_\nu$, $\mathcal{H} = \mathcal{H}_\nu\oplus(\mathcal{H}_\nu)^\perp$ satisfies all our requirements.

Now we are in a position to prove the main result of this section.

THEOREM 3. *Let* $T \in L(\mathcal{H})$ *and let* F *be the well-ordered decreasing family of all different spectra of* T. *Then* $\mathcal{H}$ *admits a decomposition* $\mathcal{H} = \bigoplus_{0\leq\nu\leq\gamma} \mathcal{H}_\nu$ *into pairwise orthogonal reducing subspaces of* T *with respect to which* $T = \bigoplus_{0\leq\nu\leq\gamma} T_\nu$ *and the following properties are satisfied:*

(i) *Unless* $\beta_1 > \aleph_0$ *and* $\Lambda_0 = \Lambda_1$, $\mathcal{H}_0$ *is a separable infinite dimensional subspace such that* $\Lambda(T_0) = \Lambda(T)$ *and* $\Lambda(T|\mathcal{H}_0^\perp) = \Lambda_1$. *If* $\beta_1 > \aleph_0$ *and* $\Lambda_0 = \Lambda_1$, *then* $\mathcal{H}_0 = \{0\}$ *and* $T_0 = 0$.

(ii) *If* $\mu > 0$ *is not a limit ordinal, then* $\dim \mathcal{H}_\mu = \beta_\mu$, $\Lambda(T_\mu) =$ $= \Lambda_{\beta_\mu}(T_\mu) = \Lambda_{\beta_\mu}(\bigoplus_{0\leq\nu\leq\mu} T_\nu) = \Lambda_\nu$ *and* $\Lambda(T|[\bigoplus_{0\leq\nu\leq\mu}\mathcal{H}_\nu]^\perp) =$

$= \Lambda_{\beta_{\mu+1}}(T|[\bigoplus_{0\leq\nu\leq\mu}\mathcal{H}_\nu]^\perp) = \Lambda_{\mu+1}$.

(iii) *If* μ *is a limit ordinal and* $\beta_\mu = \sup(\nu < \mu)\ \beta_\nu$, *then* $\mathcal{H}_\mu = \{0\}$ *and* $T_\mu = 0$. *If* $\beta_\mu > \sup(\nu < \mu)\ \beta_\nu$, *then* $\dim \mathcal{H}_\mu = \beta_\mu$ *and* $\Lambda(T_\mu) = \Lambda_{\beta_\mu}(T_\mu) = \Lambda_\mu$. *In either case,* $\Lambda_{\beta_\mu}(\oplus_{0\leq\nu\leq\mu} T_\nu) = \Lambda_\mu$ *and* $\Lambda(T|[\oplus_{0\leq\nu\leq\mu}\mathcal{H}_\nu]^\perp) = \Lambda_{\beta_{\mu+1}}(T|[\oplus_{0\leq\nu\leq\mu}\mathcal{H}_\nu]^\perp) = \Lambda_{\mu+1}$.

(iv) *If* $\nu > 0$ *and* $T_\nu \neq 0$, *then* $T_\nu = \oplus_{k\in\Gamma_\nu} T_{\nu k}$, *where* $\not c(\Gamma_\nu) = \beta_\nu$ *and* $T_{\nu k}$ *is an operator on a separable infinite dimensional reducing subspace such that* $\Lambda(T_{\nu k}) = E(T_{\nu k}) = \Lambda_\nu$ *for all* $k \in \Gamma_\nu$. *Moreover, the analogue results are also true if* $\Lambda(T_{\nu k})$, $E(T_{\nu k})$ *and* Λ_ν *are replaced by* $\Pi(T_{\nu k})$, $\Pi_{\aleph_0}(T_{\nu k})$ *and* $\Pi_{\beta_\nu}(T)$ *(or by* $\Pi(T_{\nu k}{}^*)^*$, $\Pi_{\aleph_0}(T_{\nu k}{}^*)^*$ *and* $\Pi_{\beta_\nu}(T^*)^*$*) respectively. In particular,*

(v) *If, in addition, it is assumed that* $\operatorname{ind}(\lambda - T) = 0$ *for all complex* λ, *then* $\Lambda(T_o) = \Pi(T_o) = \Pi(T_o{}^*)^*$ *and* $\Lambda(T_{\nu k}) = \Pi(T_{\nu k}) = \Pi(T_{\nu k}{}^*)^*$ *for all* ν *and* k *as in* (iv).

(vi) *The representation is not unique, unless either* $\mathcal{H}$ *is separable or* $\Lambda(T) = \Lambda_h(T)$.

Proof. Preliminary decomposition. If $\Lambda_o \neq \Lambda_1 = E(T) = \Lambda_{\beta_1}(T)$, then we can use Lemma 1 to obtain a reducing subspace $\mathcal{H}_o'$ such that, if $T_o' = T|\mathcal{H}_o'$ and $C_o' = T|(\mathcal{H}_o')^\perp$, then $\Lambda(T_o') = \Lambda(T)$ and $\Lambda(C_o') \subset \Lambda_1$. Moreover, if $\beta_1 > \aleph_o$, then we set $\mathcal{H}_o = \mathcal{H}_o'$, $T_o = T_o'$, $C_o = C_o'$ and it follows that $\Lambda(C_o) = \Lambda_{\beta_1}(C_o) = \Lambda_1$; then we continue our analysis with C_o.

If $\beta_1 = \aleph_o$, then we can use Lemma 2 to obtain a (possibly larger) separable reducing subspace $\mathcal{H}_1 \supset \mathcal{H}_o'$ such that if $T_1 = T|\mathcal{H}_1$, then $\Lambda(T_1) = \Lambda_o$, $E(T_1) = \Lambda_1$ and $\Lambda(C_1) = \Lambda_{\beta_2}(C_1) = \Lambda_2$, where $C_1 = T|\mathcal{H}_1{}^\perp$. In this case we set $\mathcal{H}_o = \{0\}$, $T_o = 0$ and continue our analysis with C_1.

Finally, if $\Lambda_o = \Lambda_1 (= \Lambda(T) = E(T) = \Lambda_{\beta_1}(T))$ and $\beta_1 > \aleph_o$, then we use Lemma 2 to obtain a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1{}^\perp$ with respect to which $T = T_1 \oplus C_1$, $\Lambda(T_1) = E(T_1) = \Lambda_{\beta_1}(T_1) = \Lambda_1$, $\Lambda(C_1) = \Lambda_{\beta_2}(C_1) = \Lambda_2$ and (as above) we define $\mathcal{H}_o = \{0\}$, $T_o = 0$ and continue our analysis with C_1.

Now we proceed by transfinite induction. Assume that T_ν, $\mathcal{H}_\nu$ have been defined for every $\nu < \mu \leqslant \gamma$ so that the properties (i), (ii) and (iii'): *If* ν *is a limit ordinal, then either* T_ν, $\mathcal{H}_\nu$ *satisfy* (iii) *or* $\beta_\nu = \sup(\theta < \nu)\ \beta_\theta$, $\Lambda(T|[\oplus_{\theta<\nu}\mathcal{H}_\theta]^\perp)$ *is strictly larger than* $\Lambda_{\nu+1}$, $\dim \mathcal{H}_\nu = \beta_\nu$, $\bigcap_{\theta<\nu} \Lambda_\theta = \Lambda_\nu \supset \Lambda(T_\nu) = \Lambda_{\beta_\nu}(T_\nu) \supset \Lambda_{\nu+1}$ *and*

$\Lambda(T|[\oplus_{\theta\leq\nu}\mathcal{H}_\theta]^\perp) = \Lambda_{\beta_{\nu+1}}(T|[\oplus_{\theta\leq\nu}\mathcal{H}_\theta]^\perp) = \Lambda_{\nu+1}$, *hold for these indices.*
(a) If $\alpha_\mu = \sup(\nu<\mu)\,\beta_\nu < \beta_\mu$, then $\dim[\oplus_{\nu<\mu}\mathcal{H}_\nu] = \alpha_\mu < \beta_\mu$ and the restriction $C_\mu' = T|[\oplus_{\nu<\mu}\mathcal{H}_\nu]^\perp$ satisfies $\Lambda(C_\mu') = \Lambda_{\beta_\mu}(C_\mu') = \Lambda_\mu$, so we can apply Lemma 2 to obtain a decomposition $[\oplus_{\nu<\mu}\mathcal{H}_\nu]^\perp = \mathcal{H}_\mu \oplus \mathcal{H}_\mu'$ such that $\dim \mathcal{H}_\mu = \beta_\mu$, the restriction $T_\mu = C_\mu'|\mathcal{H}_\mu$ has the properties $\Lambda(T_\mu) = \Lambda_{\beta_\mu}(T_\mu) = \Lambda_\mu$ and the restriction $C_\mu = C_\mu'|\mathcal{H}_\mu'$ has the properties $\Lambda(C_\mu) = \Lambda_{\beta_{\mu+1}}(C_\mu) = \Lambda_{\mu+1}$. This case includes, in particular, the one where $\mu = \nu+1$ for some ordinal ν.

(b) If $\alpha_\mu = \beta_\mu$, then μ is necessarily a limit ordinal and β_μ is an $\aleph_0$-irregular cardinal. Let $\Lambda_\mu' = \Lambda_{\beta_\mu}(T|[\oplus_{\nu<\mu}\mathcal{H}_\nu]^\perp)$; it is immediate that $\Lambda_{\mu+1} \subset \Lambda_\mu' \subset \Lambda_\mu$ and very simple examples show that both inclusions could be proper. If $\Lambda_\mu' = \Lambda_{\mu+1}$, we simply set $\mathcal{H}_\mu = \{0\}$ and $T_\mu = 0$ and continue our analysis with $C_\mu = T|[\oplus_{\nu<\mu}\mathcal{H}_\nu]^\perp)$ and $\mu+1$ (Roughly speaking, we "forget" μ). If $\Lambda_{\mu+1}$ is a proper subset of Λ_μ', then we proceed as in (a) in order to obtain a decomposition of the usual type (via Lemma 2), which defines T_μ and $\mathcal{H}_\mu$, such that $\dim \mathcal{H}_\mu = \beta_\mu$, $\Lambda(T_\mu) = \Lambda_{\beta_\mu}(T_\mu) = \Lambda_\mu'$ and $\Lambda(C_\mu) = \Lambda_{\beta_{\mu+1}}(C_\mu) = \Lambda_{\mu+1}$, where $C_\mu = T|[\oplus_{\nu\leq\mu}\mathcal{H}_\nu]^\perp$.

Now T_ν and $\mathcal{H}_\nu$ can be defined for all ν, $0 \leqslant \nu \leqslant \gamma$, and it is clear from the above construction that the properties (i), (ii) and (iii') are fulfilled.

Final decomposition. Let $T = \oplus_{0\leq\nu\leq\gamma} T_\nu$, $\mathcal{H} = \oplus_{0\leq\nu\leq\gamma}\mathcal{H}_\nu$ be the decomposition obtained in the first part of the proof. Assume that μ is a limit ordinal such that T_μ and $\mathcal{H}_\mu$ satisfy (iii'), but not (iii). Then $\beta_\mu = \dim \mathcal{H}_\mu$ is $\aleph_0$-irregular, so we can write $\beta_\mu = \sum_{n=1}^\infty \beta_{\nu_n}(\mu)$ for a suitable increasing sequence of cardinals $\{\beta_{\nu_n}(\mu)\}_{n=1}^\infty$ corresponding to a decreasing sequence of different spectra $\{\Lambda_{\nu_n}(\mu)\}$ such that $\Lambda_\mu = \bigcap_{n=1}^\infty \Lambda_{\nu_n}(\mu)$. By replacing, if necessary, ν_n by $\nu_n + 1$ we can directly assume that NONE OF THE ν_n's IS A LIMIT ORDINAL. Moreover, according to this decomposition of β_μ we can also write $\mathcal{H}_\mu$ as a denumerable orthogonal direct sum $\mathcal{H}_\mu = \oplus_{n=1}^\infty \mathcal{H}_{\mu n}$, where $\dim \mathcal{H}_{\mu n} = \beta_{\nu_n}(\mu)$.
We shall say that a μ in the above conditions is "irregular".

Let μ be an irregular ordinal and let $\mathcal{H}_\mu = \oplus_{n=1}^\infty \mathcal{H}_{\mu n}$ be as above. Without loss of generality, we can assume that $\mathcal{H}_{\mu n}$ reduces T_μ for every $n=1,2,\ldots$. Let $T_\mu = \oplus_n T_{\mu n}$ be the corresponding decom separable (then the Theorem is trivial) or $\Lambda_h(T) = \Lambda(T)$ (then the family F is trivial!), this Corollary can be used to change the definition on a large family of separable reducing subspaces. The proof is complete now.

We shall give two examples to illustrate the need of a redefined decomposition in the case when F is "large".

EXAMPLE 1. Let E_n be the ellipse of vertices $\{\pm(1+1/n);\pm i/n\}$; let N_n be the normal operator "multiplication by z" acting on $L^2(E_n, d|z|)$ ($d|z|$ denotes the "arc length" measure on E_n) and let $A = \bigoplus_{n=1}^{\infty}(N_n \otimes I_n)$, where I_n is the identity map on a Hilbert space $\mathcal{R}_n$ of dimension $\aleph_n$, acting in the obvious way on $\mathcal{H} = \bigoplus_{n=1}^{\infty}(L^2{}_{(n)} \otimes \mathcal{R}_n)$. (Since $L^2{}_{(n)}$ is separable, it is clear that $\dim(L^2{}_{(n)} \otimes \mathcal{R}_n) = \aleph_n$).

It is easy to see that: (1) $h = \dim \mathcal{H} = \aleph_\omega = \sup_n \aleph_n$; (2) $\Lambda_o = \Lambda_1 = \Lambda_{\aleph_o}(A) = \Lambda_{\aleph_1}(A) = \{\bigcup_{n=1}^{\infty} E_n\}^- = \{\bigcup_{n=1}^{\infty} E_n\} \cup [-1,+1]$; (3) For every $m > 1$, $\Lambda_m = \Lambda_{\aleph_m}(A) = \{\bigcup_{n=m}^{\infty} E_n\}^-$; (4) $\Lambda_\omega(A) = \bigcap_{n=1}^{\infty} \Lambda_n = \Lambda_h(A) = [-1,1]$; (5) Let $A = \int \lambda \, dE(\lambda)$ be the spectral decomposition of A; then $E([-1,1]) = 0$; (6) In the notation of Theorem 3, $\mathcal{H}_o = \{0\}$, $\mathcal{H}_m = \bigoplus_{n\geq m} L^2{}_{(n)} \otimes \mathcal{R}_{n(m)}$, where $\mathcal{R}_{n(m)}$ is a subspace of dimension $\aleph_m$ of $\mathcal{R}_n$, and $\mathcal{R}_n = \bigoplus_{m=1}^{n} \mathcal{R}_{n(m)}$; therefore, $\Lambda_\omega' = \emptyset$!.

EXAMPLE 2. Let A and $\mathcal{H}$ be as above and define $B \in L(\mathcal{H}\oplus\mathcal{H})$ by $B(x,y) = (Ax,Cy)$, where C is a normal operator such that $\Lambda(C) = \Lambda_h(C) = [0,1]$. Then B has the properties (1) - (4) of Example 1, and (5') If $B = \int \lambda \, dF(\lambda)$ is the spectral decomposition of B, then $F([-1,1]) = F([0,1])$ is an orthogonal projection of rank and corank h; (6') $\mathcal{H}_o$, $\mathcal{H}_m$ can be chosen as in (6) or $\mathcal{H}_m$ can be replaced by $\mathcal{H}_m' = \mathcal{H}_m \oplus \mathcal{H}_m''$, where $\{\mathcal{H}_m''\}_{m=1}^{\infty}$ is a suitably chosen family of subspaces of $\mathcal{H}$ such that $\dim \mathcal{H}_m'' = \aleph_m$ ($m=1,2,\ldots$) and $\mathcal{H} = \bigoplus_{m=1}^{\infty} \mathcal{H}_m''$. In the first case we have $\Lambda_\omega' = [0,1]$ (properly contained in $\Lambda_h(B)$) and the decomposition must be modified. In the second one we have the desired kind of decomposition.

REMARK. Theorem 3 remains true if the single operator A is replaced by a denumerable family of operators.

The next corollary affirmatively answers a question raised in [6] for the case of an $\aleph_o$-regular cardinal.

COROLLARY 3. *Let* $A \in L(\mathcal{H})$ *and let* $\aleph_o \leq \alpha \leq h$. *Then:*

position of T_μ. Now we are going to modify the preliminary decomposition to obtain the final one. Recall that there are only countably many different spectra of T and, a fortiori, *there are only countably many irregular ordinals*. Furthermore, each irregular ordinal is associated with a *denumerable family of subspaces*. Hence, the family of all subspaces $\{\mathcal{H}_{\mu n}: \mu$ is irregular; $n=1,2,\ldots\}$ is countable.

The final decomposition is then defined as follows:

$$T = \bigoplus_{0\leq\nu\leq\gamma} L_\nu \quad , \quad \mathcal{H} = \bigoplus_{0\leq\nu\leq\gamma} M_\nu \quad , \quad L_\nu \in L(M_\nu)$$

where $M_\nu = \mathcal{H}_\nu \oplus [\oplus\{\mathcal{H}_{\mu n}: \beta_{\nu_n}(\mu) = \nu\}]$, whenever ν is a non-irregular ordinal for the first decomposition, and $M_\nu = \{0\}$ if ν is an irregular ordinal, and $L_\nu = T_\nu \oplus [\oplus\{T_{\mu n}: \beta_{\nu_n}(\mu) = \nu\}]$, if ν is not irregular, and $L_\nu = 0$, if ν is irregular.

It is straightforward to check that the final decomposition has the properties (i), (ii) and (iii).

The first part of (iv) follows by applying Corollary 1 to L_ν acting on M_ν, for $\nu > 0$, ν a regular ordinal. The remaining statements of (iv) can be proved by proved by using Corollary 2. We shall give here the proof for the case considered in (v) and the other cases are left to the reader. Assume that $\text{ind}(\lambda - T) = 0$ for all complex λ. It readily follows that $\text{ind}(\lambda - L_\nu) = 0$ for all $\lambda \notin \Lambda_{\nu+1}$. If $\lambda \in \Lambda_\nu \setminus \Lambda_{\nu+1}$, then $\text{ind}(\lambda - L_{\nu+1}) = 0$ and, by Corollary 2, $L_{\nu+1} = \bigoplus_{k\in\Gamma_{\nu+1}} L_{(\nu+1)k}$, where $\lambda \in \Pi(L_{(\nu+1)k}) \cap \Pi(L^*_{(\nu+1)k})^* = \Pi_{\aleph_0}(L_{(\nu+1)k}) \cap \Pi_{\aleph_0}(L_{(\nu+1)k}^*)^*$ for all $k \in \Gamma_{\nu+1}$, and $\not c(\Gamma_{\nu+1}) = \beta_{\nu+1} > \beta_\nu$. Thus, we can separate a subset of cardinal β_ν of the index set $\Gamma_{\nu+1}$ and use the corresponding $L_{(\nu+1)k}$'s to redefine L_ν so that $\text{ind}(\lambda - L_\nu) = 0$ for all $\lambda \in \Lambda_{\nu+2}$. By a double process of transfinite induction we can redefine the final decomposition of T so that $\text{ind}(\lambda - L_\nu) = 0$ for all complex λ and for all ν, $0 \leqslant \nu \leqslant \gamma$, (As in the re-definition of the preliminary decomposition to obtain the final decomposition, there are only countably many subspaces of dimension not greater than β_ν which contribute to the modification of $\mathcal{H}_\nu$; $\dim \mathcal{H}_\nu = \beta_\nu \geqslant \aleph_0$ unless $\nu = 0$ or ν an irregular ordinal and $\mathcal{H}_\nu = \{0\}$, but in these cases the original and the modified $\mathcal{H}_\nu$ are both equal to $\{0\}$, because it is trivially true that $\text{ind}(\lambda - L_\nu) = 0$ for all complex λ in these cases).

Finally, (vi) follows immediately from Corollary 1: unless $\mathcal{H}$ is

(i) *If* α *is* $\aleph_0$*-regular, there exists* $K \in J_\alpha$ *such that* $\Lambda(A+K) = \Lambda_\alpha(A)$.

(ii) *If* α *is* $\aleph_0$*-irregular, given* $\varepsilon > 0$ *there exists* $K_\varepsilon \in J_\alpha$ *such that* $\Lambda(A+K_\varepsilon) \subset (\Lambda_\alpha(A))_\varepsilon$, *where* $(\Lambda_\alpha(A))_\varepsilon = \{\lambda : \mathrm{dist}[\lambda, \Lambda_\alpha(A)] \leqslant \varepsilon\}$.

(iii) *In either case,* $\Lambda_\alpha(A) = \cap \{\Lambda(A+K) : K \in J_\alpha\}$.

Proof. (i) Let $A = \bigoplus_{0 \leq \nu \leq \gamma} A_\nu$ be the decomposition of A given by Theorem 3 and define $K = [\oplus\{A_\nu : \beta_\nu < \alpha\}] \oplus 0$. Clearly, $K \in J_\alpha$ because Rank $K \leqslant \sup(\beta_\nu < \alpha) < \alpha$ and $\Lambda(A+K) = \Lambda_\mu = \Lambda_\alpha(A)$, where μ is the smaller index such that $\alpha \leqslant \beta_\mu$.

(ii) In the case when α is $\aleph_0$-irregular but $\sup(\beta_\nu < \alpha) < \alpha$, we still can find a $K \in J_\alpha$ such that $\Lambda(A+K) = \Lambda_\alpha(A)$.

Assume that $\sup(\beta_\nu < \alpha) = \alpha = \beta_\mu$. Then Theorem 3 shows that $\Lambda_\alpha(A) = \cap \{\Lambda_\nu : \nu < \mu\}$. Since $\{\Lambda_\nu\}$ is a decreasing family of compact sets, it follows that $\Lambda_\nu \subset (\Lambda_\alpha(A))_\varepsilon$ for all $\nu \geqslant \nu(\varepsilon)$. Defining $K_\varepsilon = [\oplus\{A_\nu : \nu < \nu(\varepsilon)\}] \oplus 0$, it follows as in (i) that $\Lambda(A+K_\varepsilon) = \Lambda_{\nu(\varepsilon)} \subset (\Lambda_\alpha(A))_\varepsilon$.

(iii) This is an immediate consequence of (i) and (ii).

In ([6], Theorem 4.6) it is claimed that the result of Corollary 3(i) remains true for $\aleph_0$-irregular, provided A is a *normal* operator; however, our Examples 1 and 2 seem to contradict the proof given there. Anyway, the result is actually true. In fact, we have:

COROLLARY 4. *Let* A *be a normal operator on a Hilbert space of infinite dimension* h. *Let* α *be a cardinal,* $\aleph_0 \leqslant \alpha \leqslant h$. *Then there exists a normal operator* $K \in J_\alpha$ *such that* K *commutes with* A *and* $\Lambda(A+K) = \Lambda_\alpha(A)$.

Proof. Let $A = \int \lambda dE(\lambda)$ be the spectral decomposition of A. If α is not equal to the supremum of the cardinals β_ν such that $\beta_\nu < \alpha$, then the answer is given by Corollary 3. On the other hand, if $\alpha = \beta_\mu$ is the supremum of those cardinals, then $\Lambda_\alpha(A) = \cap \{\Lambda_\nu : \nu < \mu\}$. Hence, if Ω is a Borel set such that $\Lambda_\alpha(A) \cap \Omega^- = \emptyset$, then $E(\Omega)$ is an orthogonal projection of rank strictly smaller than α and therefore $E(\Omega) \in J_\alpha$.

Let $A = A_\alpha \oplus A_1' \oplus A_2'$, where $A_\alpha = [\int_{\Lambda_\alpha(A)} \lambda \, dE(\lambda)] | E(\Lambda_\alpha(A))$, $A_1' = [\int_\Omega \lambda \, dE(\lambda)] | E(\Omega)$ and $\Omega = [\mathbb{C} \setminus \Lambda_\alpha(A)] \cap \{\lambda : |\lambda| \leqslant \|A\|\}$.

Decompose Ω as a denumerable union $\bigcup_{n=1}^{\infty} \Delta_n$ of pairwise disjoint Borel sets such that $\Lambda_\alpha(A) \cap \Delta_n^- = \emptyset$, diameter $(\Delta_n) \to 0$ and

$\text{dist}[\lambda_n, \Delta_n] \to 0$, as $n \to \infty$, for a suitable sequence $\{\lambda_n\}$ with values in $\Lambda_\alpha(A)$ (The construction of $\{\Delta_n\}$ and $\{\lambda_n\}$ follows by standard arguments).

Define $B = A_\alpha \oplus [\bigoplus_{n=1}^{\infty} \lambda_n E(\Delta_n)]$. Then $\Lambda(B) = \Lambda_\alpha(B) = \Lambda_\alpha(A)$ and $K = $ $= A-B = 0 \oplus [\bigoplus_{n=1}^{\infty} \int_{\Delta_n} (\lambda - \lambda_n)\, dE(\lambda)]$ belongs to J_α. It is clear that A commutes with K.

3. CONSEQUENCES OF THE DECOMPOSITION THEOREM.

Theorem 3 is a very useful tool to obtain non-separable versions of well-known separable results. Namely, we can obtain the following extension of a result due to C.Pearcy and N.Salinas.

COROLLARY 5. *Let $T \in L(\mathcal{H})$ be a hyponormal operator and let $N \in L(\mathcal{H})$ be any normal operator such that $\Lambda(N) = E(N)$ and $\Lambda_\alpha(N) \subset \Pi_\alpha(T)$ for all α, $\aleph_0 \leqslant \alpha \leqslant h$. Then, given $\varepsilon > 0$, there exists an operator K, $\|K\| < \varepsilon$, such that $T+K \approx T\oplus N$ (the symbol $\approx$ means that the two operators are unitarily equivalent). Moreover, if h is $\aleph_0$-irregular, then K can be chosen to be an h-Hilbert-Schmidt operator as defined in* [6].

Proof. According to Theorem 3, T can be written as $T = \bigoplus_{0 \leq \nu \leq \gamma} T_\nu$, $T_\nu = 0$ if ν is irregular and $T_\nu = \bigoplus_{k \in \Gamma_\nu} T_{\nu k}$, where $T_{\nu k}$ acts on a separable reducing subspace; $\Lambda(T_{\nu k}) = E(T_{\nu k}) = \Lambda_\nu$, $\Pi(T_{\nu k}) = $ $= \Pi_{\aleph_0}(T_{\nu k}) = \Pi_{\beta_\nu}(T)$, for all $k \in \Gamma_\nu$ and $\not{c}(\Gamma_\nu)$.

Applying the separable theorem ([18], Theorem 1) to $T_{\nu k}$ for all ν and for all k in Γ_ν, we can find an operator $K \in L(\mathcal{H})$, $K = \oplus_\nu K_\nu$, $K_\nu = \bigoplus_{k \in \Gamma_\nu} K_{\nu k}$, where $K_{\nu k}$ is a compact operator of norm smaller than $\varepsilon/2$, such that $T+K = \oplus_\nu[\oplus_k (T_{\nu k}+K_{\nu k})] \approx \oplus_\nu[\oplus_k(T_{\nu k}\oplus N_{\nu k})]$ and the $N_{\nu k}$'s are normal operators such that $\oplus_k N_{\nu k} = N_\nu$ and $\oplus_\nu N_\nu$ is the given operator N. Since $\|K\| \leqslant \varepsilon/2 < \varepsilon$, the proof of the first statement is complete.

Now consider the case when h is $\aleph_0$-irregular. Let $\{\varepsilon_\nu\}$ be a family of positive numbers such that $\sum_\nu \varepsilon_\nu < \varepsilon/2$, where the sum is extended over all $\nu < \gamma$. For these values of ν, instead of $\|K_{\nu k}\| < \varepsilon/2$ we require the inequality $\|K_{\nu k}\| < \varepsilon_\nu$ for all $k \in \Gamma_\nu$. Thus we obtain $\|K_\nu\| \leqslant \varepsilon_\nu$ for all $\nu < \gamma$, and it readily follows that $\bigoplus_{\nu<\gamma} K_\nu$ is an h-Hilbert-Schmidt operator of norm smaller than $\varepsilon/2$. If γ is an irregular ordinal (in the sense of the proof of Theorem 3), then $\mathcal{H} = \bigoplus_{\nu<\gamma} \mathcal{H}_\nu$ and the result is immediate. If γ is not irregular, then $\sum_{\nu<\gamma} \beta_\nu < \beta_\gamma = h$ and $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{H}_\gamma \oplus \mathcal{H}_\gamma'$, where $\dim \mathcal{H}_\gamma' < h$. By using that h is $\aleph_o$-irregular, it is not difficult to obtain a decomposition $\mathcal{H}_\gamma = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ into reducing subspaces $\mathcal{H}_n$, $\dim \mathcal{H}_n = \alpha_n < h$, such that $\sum_{n=1}^{\infty} \alpha_n = h$, $\Lambda(T|\mathcal{H}_n) = \Lambda_{\alpha_n}(T|\mathcal{H}_n) = \Lambda_\gamma$ and $\Pi(T|\mathcal{H}_n) = \Pi_{\alpha_n}(T|\mathcal{H}_n) = \Pi_h(T)$, for all $n=1,2,\ldots$. Now define $K_n \in L(\mathcal{H}_n)$ such that $\|K_n\| < \varepsilon/2^{(n+1)}$, $n=1,2,\ldots$, and $T_\gamma + (\oplus_n K_n) \approx T_\gamma \oplus N_\gamma$. Then, if $K = (\bigoplus_{\nu<\gamma} K_\nu) \oplus (\oplus_n K_n)$, it is easy to see that K is actually an h-Hilbert-Schmidt operator of norm less than ε and $T+K \approx T \oplus N$.

Our next result extends F.Wolf's theorem (see [4];[18];[24]).

THEOREM 4. *If* h *is* $\aleph_o$*-irregular and* $T \in L(\mathcal{H})$ *is not left invertible modulo* J_h, *then given* $\varepsilon > 0$ *there exists an h-Hilbert-Schmidt operator* K *of norm less than* ε *such that* $\dim \ker(T+K) = h$. *On the other hand, if* h *is* $\aleph_o$*-regular, then there exists a non-negative hermitian operator* $H \in L(\mathcal{H})$ *such that* $\Lambda(H) = \Lambda_h(H) = [0,1]$, *but* $\dim \ker(H+K) < h$ *for any* $K \in J_h$ *(a fortiori, for any h-Hilbert-Schmidt operator* K*).*

Proof. Assume that h is $\aleph_o$-irregular and let $T = VH$ be the polar decomposition of T. According to ([6], Theorem 5.6) and its proof, there exists $H' \in L(\mathcal{H})$, $H' \geqslant 0$ such that $H'-H$ is an h-Hilbert-Schmidt operator of norm smaller than ε and $\dim \ker H' = h$. Then the operator $K = V(H'-H)$ satisfies our requirements.

Conversely, if h is $\aleph_o$-regular and $H \approx \oplus\{qI : q \text{ is rational and } 0 < q < 1\}$, then $\Lambda(H) = \Lambda_h(H) = [0,1]$, but $\dim \ker(H+K) < h$ for any $K \in J_h$ (see Example in [6]).

The following corollary is the non-separable version of ([1], Theorem 2.2) and it can be proved by following the same arguments and by using Theorem 4 instead of Wolf's theorem. The proof is left to the reader.

COROLLARY 6. *If* h *is* $\aleph_o$*-irregular and* $T \in L(\mathcal{H})$, *given* $\varepsilon > 0$ *there exists an h-Hilbert-Schmidt operator* K_ε *of norm less than* ε *and a subspace* $\mathcal{H}_\varepsilon$ *of dimension* h *such that* $(T-K_\varepsilon)\mathcal{H}_\varepsilon \subset \mathcal{H}_\varepsilon$, $T_\varepsilon = (T-K_\varepsilon)|\mathcal{H}_\varepsilon$ *is normal and* $\Lambda(T_\varepsilon) = \Lambda_h(T_\varepsilon) = \Pi_h(T)$.

The next step will be a non-separable version of Propositions 3.2 and 3.3 of [23] (Corollary 7, below) about bi-quasitriangular operators. *Quasitriangular* and *bi-quasitriangular* operators are defined in ([1];[10]). According to the celebrated characterization of quasitriangularity given in [1], an operator A is bi-quasitriangular if and only if $\mathrm{ind}(\lambda-A) = 0$ for all complex λ.

We shall need the following elementary result.

LEMMA 3. *Let $\{\Lambda_\nu\}$ be an arbitrary family of nonempty compact subsets of the complex plane, contained in a fixed compact Λ_o, and let $\varepsilon > 0$ be given. Then there exists a finite family $\{G_1, G_2, \ldots, G_m\}$ of open sets such that $\Lambda_\nu \subset G_j \subset (\Lambda_\nu)_\varepsilon$ for at least one value of j depending on ν.*

Proof. Let $\{D_n\}_{n=1}^N$ be a finite covering of Λ_o by open discs of radii $\varepsilon/2$. Then to each Λ_ν we associate the open set $G_\nu = \cup\{D_n : D_n \cap \Lambda_\nu \neq \emptyset\}$. Clearly, the family $\{G_\nu\}$ cannot have more than 2^N-1 different elements, whence the result follows.

The following result was conjectured by the author in [15]. The proof follows easily from [23].

THEOREM 5. *Let $A \in L(\mathcal{H})$ be a bi-quasitriangular operator acting on the separable infinite dimensional Hilbert space $\mathcal{H}$ and let $\varepsilon > 0$ be given. Then there exists $A' \in L(\mathcal{H})$ such that $A-A'$ is a compact operator of norm smaller than ε such that for a suitable orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into two infinite dimensional subspaces, $A' = \begin{bmatrix} N & L \\ 0 & B \end{bmatrix}$, where N is normal, B is bi-quasitriangular and $\Lambda(N) = E(N) = E(B) = E(A)$. Furthermore, there also exists a bi-quasitriangular operator $A'' = \begin{bmatrix} N & L \\ 0 & B' \end{bmatrix}$ such that $\|B-B'\| < \varepsilon$, $E(B') \subset E(A)$ and $\mathcal{H}_1$ admits an algebraic complement $\mathcal{H}_1'$ invariant under A'' such that, with respect to the algebraic (not necessarily orthogonal) direct sum $\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{H}_1'$, $A'' = N \dotplus B''$ with B'' similar to B'.*

Proof. According to ([23],Proposition 3.2), there exists a compact operator $K \in L(\mathcal{H})$ of norm smaller than ε and an orthogonal direct sum decomposition $\mathcal{H} = \oplus_{j=1}^{8} \mathcal{H}_j$ into eight infinite dimensional subspaces with respect to which A+K can be written as the upper triangular operator matrix

$$A' = A+K = \begin{bmatrix} N_1 & 0 & & & & & & \\ 0 & D_1 & & & & * & & \\ & & N_2 & 0 & & & & \\ & & 0 & D_2 & & & & \\ & & & & N_3 & 0 & & \\ & 0 & & & 0 & D_3 & & \\ & & & & & & N_4 & 0 \\ & & & & & & 0 & D_4 \end{bmatrix} \qquad \text{(I)}$$

where N_k is a normal operator and D_k is a block-diagonal operator such that $E(D_k) \subset E(N_k) = E(T)$, $k=1,2,3,4$. (The * denotes unspecified entries; see ([23],Proposition 3.2) for the proof and definition of block-diagonal operators). Block-diagonal operators are particular cases of bi-quasitriangular operators. It is clear that, with respect to the orthogonal direct sum $\mathcal{R} = \mathcal{R}_1 \oplus [\bigoplus_{j=2}^{8} \mathcal{R}_j]$, $A' = \begin{bmatrix} N_1 & C \\ 0 & B \end{bmatrix}$, where B is bi-quasitriangular and $E(B) = E(N_1) = E(T)$. This proves the first statement.

Since B is bi-quasitriangular, it follows from ([23],Proposition 3.3) that B is a norm limit of algebraic operators. Hence, there exists an algebraic operator $B' \in L(\mathcal{R}_2')$ (where $\mathcal{R}_2' = \bigoplus_{j=2}^{8} \mathcal{R}_j$) such that $\|B-B'\| < \varepsilon/3$; moreover, B' can be chosen so that $E(B') \subset \subset E(B)$. Since $\Lambda(B')$ is finite, we can use the spectral decomposition of N_1 to obtain a normal operator M_1 such that $\|N_1-M_1\| < < \varepsilon/3$ and $\Lambda(M_1) = E(M_1)$ is a finite set disjoint from $\Lambda(B')$. Then, by Riesz' decomposition theorem ([19], Chapter XI), the operator $L' = \begin{bmatrix} M_1 & C \\ 0 & B' \end{bmatrix}$ admits two complementary invariant subspaces $\mathcal{R}_1$ and $\mathcal{R}_1'$ such that, with respect to the algebraic direct sum $\mathcal{R} = \mathcal{R}_1 \dotplus \mathcal{R}_1'$, L' can be written as $L' = M_1 \dotplus B''$, with B'' similar to B'.

Finally, let $A'' = N \dotplus B''$, where N is a normal operator in $L(\mathcal{R}_1)$, $N \approx N_1$, such that $\|N-N_1\| < \varepsilon/3$ (i.e., A'' is the operator obtained from L' by replacing M_1 by N). Then $\|A-A''\| < \varepsilon$ and A'' has the desired properties. The proof is complete now.

Now we are in a position to prove a very important consequence of Theorem 3.

COROLLARY 7. *The set of all operators in $L(\mathcal{H})$ with only finitely many different spectra is norm dense. Furthermore, given* $T \in L(\mathcal{H})$

and $\varepsilon > 0$, *there exists* $T' \in L(\mathcal{H})$ *such that* $\|T-T'\| < \varepsilon$ *and*

(i) $\Lambda(T) \subset \Lambda(T') \subset (\Lambda(T))_\varepsilon$ *and* $\Lambda_\alpha(T) \subset \Lambda_\alpha(T') \subset (\Lambda_\alpha(T))_\varepsilon$ *for all* α, $\aleph_o \leqslant \alpha \leqslant h$.

(ii) $T' = T_o' \oplus [\bigoplus_{j=1}^{m} T_j']$, *where* $T_j' \in L(\mathcal{H}_j)$, $\mathcal{H}_j = \mathcal{H}_{j,1} \oplus \mathcal{H}_{j,2}$, $\dim \mathcal{H}_{j,1} = \dim \mathcal{H}_{j,2} = \beta_j$, $\aleph_o < \beta_1 < \beta_2 < \ldots < \beta_m = h$, *and*

(iii) *With respect to the above orthogonal direct sum,* T_j *is the* 2x2 *operator matrix* $T_j = \begin{bmatrix} N_j'' & T_{j1} \\ 0 & T_{j2} \end{bmatrix}$, *where* N_j'' *is a normal operator of uniform multiplicity* β_j *such that* $\Lambda(N_j) = \Lambda_{\beta_j}(N_j) = G_j^-$ *for some open neighborhood* G_j *of* $\Pi_{\beta_j}(T)$ *bounded by finitely many pairwise disjoint rectifiable Jordan curves,* $j=1,2,\ldots,m$.

(iv) $\Pi(T_{j2}) = \Pi_{\beta_j}(T_{j2}) \subset \text{interior } \Lambda(N_j)$, $j=1,2,\ldots,m$.

(v) $\Lambda_{\beta_1}(T) \subset \Lambda_\alpha(T) \subset \Lambda(N_1) \cup \Lambda_{\beta_j}(T_{12}) \subset (\Lambda_{\beta_1}(T))_\varepsilon$ *for all* $\alpha \leqslant \beta_1$, *and* $\Lambda_{\beta_j}(T) \subset \Lambda_\alpha(T) \subset \Lambda(N_j) \cup \Lambda_{\beta_j}(T_{j2}) \subset (\Lambda_{\beta_j}(T))_\varepsilon$ *for all* α, $\beta_{j-1} < \alpha \leqslant \beta_j$, $j=2,3,\ldots,m$.

(vi) *If, in addition, it is assumed that* $\text{ind}(\lambda-T) = 0$ *for all* $\lambda \in \mathbf{C}$, *then* $\text{ind}(\lambda-T') = 0$ *for all* $\lambda \in \mathbf{C}$.

Proof. Let $T = \bigoplus_{0\leq\nu\leq\gamma} T_\nu$, $T_\nu = \bigoplus_{k\in\Gamma_\nu} T_{\nu k}$ be the decomposition of T given by Theorem 3. If either $\Lambda(T) \neq E(T)$ or $\Lambda(T) = E(T)$ and $\beta_1 = \aleph_o$, then the decomposition's first term (T_o, or T_1, if $T_o = 0$) is kept unchanged: $T_o' = T_o$ (or $T_o' = T_1$, if $T_o = 0$). Thus, we can restrict our attention to the case when $T = \bigoplus_{0\leq\nu\leq\gamma} T_\nu$, $\Lambda(T) = E(T) = \Lambda_1$ and $\beta_1 > \aleph_o$.

By ([1], Theorem 2.2), there exist operators $T_{\nu k}' \in L(\mathcal{H}_{\nu k})$ such that $T_{\nu k} - T_{\nu k}'$ is a compact operator of norm smaller than $\varepsilon/4$ and $\mathcal{H}_{\nu k}$ admits a decomposition $\mathcal{H}_{\nu k} = \mathcal{H}_{\nu k,1} \oplus \mathcal{H}_{\nu k,2}$ with respect to which $T_{\nu k}'$ is the 2x2 operator matrix $T_{\nu k}' = \begin{bmatrix} N_{\nu k} & T_{\nu k,1} \\ 0 & T_{\nu k,2} \end{bmatrix}$, where $N_{\nu k}$ is a normal operator such that $\Lambda(N_{\nu k}) = E(N_{\nu k}) = \Pi_{\aleph_o}(T) \supset \Pi_{\aleph_o}(T_{\nu k,2})$; moreover, with a minor modification, it can be obtained that $\Pi_{\aleph_o}(T_{\nu k,2}) = \Lambda(N_{\nu k})$. Let $R = \bigoplus_{1\leq\nu\leq\gamma}(\bigoplus_{k\in\Gamma_\nu} T_{\nu k}')$; then $\|T-R\| \leqslant \varepsilon/4$, $\Lambda_\alpha(T) \subset \Lambda_\alpha(R)$ for all α and (by the upper semi-continuity of the spectrum; see, e.g., [9]) we can assume that $\|T-R\|$ is small enough to insure that $\Lambda(R) \subset (\Lambda(T))_{\varepsilon/4}$ and $\Lambda_\alpha(R) \subset$

$\subset(\Lambda_\alpha(T))_{\varepsilon/4}$ for all α, $\aleph_o \leqslant \alpha \leqslant h$.

By Lemma 3 there exists an increasing finite family $\{G_1, G_2, \ldots, G_m\}$ of open sets such that for every α, $[\Pi_\alpha(T)_{\varepsilon/4}]^- \subset G_j \subset$ $\subset [\Pi_\alpha(T)_{\varepsilon/2}]$ for some $j=j(\alpha)$ and $j=j(\alpha)$ for at least one value of α. Moreover, G_j^- is bounded by finitely many pairwise disjoint rectifiable Jordan curves. To every G_j we associate a normal operator M_j of uniform multiplicity $\aleph_o$ such that $\Lambda(M_j) = E(M_j) = G_j^-$. Let $T_{\nu k}''$ be the operator obtained from $T_{\nu k}'$ by replacing $N_{\nu k}$ by $N_{\nu k}'' \approx M_{j(\alpha)}$ (where $\Lambda_\alpha(T) = \Lambda_\nu!$) such that $\|N_{\nu k} - N_{\nu k}''\| < 3\,\varepsilon/4$ (The existence of such operators $N_{\nu k}''$ is guaranteed by standard arguments based on the spectral theorem for normal operators; see [8]). Let $T' = \oplus_\nu[\oplus_k T_{\nu k}'']$. It readily follows that $\|T-T'\| <$ $< \varepsilon$.

Observe that $\mathcal{N} = \oplus_\nu[\oplus_k \mathcal{H}_{\nu k,1}]$ is invariant under R and, with respect to the orthogonal direct sum $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$, R can be written as the operator matrix $R = \begin{bmatrix} N & T_1 \\ 0 & T_2 \end{bmatrix}$. With respect to this decomposition, $T' = \begin{bmatrix} N'' & T_1 \\ 0 & T_2 \end{bmatrix}$, where $N'' = \overset{m}{\underset{j=1}{\oplus}} N_j''$ and $T_2 = \overset{m}{\underset{j=1}{\oplus}} T_{2,j}$, N_j'' is a normal operator uniterily equivalent to the direct sum of β_j copies of M_j such that $\Pi_{\beta_j}(T_{2,j}) = \Pi(T_{2,j})$ is contained in the *interior* of $G_j^- = \Lambda(N_j'')$, $j = 1,2,\ldots,m$.

It is not difficult to check that T' satisfies the conditions (i)-(v).

Finally, consider the case when $\mathrm{ind}(\lambda - T) = 0$ for all complex λ. Then, by Theorem 3, the operators T_o, $T_{\nu k}$ can be chosen so that $\mathrm{ind}(\lambda - T_o) = 0$, $\mathrm{ind}(\lambda - T_{\nu k}) = 0$ for all complex λ, i.e., T_o and $T_{\nu k}$ are bi-quasitriangular operators. Since the class of all bi-quasitriangular operators is invariant under compact perturbations (see [10]) it follows that $\mathrm{ind}(\lambda - T_{\nu k}') = 0$ for all complex λ and, a fortiori, the same result holds for R. Moreover, Theorem 5 shows that $T_{\nu k}'$ can be chosen so tnat $T_{\nu k,2}$ is also bi-quasitriangular and $E(T_{\nu k,2}) = E(N_{\nu k})$. Then, our previous arguments show that T' actually satisfies the condition (vi) too. In fact, $\mathrm{ind}(\lambda - T_2) = 0$ for all λ.

REMARK. If $\dim \mathcal{H} = h \geqslant \aleph_\omega$, then we have the following "converse" to Corollary 7: The set of all operators with infinitely many different spectra is norm dense. The proof can be carried out by using the same arguments as in Corollary 7, and Example 1.

4. PROOF OF THEOREM 1.

Necessity. First of all, observe that the conditions of (i) and (ii) are necessary. In fact, if A is an algebraic operator , then $\operatorname{ind}(\lambda-A) = 0$ for all complex λ. By the stability properties of the Fredholm index, it readily follows that, if $T \in \underset{\sim}{A}(\mathcal{H})^-$, then $\operatorname{ind}(\lambda-T) = 0$ for every λ such that $\lambda-T$ is a semi-Fredholm operator, On the other hand, T must satisfy the following property: The left and the right spectra of $\Pi_\alpha(T)$ in $L(\mathcal{H})/J_\alpha$ coincide, i.e., $\Lambda_\alpha(T) = \Pi_\alpha(T) = \Pi_\alpha(T^*)^*$ for all α, $\aleph_0 \leq \alpha \leq h$. It is clear from the results quoted in the Introduction that this property is equivalent to say that $\delta(\lambda-T) = \delta(\bar{\lambda}-T^*)$ for every λ (These very simple necessary conditions have been proved by several authors; see [2]; [12];[13];[15];[20]). Thus,we conclude that if $T \in \underset{\sim}{A}(\mathcal{H})^-$, then $\operatorname{ind}(\lambda-T) = 0$ for all complex λ.

If, moreover, $T \in \underset{\sim}{N}(\mathcal{H})^-$, then $\Lambda(T)$ and $\Lambda_\alpha(T)$ ($\aleph_0 \leq \alpha \leq h$) must be connected sets containing the origin (see [13], Theorem 3). Hence, the conditions given in Theorem 1 (i) and (ii) are necessary.

Sufficiency for the case (i). Now we are going to prove the sufficiency of those conditions. First we consider the case (i). Observe that many of the arguments used in([15];[23])do not depend on the separability of the underlying Hilbert space. We can say even more: (With the notation of [15]). Let $T \in L(\mathcal{H})$ be an operator such that $\Lambda(T) \subset \Omega$, where Ω is an open set bounded by finitely many pairwise disjoint rectifiable Jordan curves and let M_T be the Rota subspace of T ($M_T \subset R_{\mathcal{H}}{}^2$ and T is similar to $S_{\mathcal{H}}{}^* | (M_T)^\perp$); then $M_T = (z-T^*)R_{\mathcal{H}}{}^2$ and therefore $S_{\mathcal{H}}|M_T$ is actually *similar* to $S_{\mathcal{H}}$ acting on the whole space $R_{\mathcal{H}}{}^2$. This result can be proved by using the same kind of arguments as in [21] or in [23] and it introduces strong simplifications in the results of [15]. Namely, in ([15],Theorem 3), "simply connected" can be replaced by "connected", etc., etc. (There is a second way to prove our results: instead of the generalized universal model given in [14], we can use the equivalent model given in [23]). This shows, in particular, that if $N,B \in L(\mathcal{H})$, where N is a normal operator such that

$\Lambda(N) = \Lambda_h(N) \supset \Lambda(B)$, then $N \oplus B \in \underset{\sim}{A}(\mathcal{H} \oplus \mathcal{H})^-$. If, in addition, it is assumed that $\Lambda(N)$ is a connected set containing the origin, then $N \oplus B$ is actually a norm limit of nilpotent operators (see [2];[15], Theorem 3; [23], Proposition 3.3).

Let $T \in L(\mathcal{H})$ be such that $\mathrm{ind}(\lambda - T) = 0$ for all complex λ and let $T = T_o \oplus \{\oplus_\nu [\oplus_k T_{\nu k}]\}$ be the decomposition given by Theorem 3, such that $\mathrm{ind}(\lambda - T_o) = 0$ and $\mathrm{ind}(\lambda - T_{\nu k}) = 0$ for all ν, for all k in Γ_ν and for all complex λ.

Given $\varepsilon > 0$, let $T' \in L(\mathcal{H})$ be such that $\|T-T'\| < \varepsilon$ and T' satisfies the conditions (i) - (vi) of Corollary 7. It will be enough to prove that $T' \in \underset{\sim}{A}(\mathcal{H})^-$. (Then we can use this result to show that $\mathrm{dist}[T, \underset{\sim}{A}(\mathcal{H})] < \varepsilon$; since $\varepsilon > 0$ can be taken arbitrarily small, it will follow that $T \in \underset{\sim}{A}(\mathcal{H})^-$). Moreover, T' has the form $T_o' \oplus [\overset{m}{\underset{j=1}{\oplus}} T_j]$, where T_o' is either 0 or a bi-quasitriangular operator on a separable Hilbert space and $T_j \in L(\mathcal{H}_j)$, where $\dim \mathcal{H}_j = \beta_j > \aleph_o$ and $\Lambda(T_j) = \Lambda_{\beta_j}(T_j) = \Pi_{\beta_j}(T_j) = \Pi_{\beta_j}(T_j^*)^*$. Therefore, it suffices to show that $T_o', T_1, \ldots, T_m$ are norm limits of algebraic operators in their respective spaces. Now, Proposition 3.3 of [23] takes care of T_o'. Thus, we have reduced our problem to show that:

(i') If $A \in L(\mathcal{H})$, $\dim \mathcal{H} = h > \aleph_o$. $\Lambda(A) = \Lambda_h(A)$ and $\mathrm{ind}(\lambda - A) = 0$ for all complex λ, then $A \in \underset{\sim}{A}(\mathcal{H})^-$.

The proof of (i') follows as in [23]. In fact, by using once again Proposition 3.2 of [23], we can proceed as in the proof of Corollary 7 to show that, given $\varepsilon > 0$, there exists $A' \in L(\mathcal{H})$ such that $\|A-A'\| < \varepsilon$, $\Lambda(A) \subset \Lambda(A') = \Lambda_h(A') \subset (\Lambda(A))_\varepsilon$, $\Lambda(A')$ is the closure of an open set bounded by finitely many pairwise disjoint rectifiable Jordan curves and $\mathcal{H}$ can be decomposed as an orthogonal direct sum of eight subspaces of dimension h, $\mathcal{H} = \overset{8}{\underset{j=1}{\oplus}} \mathcal{H}_j$ with respect to which A' has the form (I), where N_k is normal, $\Lambda(N_k) = \Lambda_h(N_k) = \Lambda(A')$ and $\Lambda(D_k) \subset$ interior $(\Lambda(N_k))$. Therefore

$$A' = \begin{bmatrix} N_1 \oplus D_1 & & & * \\ & N_2 \oplus D_2 & & \\ & & N_3 \oplus D_3 & \\ 0 & & & N_4 \oplus D_4 \end{bmatrix}$$

where the diagonal entries are norm limits of algebraic operators.

It readily follows (see [2];[15];[20];[23]) that $A' \in \underset{\sim}{A}(\mathcal{H})^-$. This proves (i').

By applying (i') to $T_1, T_2, \ldots, T_m$, it follows that $T' \in \underset{\sim}{A}(\mathcal{H})^-$.

Sufficiency for the case (ii). Theorem 1 (i) yields the following result:

COROLLARY 8. *Let* $T \in \underset{\sim}{A}(\mathcal{H})^-$. *Then* T *is the norm limit of operators* T' *of the form* $T' = N \dotplus T''$, *where* N *is normal*, $\Lambda(N) = E(N)$ *and* $\Lambda_\alpha(T'') = \Lambda_\alpha(N) = \Lambda_\alpha(T)$ *for all* α, $\aleph_0 \leqslant \alpha \leqslant h$.

The proof is a formal repetition of the one given for Theorem 5, by using Theorem 1 (i) and Corollary 7 instead of ([23], Proposition 3.2 and 3.3). The details are left to the reader.

Let $T \in \underset{\sim}{A}(\mathcal{H})^-$ and assume that $\Lambda(T)$, $\Lambda_\alpha(T)$ are connected sets containing the origin. As in the proof of the case (i), we can restrict our attention to the case when T has the form $T = T_o' \oplus [\overset{m}{\underset{j=1}{\oplus}} T_j]$, where T_o' acts on a separable reducing subspace and $T_j \in L(\mathcal{H}_j)$, $\dim \mathcal{H}_j = \beta_j > \aleph_o$, and $\Lambda(T_j) = \Lambda_{\beta_j}(T_j) = \Pi_{\beta_j}(T_j) = \Pi_{\beta_j}(T_j^*)^*$, $j = 1, 2, \ldots, m$.

Then it suffices to show that $T_o', T_1, \ldots, T_m$ are norm limits of nilpotents. The result of [2] takes care of T_o' and we have reduced the problem to show that:

(ii') If $A \in L(\mathcal{H})$, $\dim \mathcal{H} = h > \aleph_o$, $\Lambda(A) = \Lambda_h(A)$ is a connected set containing the origin and $\operatorname{ind}(\lambda - A) = 0$ for all complex λ, then A is a norm limit of nilpotents.

Assume that A satisfies (ii'). Then, by Corollary 8, given $\varepsilon > 0$ there exists A' similar to $N \oplus A''$, where N is a normal operator such that $\Lambda(A) = \Lambda(N) = \Lambda_h(N) \supset \Lambda(A'')$. By the previous comments in the proof of sufficiency for the case (i), $N \oplus A'' \in \underset{\sim}{N}(\mathcal{H})^-$, whence it readily follows that $A \in \underset{\sim}{N}(\mathcal{H})^-$.

Applying this result to $T_1, T_2, \ldots, T_m$, it follows that $T \in \underset{\sim}{N}(\mathcal{H})^-$ and the proof of Theorem 1 is complete.

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