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INVERTING A COVARIANCE MATRIX OF TOEPLITZ TYPE BY THE METHOD OF COFACTORS

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SUMMARY. A matrix is said to be of Toeplitz type if it has equal elements along diagonals. These matrices, with the additional property of symmetry, arise frequently in statistical work, as covariance matrices of wide-sense stationary stochastic processes, in nonparametric theory, etc. The inverse is often of interest, and a method is used to find its components when the TxT matrix has only 5 nonvanishing central diagonals. The method is to express the cofactors of the components of the given matrix in terms of some determinants, that are shown to satisfy certain linear difference equations, and to solve these explicitly. The complexity of the resulting expressions for the components of the inverse is comparable to those known in the literature.

1. INTRODUCTION.

A matrix $\underset{\sim}{A} = (a_{ij})$ is called a *Toeplitz matrix* if $a_{ij} = a_{i-j}$.
In mathematical statistics Toeplitz matrices arise in several contexts; see, for example, Grenader and Szegö [3]. They appear as covariance matrices of wide-sense stationary stochastic processes, in which case they are symmetric and positive semidefinite. In the present paper we assume throughout that they are positive definite and symmetric (i.e. are covariance matrices in nonsingular cases), even when some of our results hold for nonsingular symmetric matrices in general.

If $\underset{\sim}{\Sigma} = (\sigma_{ij}) = (\sigma_{|i-j|})$ is the given matrix, the underlying assumptions often imply that the components vanish if $|i-j| > m$, where $m \geq 0$ is an integer. We may call the corresponding processes finitely correlated of order m, and m=0 is the case of lack of correlation. This occurs, for example, in the moving average model of order m, $x_t = v_t + a_1 v_{t-1} + \dots + a_m v_{t-m}$, the v_t being uncorrelated random variables with common means and (finite) variances.

There exists wide interest in finding either exact or approximate forms for the components of the inverses of these Toeplitz matrices, since that knowledge can be used to derive the statistical theory of procedures defined in terms of them. For example, the author's interest in the inverse matrix studied here stems from the analysis of Walker's [12] estimation procedure for the moving average time-series model.See Mentz [6].

For the case of m=1, explicit forms for the components of $\underset{\sim}{\Sigma}_T^{-1}$ obtained by means of the method of cofactors are well known; see, for example, Shaman [8]. It has been conjectured that it would be feasible to extend the procedure to $m > 1$. In the present work we deal with the case of m=2 in detail.

In the case of m=1 several other approaches also solved the problem of finding the components of $\underset{\sim}{\Sigma}_T^{-1}$. For higher values of m the problem proved difficult. Shaman [9] exhibited a close-form expression for the components of the inverse matrix when m=2, and for that case also Mentz [5] has an expression. For general Toeplitz matrices there is a paper by Calderon, Spitzer and Widom [1], but it appears as a hard problem to deduce from their solution an useful explicit form for the components of $\underset{\sim}{\Sigma}_T^{-1}$ for finite T.

A useful notation is $\underset{\sim}{\Sigma} = \sum_{k=0}^{m} \sigma_k \underset{\sim}{G}_k$, where $\underset{\sim}{G}_k$ has components $g_{ij}^{(k)}=1$ for $|i-j| = k$, and otherwise equal to 0. For a solution of a similar problem with different $\underset{\sim}{G}_j$ see Mustafi [7]. In this notation $\underset{\sim}{\Sigma} = \sigma_0 \sum_{k=0}^{m} \rho_k \underset{\sim}{G}_k$, where $\rho_j = \sigma_j/\sigma_0$ ($\sigma_0 > 0$ because $\underset{\sim}{\Sigma}$ is positive definite),and we see that there is no loss of generality in taking the coefficient of $\underset{\sim}{G}_0$ to be equal to 1, as will be done below.

In Section 2 we show that the cofactors of the components of $\underset{\sim}{\Sigma}_T^{-1}$ can be written in terms of some determinants that in turn satisfy linear difference equations that we propose to solve explicitly. Then in Section 3 we use the analysis of Section 2 to derive some close-form and recursive expressions that are comparatively easy to apply analitically and computationally. However, we do not study the computational merits of the proposals as compared with computer rutines prepared for certain Toeplitz matrices; see, for example, Trench [10].

2. THE INVERSE OF $\underset{\sim}{I} + \rho_1 \underset{\sim}{G}_1 + \rho_2 \underset{\sim}{G}_2$ BY EVALUATION OF COFACTORS.

Let $\underset{\sim}{\Sigma}_T = (\sigma_{ij}) = \underset{\sim}{I} + \rho_1 \underset{\sim}{G}_1 + \rho_2 \underset{\sim}{G}_2$, $\rho_2 \neq 0$, and $\underset{\sim}{\Sigma}_T^{-1} = \underset{\sim}{W}_T = (w_{ij}^{(T)})$.

The components of $\underset{\sim}{W}_T$ can be computed from

$$(2.1)\qquad w_{ij}^{(T)} = \frac{\text{cofactor of } \sigma_{ji}}{|\underset{\sim}{\Sigma}_T|}$$

In this section we use the following notation, where a subscript denotes the order of the corresponding matrix or determinant, and we omit the superscripts in the components to simplify the writing. We also use the notation of partitioned matrices:

$$(2.2)\qquad \Sigma_s = |\underset{\sim}{\Sigma}_s|$$

$$(2.3)\qquad \underset{\sim}{L}_s = \begin{bmatrix} \rho_1 & 1 & \rho_1 & \rho_2 & 0 & \cdots & 0 & 0 & 0 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & 0 & 0 & 0 \\ 0 & \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_2 & \rho_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \rho_2 & \rho_1 \end{bmatrix} = \left[\begin{array}{c|c} \begin{matrix} \rho_1 \\ \rho_2 \\ 0 \\ \vdots \\ 0 \end{matrix} & \underset{\sim}{\Sigma}_{s-1} \\ \hline 0 & 0 \;\cdots\; 0 \;\; \rho_2 \;\; \rho_1 \end{array}\right]$$

$$L_s = |\underset{\sim}{L}_s|$$

$$(2.4)\qquad \underset{\sim}{K}_s = \begin{bmatrix} \rho_1 & \rho_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \rho_1 & 1 & \rho_1 & \rho_2 & 0 & \cdots & 0 & 0 & 0 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_1 & 1 & \rho_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_2 & \rho_1 & 1 \end{bmatrix} = \left[\begin{array}{c|c} \rho_1 & \rho_2 \;\; 0 \;\cdots\; 0 \\ \hline \begin{matrix} \rho_1 \\ \rho_2 \\ 0 \\ \vdots \\ 0 \end{matrix} & \underset{\sim}{\Sigma}_{s-1} \end{array}\right]$$

$$K_s = |\underset{\sim}{K}_s|$$

By expanding Σ_T in terms of the components in its first row, Durbin ([2], p. 315) found that the determinants satisfy the linear, homogeneous, fifth-order difference equation

$$(2.5)\qquad -\Sigma_n+(1-\rho_2)\Sigma_{n-1}+(\rho_2-\rho_1^2)\Sigma_{n-2}+\rho_2(\rho_1^2-\rho_2)\Sigma_{n-3}+\rho_2^3(\rho_2-1)\Sigma_{n-4}+ +\rho_2^5\Sigma_{n-5} = 0$$

The associated polynomial equation

$$(2.6)\qquad -z^5+(1-\rho_2)z^4+(\rho_2-\rho_1^2)z^3+\rho_2(\rho_1^2-\rho_2)z^2+\rho_2^3(\rho_2-1)z+\rho_2^5 = 0$$

can be written in a symmetric way using the substitution $-\rho_2 x = z$;

after division by ρ_2^5 we obtain

$$(2.7)\quad x^5 + \frac{1-\rho_2}{\rho_2} x^4 + \frac{\rho_1^2-\rho_2}{\rho_2^2} x^3 + \frac{\rho_1^2-\rho_2}{\rho_2^2} x^2 + \frac{1-\rho_2}{\rho_2} x + 1 = 0$$

Zero is not a root of (2.7), and if x* is a root so is 1/x*. Since there must be five roots, +1 or -1 must be one of them. (They are the only "self inverses".) By inspection we see it is -1. Then

$$(2.8)\quad \begin{aligned} z_1 &= \rho_2 , \quad z_2 = \frac{-\rho_2}{2}(d_1 + \sqrt{d_1^2-4}) , \quad z_3 = \frac{-2\rho_2}{d_1+\sqrt{d_1^2-4}} \\ z_4 &= \frac{-\rho_2}{2}(d_2+\sqrt{d_2^2-4}) , \quad z_5 = \frac{-2\rho_2}{d_2+\sqrt{d_2^2-4}} \end{aligned}$$

where

$$(2.9)\qquad d_1 , d_2 = (2\rho_2)^{-1}[2\rho_2 - 1 \pm \sqrt{(2\rho_2+1)^2-4\rho_1^2}\,]$$

In general the roots (2.8) can be real or complex, and some or all can be identical. Hence the solution of (2.5) will take different forms depending on this fact. As an example, which will be also used as illustration in subsequent derivations, if all roots are distinct then (2.5) has solution

$$(2.10)\qquad \Sigma_n = \sum_{i=1}^{5} C_i z_i^n$$

where the z_i are the roots given in (2.8).

Since Σ_n is defined only for $n \geq 1$, (2.5) holds for $n \geq 6$, and the sequence satisfying the difference equation and for which (2.10) is the general solution is $\Sigma_1, \Sigma_2, \ldots$. The boundary conditions to determine C_i, $i = 1,\ldots,5$, can be taken to be (2.10) for $n = 1,\ldots,5$, with the left-hand sides evaluated explicitly as

$$(2.11)\quad \begin{aligned} \Sigma_1 &= 1 \\ \Sigma_2 &= 1-\rho_1^2 \\ \Sigma_3 &= (1-\rho_2)(1+\rho_2-2\rho_1^2) \\ \Sigma_4 &= \Sigma_3-(\rho_1^2+\rho_2^2)+(\rho_1^4+\rho_2^4)+2\rho_1^2\rho_2-2\rho_1^2\rho_2^2 \\ \Sigma_5 &= \Sigma_4-\rho_1^2\Sigma_3+2\rho_1^2\rho_2(1-\rho_1^2-\rho_2+\rho_2^2)-\rho_2^2(1-\rho_1^2-\rho_2^2) \end{aligned}$$

Following the same approach we expand L_T in terms of the components in its first row and find that

$$L_n - \rho_1 L_{n-1} + \rho_2 L_{n-2} - \rho_1\rho_2^2 L_{n-3} + \rho_2^4 L_{n-4} = 0 \tag{2.12}$$

The polynomial equation is

$$y^4 - \rho_1 y^3 + \rho_2 y^2 - \rho_1\rho_2^2 y + \rho_2^4 = 0 \tag{2.13}$$

and after replacing $\rho_2 x = y$ it becomes

$$\rho_2^4 x^4 - \rho_1\rho_2^3 x^3 + \rho_2^3 x^2 - \rho_1\rho_2^3 x + \rho_2^4 = 0 \tag{2.14}$$

which has symmetric coefficients and can be studied in the same way as equation (2.7). The roots y_s, $s = 1,2,3,4$, of (2.13) are obtained from

$$d_1 = \frac{1}{2\rho_2}(\rho_1+\sqrt{\rho_1^2-4\rho_2+8\rho_2^2}) \quad , \quad d_2 = \frac{1}{2\rho_2}(\rho_1-\sqrt{\rho_1^2-4\rho_2+8\rho_2^2})$$

$$y_1 = \frac{\rho_2}{2}(d_1+\sqrt{d_1^2-4}) \quad , \quad y_2 = \frac{2\rho_2}{d_1+\sqrt{d_1^2-4}} \quad , \tag{2.15}$$

$$y_3 = \frac{\rho_2}{2}(d_2+\sqrt{d_2^2-4}) \quad , \quad y_4 = \frac{2\rho_2}{d_2+\sqrt{d_2^2-4}}$$

The particular case of all roots distinct leads to solving (2.12) by the sequence

$$L_n = \sum_{i=1}^{4} C_i^* y_i^n \qquad n = 1,2,\ldots . \tag{2.16}$$

The four boundary conditions needed to determine the C_i^*'s can be taken to be (2.16) for $n = 1,2,3,4$ with the left-hand sides evaluated explicitly as

$$\begin{aligned} L_1 &= \rho_1 \\ L_2 &= \rho_1^2-\rho_2 \\ L_3 &= \rho_1^3+\rho_2^2\rho_1-2\rho_1\rho_2 \\ L_4 &= \rho_1 L_3-\rho_2[(\rho_1^2-\rho_2)-\rho_2(\rho_1^2-\rho_2^2)] \end{aligned} \tag{2.17}$$

Expanding K_T by the components in its first row we have

$$K_n = \rho_1\Sigma_{n-1}-\rho_2 K_{n-1} \quad , \quad n = 2,3,\ldots . \tag{2.18}$$

In the special case that Σ_n is given by (2.10),

$$K_n + \rho_2 K_{n-1} = \rho_1 \sum_{i=1}^{5} C_i z_i^{n-1} \tag{2.19}$$

which is a first-order, inhomogeneous, linear difference equation. Provided that only one root (for Σ_T) equals ρ_2,

$$K_n = \tilde{C}(-\rho_2)^n + \frac{1}{2}\rho_1 C_1 \frac{1}{\rho_2}\rho_2^n + \rho_1 \sum_{j=2}^{5} C_j \frac{1}{z_j+\rho_2} z_j^n \tag{2.20}$$

The second summand corresponds to the root $z_1 = \rho_2$; no other z_j can be equal to ρ_2 in (2.20); if more than one root equals ρ_2, instead of the factor $1/(z_j+\rho_2)$ we have to use $1/2\rho_2$.
The new constant $\tilde{C}$ in (2.20) will be evaluated from (2.20) for $n=2$, with $K_1 = \rho_1$. Note that

$$\begin{aligned} K_2 &= \rho_1(1-\rho_2) \\ K_3 &= \rho_1(1-\rho_1^2) - \rho_1\rho_2(1-\rho_2) \\ K_4 &= \rho_1\Sigma_3 - \rho_2 K_3 = \rho_1(1-\rho_2)(1+\rho_2-2\rho_1^2) - \rho_2 K_3 \end{aligned} \tag{2.21}$$

With this background we now find expressions for the components w_{ij} of $\underset{\sim}{W}_T = \underset{\sim}{\Sigma}_T^{-1}$. Since $\underset{\sim}{W}_T$ is symmetric we restrict attention to the components on and above the main diagonal.

1st case: $i=j$. Then $w_{ii} = B_{ii}/\Sigma_T$, where B_{ii} is the cofactor of σ_{ii}. In terms of submatrices

$$B_{ii} = \begin{bmatrix} \underset{\sim}{\Sigma}_{i-1} & \rho_2 \underset{\sim}{E}^{*\prime} \\ \rho_2 \underset{\sim}{E}^* & \underset{\sim}{\Sigma}_{T-i} \end{bmatrix} \tag{2.22}$$

where $\underset{\sim}{E}^*$ has its upper right-hand element equal to 1 and all other elements equal to zero. We use Laplace's expansion in terms of minors of the first i-1 columns; then

$$B_{ii} = \Sigma_{i-1}\Sigma_{T-i} - \rho_2^2 \Sigma_{i-2}\Sigma_{T-i-1} \tag{2.23}$$

To make (2.23) valid for all i, we define $\Sigma_0 = 1$, $\Sigma_{-1} = 0$.

2nd case: $i < j$.

$$(-1)^{i+j} B_{ij} = \begin{bmatrix} \underset{\sim}{\Sigma}_{i-1} & \underset{\sim}{F} & \underset{\sim}{0} \\ \rho_2 \underset{\sim}{E}^* & \underset{\sim}{L}_{j-i} & \underset{\sim}{F} \\ \underset{\sim}{0} & \rho_2 \underset{\sim}{E}^* & \underset{\sim}{\Sigma}_{T-j} \end{bmatrix} \tag{2.24}$$

where $\underset{\sim}{F}$ has its lower left-hand element equal to ρ_1, the two adjacent elements equal to ρ_2, and all other elements equal to 0. We expand (2.24) by Laplace's formula in terms of minors of the first i-1 columns. In these columns there are three non-vanishing minors with non-zero complementary minors, namely

(2.25) $$|\underset{\sim}{\Sigma}_{i-1}| = \Sigma_{i-1}$$

(2.26) $$\left|\begin{array}{c|c} & 0 \\ & \vdots \\ \underset{\sim}{\Sigma}_{i-2} & \rho_2 \\ & \rho_1 \\ \hline 0 \;\ldots\ldots\; 0 & \rho_2 \end{array}\right| = \rho_2\Sigma_{i-2}$$

and

(2.27) $$\left|\begin{array}{c|c} & 0 \\ & \vdots \\ \underset{\sim}{K}^*_{i-2} & \rho_2 \\ & 1 \\ \hline 0 \;\ldots\ldots\; 0 & \rho_2 \end{array}\right| = \rho_2 K_{i-2}$$

where $\underset{\sim}{K}^*_s$ is $\underset{\sim}{K}_s$ flipped about its secondary diagonal, so that $|\underset{\sim}{K}^*_s| = K_s$. If we denote by $A_s(i,j)$, s = 1,2,3, the corresponding cofactors, then

(2.28) $$(-1)^{i+j}B_{ij} = \Sigma_{i-1}A_1(i,j)-\rho_2\Sigma_{i-2}A_2(i,j)+\rho_2 K_{i-2}A_3(i,j)$$

The $A_s(i,j)$'s are computed using Laplace's expansion in terms of the last T-j columns. Then

(2.29)
$$A_1(i,j) = \Sigma_{T-j}L_{j-i}-\rho_2 K_{T-j}L_{j-i-1}+\rho_2^3\Sigma_{T-j-1}L_{j-i-2}$$

$$A_2(i,j) = \Sigma_{T-j}(\rho_1 L_{j-i-1}-\rho_2^2 L_{j-i-2})-\rho_2 K_{T-j}(\rho_1 L_{j-i-2}-\rho_2^2 L_{j-i-3})+ +\rho_2^3\Sigma_{T-j-1}(\rho_1 L_{j-i-3}-\rho_2^2 L_{j-i-4})$$

$$A_3(i,j) = \rho_2\Sigma_{T-j}L_{j-i-1}-\rho_2^2 K_{T-j}L_{j-i-2}+\rho_2^4\Sigma_{T-j-1}L_{j-i-3}$$

For j = [T/2]+1,...,T, say, these formulas are valid for all $i < j$, provided we define $\Sigma_0 = L_0 = 1$, $K_0 = 0$, $\Sigma_{-s} = L_{-s} = K_{-s} = 0$ for $s > 0$. For $j < [T/2]+1$ similar arrangements could be made. In fact, due to the structure of $\underset{\sim}{W}_T$ we only need to compute those components of the last [(T+1)/2] columns on and between the principal and secondary diagonals, and then deduce the remaining components using the symmetry of $\underset{\sim}{W}_T$ and its *persymmetry* (symmetry with respect to its secondary diagonal). They lead for example to

$$(2.30) \qquad w_{ij} = w_{T-i+1,T-j+1} \quad , \quad i=j,\ldots,T-j; \quad j=1,2,\ldots,[T/2]$$

We summarize these results as follows:

PROPOSITION 2.1. *Let* $\underset{\sim}{\Sigma}_T = \underset{\sim}{I}+\rho_1\underset{\sim}{G}_1+\rho_2\underset{\sim}{G}_2$, *with* $\rho_2 \neq 0$, *and* $\underset{\sim}{\Sigma}_T^{-1} = \underset{\sim}{W}_T = (w_{ij}^{(T)})$. *Then*

$$(2.31) \qquad w_{ij}^{(T)} = (-1)^{i+j}\,\frac{B_{ij}}{\Sigma_T} \quad , \quad i=j,\ldots,T-j+1; \quad j=[T/2]+1,\ldots,T$$

where the B_{ij} *are given in* (2.23) *when* i=j *and in* (2.28) *and* (2.29) *when* $i < j$, *in terms of the determinants* Σ_s, L_s, K_s, *which are defined in* (2.2)-(2.4) *and satisfy the difference equations* (2.5), (2.12) *and* (2.18), *respectively. The remaining elements of* $\underset{\sim}{W}_T$ *are obtained using* $w_{ij}^{(T)} = w_{ji}^{(T)}$ *and* (2.30).

If $\rho_2 = 0$ but $\rho_1 \neq 0$, then $L_s = \rho_1^s$, $K_s = \rho_1\Sigma_{s-1}$, $(-1)^{i+j}B_{ij} = \Sigma_{i-1}A_1(i,j)$, $A_1(i,j) = \rho_1^{j-i}\Sigma_{T-j}$, and the solution reduces to

$$(2.32) \qquad w_{ij}^{(T)} = (-\rho_1)^{j-i}\,\frac{\Sigma_{i-1}\Sigma_{T-j}}{\Sigma_T}$$

Here Σ_n satisfies the corresponding version of (2.5), namely

$$(2.33) \qquad \Sigma_n - \Sigma_{n-1} + \rho_1^2\,\Sigma_{n-2} = 0$$

with boundary conditions $\Sigma_1 = 1$, $\Sigma_2 = 1 - \rho_1^2$ instead of (2.11).

3. ALTERNATIVE FORMS OF THE COMPONENTS OF THE INVERSE MATRIX.

In this section we want to use the analysis of Section 2 to obtain other forms for the $w_{ij}^{(T)}$ that can be of greater use. We shall find useful the following result:

DEFINITION. *A matrix* $\underset{\sim}{A} = (a_{ij})$ *is said to be "diagonal of type* r" *if* $a_{ij} = 0$ *whenever* $|i-j| > r$.

PROPOSITION 3.1. *Let* $\underset{\sim}{A} = (a_{ij})$ *be a* $T\times T$ *symmetric and positive definite matrix. A necessary and sufficient condition that* $\underset{\sim}{A}^{-1}$ *be diagonal of type* r *is that there exist constants* b_{ts} *such that for* $t = 1,2,\dots,T-r+1$

$$(3.1)\quad a_{tt'} + b_{t1}a_{t+1,t'} + \dots + b_{t,r-1}a_{t+r-1,t'} = 0,\ t'=t+1,\dots,T$$

This result was apparently originated with Guttman [4] and Ukita [11], and a detailed proof is given in Mentz [5].

Condition (3.1) states the existence of a linear relation between succesive sets of r adjacent rows of $\underset{\sim}{A}$; an equivalente formulation (to be used below) is to relate the first r-1 rows to each of the remaining ones.

We now proceed to derive a closed-form expression for the $w_{ij}^{(T)}$. From (2.28) and (2.29) we derive the components in rows 1 and 2 (or columns T and T-1, respectively) of $\underset{\sim}{W}_T = \underset{\sim}{\Sigma}_T^{-1}$, as follows:

$$(3.2)\quad w_{1j}=w_{T-j+1,T} = \frac{(-1)^{j+1}}{\Sigma_T}(\Sigma_{T-j}L_{j-1}-\rho_2K_{T-j}L_{j-2}+\rho_2^3\Sigma_{T-j-1}L_{j-3}),\quad j=2,\dots,T$$

$$(3.3)\quad \begin{aligned} w_{2j}=w_{T-j+1,T-1}=\frac{(-1)^{j+2}}{\Sigma_T}[&\Sigma_{T-j}L_{j-2}-\rho_2(\rho_1\Sigma_{T-j}+K_{T-j})L_{j-3} + \\ &+\rho_2^2(\rho_2\Sigma_{T-j}+\rho_2\Sigma_{T-j-1}+\rho_1K_{T-j})L_{j-4}-\rho_2^4(\rho_1\Sigma_{T-j-1}+ \\ &+K_{T-j})L_{j-5}+\rho_2^6\Sigma_{T-j-1}L_{j-6}]\ ,\quad j=3,\dots,T\end{aligned}$$

Since $\underset{\sim}{\Sigma}_T$ is "diagonal of type 3", it follows that there exist constants θ_{1i} and θ_{2i} such that

$$(3.4)\quad w_{ij} = \theta_{1i}w_{1j} + \theta_{2i}w_{2j}\ ,\quad i=3,\dots,T;\ \ j \geqslant i$$

Using these relations for j = T-1, T we form the systems

$$(3.5)\quad \begin{aligned} w_{i,T-1} &= \theta_{1i}w_{1,T-1} + \theta_{2i}w_{2,T-1} \\ w_{i,T} &= \theta_{1i}w_{1T} + \theta_{2i}w_{2T}\ ,\qquad i = 3,\dots,T\end{aligned}$$

that can be solved for the θ_{1i} and θ_{2i}; replacing the resulting values in (3.4) we obtain the following:

PROPOSITION 3.2. *Under the conditions of Proposition 2.1,*

$$w_{ij}^{(T)} = \frac{(w_{2,T-1}w_{iT}-w_{2T}w_{i,T-1})w_{T-j+1,T}-(w_{1,T-1}w_{iT}-w_{1T}w_{i,T-1})w_{T-j+1,T-1}}{w_{1T}w_{2,T-1} - w_{1,T-1}w_{2T}} =$$

(3.6)

$$= \frac{(w_{2,T-1}w_{1,T-i+1}-w_{1,T-1}w_{2,T-i+1})w_{1j}-(w_{2T}w_{1,T-i+1}-w_{1T}w_{2,T-i+1})w_{2j}}{w_{1T}w_{2,T-1} - w_{2T}w_{1,T-1}} \qquad i \leqslant j$$

where the necessary components are given in (3.2) *and* (3.3).

It is easily checked that (3.6) holds for all $i \leqslant j$. As in the case of (2.31), it suffices to compute w_{ij} for $i = j,\ldots,T-j+1$, $j = [T/2]+1,\ldots,T$.

Expression (3.6) exhibits the components of the inverse matrix as functions of the components in columns T and T-1 (or rows 1 and 2), and in turn, using (3.2) and (3.3), as functions of the roots of (2.6), (2.13) and $-\rho_2$, the latter being the root of the polynomial equation associated with the sequence of K_n's.

While (3.6) may be useful for analytic purposes, the following recursive approach may be simpler, since the $w_{ij}^{(T)}$ will be given as functions of the determinants Σ_s^{ij} and K_s only, not of the L_s.

The components along the diagonals near the main diagonal give rise to some interesting simplifications. In effect from (2.28) we have that

$$\begin{aligned}(-1)B_{i,i+1} &= \Sigma_{i-1}(\Sigma_{T-i-1}L_1-\rho_2K_{T-i-1}L_0)-\rho_2\Sigma_{i-2}\Sigma_{T-i-1}\rho_1L_0 + \\ &+ \rho_2K_{i-2}\rho_2\Sigma_{T-i-1} = \Sigma_{i-1}(\rho_1\Sigma_{T-i-1}-\rho_2K_{T-i-1}) - \\ &- \rho_1\rho_2\Sigma_{i-2}\Sigma_{T-i-1}+\rho_2^2K_{i-2}\Sigma_{T-i-1}\end{aligned} \qquad (3,7)$$

using the explicit values for some L_s's given in (2.17). Similarly

$$\begin{aligned}(-1)^2B_{i,i+2} &= \Sigma_{i-1}[(\rho_1^2-\rho_2)\Sigma_{T-i-2}-\rho_1\rho_2K_{T-i-2}+\rho_2^3\Sigma_{T-i-3}]-\rho_2\Sigma_{i-2} \\ &[(\rho_1^2-\rho_2^2)\Sigma_{T-i-2}-\rho_1\rho_2K_{T-i-2}]+\rho_2K_{i-2}(\rho_1\rho_2\Sigma_{T-i-2}-\rho_2^2K_{T-i-2})\end{aligned} \qquad (3.8)$$

$$\begin{aligned}(-1)B_{i,i+3} &= \Sigma_{i-1}[(\rho_1^3+\rho_1\rho_2^2-2\rho_1\rho_2)\Sigma_{T-i-3}-\rho_2(\rho_1^2-\rho_2)K_{T-i-3} + \\ &+ \rho_1\rho_2^3\Sigma_{T-i-4}]-\rho_2\Sigma_{i-2}[(\rho_1^2-\rho_2-\rho_1\rho_2^2)\Sigma_{T-i-3}-\rho_2(\rho_1^2-\rho_2^2)K_{T-i-3} - \\ &-\rho_1\rho_2^3\Sigma_{T-i-4}]+\rho_2K_{i-2}[\rho_2(\rho_1^2-\rho_2)\Sigma_{T-i-3} -\end{aligned} \qquad (3.9)$$

$$- \rho_1\rho_2^2 K_{T-i-3} + \rho_2^4 \Sigma_{T-i-4}]$$

For the present case of r=3 expression (3.1) reads

$$w_{ij} = -b_{i1}w_{i+1,j} - b_{i2}w_{i+2,j} \quad , \quad i = 1,\ldots,T-2; \quad j \geqslant i \tag{3.10}$$

using this expression for columns i+2 and i+3 we form the systems

$$\begin{aligned} w_{i,i+2} &= -b_{i1}w_{i+1,i+2} - b_{i2}w_{i+2,i+2} \\ w_{i,i+3} &= -b_{i1}w_{i+1,i+3} - b_{i2}w_{i+2,i+3} \quad , \quad i = 1,\ldots,T-3 \end{aligned} \tag{3.11}$$

From these systems we derive b_{i1} and b_{i2}, substitute them back in (3.10) and obtain

$$\begin{aligned} w_{ij} = &\frac{(w_{i+2,i+3}w_{i,i+2} - w_{i+2,i+2}w_{i,i+3})w_{i+1,j}}{w_{i+1,i+2}w_{i+2,i+3} - w_{i+1,i+3}w_{i+2,i+2}} - \\ &- \frac{(w_{i+1,i+3}w_{i,i+2} - w_{i+1,i+2}w_{i,i+3})w_{i+2,j}}{w_{i+1,i+2}w_{i+2,i+3} - w_{i+1,i+3}w_{i+2,i+2}} \end{aligned} \tag{3.12}$$

The components of the inverse matrix are then computed as follows:

PROPOSITION 3.3. *Under the conditions of Proposition 2.1, the components* $w_{ij} = w_{ij}^{(T)}$ *of the inverse matrix* $\underset{\sim}{W}_T = \underset{\sim}{\Sigma}_T^{-1}$ *are determined as follows:*

(a) *Determine* $w_{i,i+s}$ *for* s = 0,1,2,3 *according to*

$$w_{ii} = \frac{1}{\Sigma_T}(\Sigma_{i-1}\Sigma_{T-i} - \rho_2^2\Sigma_{i-2}\Sigma_{T-i-1}) \quad , \quad i = 1,\ldots,T \tag{3.13}$$

$$w_{i,i+1} = \frac{(-1)}{\Sigma_T}[\Sigma_{T-i-1}(\rho_1\Sigma_{i-1} - \rho_1\rho_2\Sigma_{i-2} + \rho_2^2 K_{i-2}) - \rho_2 K_{T-i-1}\Sigma_{i-1}] , \quad i = 2,\ldots,T \tag{3.14}$$

$$\begin{aligned} w_{i,i+2} = &\frac{1}{\Sigma_T}[\Sigma_{T-i-2}\{(\rho_1^2-\rho_2)\Sigma_{i-1} - \rho_2(\rho_1^2-\rho_2^2)\Sigma_{i-2} + \rho_1\rho_2^2 K_{i-2}\} + \\ &+ \rho_2^3\Sigma_{T-i-3}\Sigma_{i-1} + K_{T-i-2}\{-\rho_1\rho_2\Sigma_{i-1} + \rho_1\rho_2^2\Sigma_{i-2} - \rho_2^3 K_{i-2}\}] , \\ &i = 3,\ldots,T \end{aligned} \tag{3.15}$$

$$\begin{aligned} w_{i,i+3} = &\frac{(-1)}{\Sigma_T}[\Sigma_{T-i-3}\{(\rho_1^3 + \rho_1\rho_2^2 - 2\rho_1\rho_2)\Sigma_{i-1} - \rho_2(\rho_1^2 - \rho_2 - \rho_1\rho_2^2)\Sigma_{i-2} + \\ &+ \rho_2^2(\rho_1^2-\rho_2)K_{i-2}\} + \Sigma_{T-i-4}(\rho_1\rho_2^3\Sigma_{i-1} + \rho_1\rho_2^4\Sigma_{i-2} + \rho_2^5 K_{i-2}) + \end{aligned} \tag{3.16}$$

$$+ K_{T-i-3}\{-\rho_2(\rho_1^2-\rho_2)\Sigma_{i-1}+\rho_2^2(\rho_1^2-\rho_2^2)\Sigma_{i-2}-\rho_1\rho_2^3K_{i-2}\}] ,$$

$$i = 4,\ldots,T$$

(b) *For column* j, $[T/2]+1 \leqslant j \leqslant T$, *find in succession* $w_{j-s,j}$ *for* $s = 4,\ldots,2j-T-1$, *using* (3.12).

(c) *Determine the remaining* w_{ij} *using the symmetry and persymmetry of* $\underset{\sim}{W}_T$.

Expression (3.13) is derived from (2.23), and expressions (3.14)-(3.16) from (3.7)-(3.9), respectively.

As was remarked above, Proposition 3.3 solves the problem of specifying the w_{ij} as functions of the determinants Σ_n and K_n of different orders. Also note [for example, see equation (2.19)] that once the Σ_n are available the K_n are easily determined.

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