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UNSTEADY, LAMINAR, INCOMPRESSIBLE FLOW THROUGH DUCTS OF ARBITRARY, DOUBLY CONNECTED CROSS SECTION

Patricio A.A. Laura

ABSTRACT. The present investigation deals with the dynamic response of an incompressible viscous fluid contained in a conduit of arbitrary cross section when subjected to an impulsive pressure gradient. The results are generalized to include any arbitrary variation of the pressure gradient with respect to time.

INTRODUCTION.

The analysis of unsteady flow in conduits of complicated cross section is of interest from both academic and practical viewpoints. Irregular shaped ducts are commonly used in space technology and nuclear engineering since they must be placed in the available space between compactly arranged components and conduits of "exotic" shape are then designed [1].

The unsteady flow of a viscous incompressible fluid in a circular duct under a time-varying pressure gradient has been studied by Mithal [2].

Following a similar approach, Mittal [3] investigated the same problem in the case of an annular duct. The Laplace transform technique was used by both investigators.

The unsteady flow in a duct of rectangular cross-section due to a sinusoidally varying pressure gradient was analysed by Drake [4].

The velocity was obtained in terms of a double Fourier series. Fan and Chao [5] investigated the same configuration but for an arbytrary prescribed pressure gradient. They obtained first the solution for an impulse pressure gradient and then the solution for an arbitrary pressure gradient f(t) was calculated by the convolution integral

$$v_z = \int_0^t f(\eta)\, v_z^* \,(t - \eta).d\eta \qquad (1)$$

where v_z^* is the axial velocity corresponding to an impulse pressure gradient.

In a recent paper [6], Daneshyar has extended Mithal's. A discussion of pertinent references has also been published [7].

Jeng [8] investigated the possibility of extending Fan and Chao's analysis to ducts of arbitrary shape by using the point-matching method. The only shape considered in his interesting paper is the square cross-section where he compared exact results obtained by means of the double Fourier series approach and approximate values calculated by use of the point-matching technique wherein a truncated series of cylindrical harmonics was used. The agreement was excellent in all cases.

Some of the advantages and disadvantages of the point-matching technique have been discussed by several investigators [9]-[12]. The main advantage of the straight point matching technique is its inherent simplicity; the principal objection to the method is its lack of stability for truly complicated shapes [13]-[15].

It is important to point out that the original, straight point-matching method has been improved by several investigators [16]-[17].

Laura and Santamarina [18] have applied the conformal mapping technique to the problem of unsteady, incompressible flows in ducts of non-conventional cross section. Since the governing partial differential equation is not invariant under the transformation, a variational method is used to solve the transformed differential equation.

The same approach has been used by the author to solve other eigenvalue and diffusion - type problems [20]-[24] and the results have been quite satisfactory.

The present study is an extension of the analysis developed in Reference [18] where only simply connected cross sections are considered. In the case of circular-annular cross sections the solution obtained turns out to be the exact solution of the problem [3].

THE MATHEMATICAL MODEL.

The dynamic behavior of an incompressible, viscous flow is governed by the Navier-Stokes' partial differential equations, which in vector form can be expressed as follows:

$$\rho \,.\frac{D\vec{V}}{D\,t} = \vec{F} - \text{grad } p + \mu \,.\nabla^2 \vec{V} \qquad (2)$$

where $\vec{V}$: velocity vector; ρ: density; p: pressure; $\vec{F}$: body force and μ is the absolute viscosity.

Expressing (2) in cylindrical coordinates and assuming [8]:

$$v_r = v_\phi = 0 \quad ; \quad \frac{\partial v_z}{\partial z} = 0 \qquad (3)$$

one obtains in the z - direction:

$$\frac{\partial v_z}{\partial t} - \nu.\nabla^2 v_z = -\frac{1}{\rho}.\frac{\partial p}{\partial z} \qquad (4)$$

where ν is the kinematic viscosity.

Let $-\frac{1}{\rho}.\frac{\partial p}{\partial t} = \delta(t)$ where $\delta(t)$ denotes the Dirac delta function.

The governing differential system is then:

$$\frac{\partial v_z}{\partial t} - \nu\, \nabla^2.v_z = \delta(t) \qquad (5a)$$

$$v_z(r,\phi,t)\Big|_{t=0} = 0 \qquad (5b)$$

$$v_z[L_i(r,\phi)=0,t] = 0 \quad (i=1,2) \qquad (5c)$$

where $L_i(r,\phi) = 0$ denotes the functional relations wich define the boundaries of the doubly connected cross section.

The differential system (5) is equivalent to the more convenient one [8]:

$$\frac{\partial v_z}{\partial t} - \nu.\nabla^2 v_z = 0 \qquad (6a)$$

$$v_z(r,\phi,t)\Big|_{t=0} = 1 \qquad (6b)$$

$$v_z[L_i(r,\phi)=0,t] = 0 \qquad (6c)$$

Since the governing system is linear, once (6b) is solved it is possible to extend the results to an arbitrary variation of the pressure input using Eq.(1).

SOLUTION OF THE DIFFERENTIAL SYSTEM.

Taking

$$v_z = U(r,\phi)\ T(t) \qquad (7)$$

and replacing Eq.(7) in Eq.(6a) results in the following two differential equations:

$$\nabla^2\ U(r,\phi) + \gamma.U(r,\phi) = 0 \qquad (8a)$$

$$\frac{d\ T(t)}{d\ t} + \gamma.\nu.T(t) = 0 \qquad (8b)$$

where γ is the separation constant.

The solution of Eq.(8b) is simply:

$$T(t) = C.e^{-\nu.\gamma.t} \tag{9}$$

Equation (8a) is the well known Helmholtz equation and for convenience is written in complex variable form:

$$4\frac{\partial^2 U}{\partial w.\partial \overline{w}} + \gamma.U = 0 \tag{10}$$

where $w = r.e^{i\phi}$.

Let $w = f(\xi)$ be the analytic function which transforms the given domain in the w-plane onto a circular annulus in the ξ-plane. The governing differential system becomes in the ξ-plane:

$$4\frac{\partial^2 U}{\partial \xi\ \partial \overline{\xi}} + \gamma.\left|\frac{dw}{d\xi}\right|^2.U = 0 \tag{11a}$$

$$U(r_1,\phi_1)\Big|_{r_1 = r_{1,i}\ ;\ r_{1,e}} = 0 \tag{11b}$$

where $\xi = r_1 e^{i\phi_1}$ (see Figure 1).

The transformed boundary configuration is quite simple now but it becomes evident the fact that it will be necessary to use an approximate method to solve the differential equation.

Let the solution of (11a) be expressed in terms of a double infinite sum of cylindrical harmonics:

$$U(r_1,\phi_1) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}[J_n(\eta_{nm}r_1)\ Y_n(\eta_{nm}r_{1,e}) - J_n(\eta_{nm}r_{1,e})\ Y_n(\eta_{nm}r_1)]\cos n\,\phi_1 \tag{12}$$

where J_n and Y_n are the Bessel functions of first and second kind respectively. The η_{nm}'s are the roots of the secular determinant

$$\begin{vmatrix} J_n(\eta_{nm}.r_{1,i}) & Y_n(\eta_{nm}.r_{1,i}) \\ J_n(\eta_{nm}.r_{1,e}) & Y_n(\eta_{nm}.r_{1,e}) \end{vmatrix} = 0 \tag{13}$$

Since only an approximate solution is sought it will be convenient to simplify Eq.12 even further. It is obvious that only a finite number of terms will be taken, as is usual with most of the techniques that follow a weighted residuals approach. The second approximation assumes that $U(r_1,\phi_1)$ is practically independent of the ϕ_1 coordinate (this is an actual fact for $r_1 = r_{1,i}$ and $r_1 = r_{1,e}$).The accuracy of this approximation has been previously demostrated in the technical literature [20]-[22]. The solution of the transformed Helmholtz equation becomes then

$$U(r_1,\phi_1) \simeq U(r_1) = \sum_{m=0}^{M} A_{om}[\,J_o(\eta_{om}.r_1)\;Y_o(\eta_{om}.r_{1,e}) - \qquad (14)$$
$$- J_o(\eta_{om}.r_{1,e})\;Y_o(\eta_{om}.r_1)]$$

Replacing Eq.(14) in Eq.(11a) results in an error or residual function $\varepsilon(r_1,\phi_1)$ which must be minimized by means of an appropriate criterium: Galerkin's method, least squares,etc. By using the corresponding mathematical condition one obtains a homogeneous system of $(M + 1)$ equations in the A_{om}'s.

For a nontrivial solution, the determinant of the coefficients of the unknowns must vanish and the eigenvalues γ_{om}'s can then be evaluated. The A_m coefficients are now found from the initial condition (6b) making use of the Fourier - Bessel expansion

$$1 \simeq \sum_{m=0}^{M} A_{om}[\,J_o(\eta_{om}.r_1)Y_o(\eta_{om}.r_{1,e})-J_o(\eta_{om}.r_{1,e})Y_o(\eta_{om}.r_1)] \qquad (15a)$$

where the A_{om}'s are given by [23]

$$A_{om} = \pi . \frac{J_o(\eta_{om}.r_{1,i})}{J_o(\eta_{om}.r_{1,i}) + J_o(\eta_{om}.r_{1,e})} \qquad (15b)$$

The following approximate expression for the unsteady velocity profile in the ξ-plane then results:

$$v_z \simeq \sum_{m=0}^{M} A_{om}[\,J_o(\eta_{om}.r_1)\;Y_o(\eta_{om}.r_{1,e}) - \qquad (16)$$
$$- J_o(\eta_{om}.r_{1,e})\;Y_o(\eta_{om}.r_1)]\,e^{-\nu.\gamma_{om}.t}$$

It is important to point out that for practical engineering calculations only a few terms of Eq.(16) are needed.

APPLICATIONS.

Consider now the doubly connected cross sections shown in Figure 1. A complete discussion regarding the determination of the mapping functions of such doubly connected cross sections is available elsewhere [24] and will not be repeated here. The eigenvalues γ_{om} are also taken from Ref.24 where Galerkin's method was used and their values are given in Table 1.

It is interesting to study the response of the fluid when f(t) is equal to a step function, e.g.

$$\frac{1}{\rho} . \frac{\partial p}{\partial z} = K_o.U(t) \qquad (17)$$

where U(t) is the unit step function. From Eqs.(1),(16) and (17) one obtains:

$$v_z \Big/ K_o . a^2/\nu \simeq \sum_{m=o}^{M} \frac{A_{om}}{\gamma'_{om}} [J_o(\eta_{om}.r_1)\ Y_o(\eta_{om}.r_{1,e}) - - J_o(\eta_{om}.r_{1,e}) Y_o(\eta_{om}.r_1)] (1-e^{-\gamma'_{om}.\tau}) \tag{18}$$

where $\tau = \nu.t/_{a}2$; $\gamma'_{om} = a^2.\gamma_{om}$ and a^2 is the apothem of the polygon. Figures 2 through 4 show the velocity profile in the ξ-plane for the configurations of Figure 1 and for different values of τ.

One can immediately observe from the graphs that the value of the ratio $r_{1,i}/r_{1,e}$ has a very distinct effect on the shape of the profile. One of the most important features of the approach presented in this paper consists in the determination of unified solutions for any type of doubly connected cross sections. For instance, the approximate solutions, Eqs.(16) and (18) are valid regardless the shape, only the eigenvalues are different for each individual configuration.

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	Geometry	$r_{1,i}/r_{1,e}$	R/a	$\gamma'_{01}=(\gamma_{01}.a^2)$	$\gamma'_{02}=(\gamma_{02}.a^2)$
CASE I	Fig.1(a)	0.10	0.108	9.3697	43.0952
CASE II	Fig.1(b)	0.10	0.105	9.9288	46.4878
CASE III	Fig.1(a)	0.40	0.432	22.4676	94.0124
CASE IV	Fig.1(b)	0.40	0.42	24.0786	102.3132

TABLE I

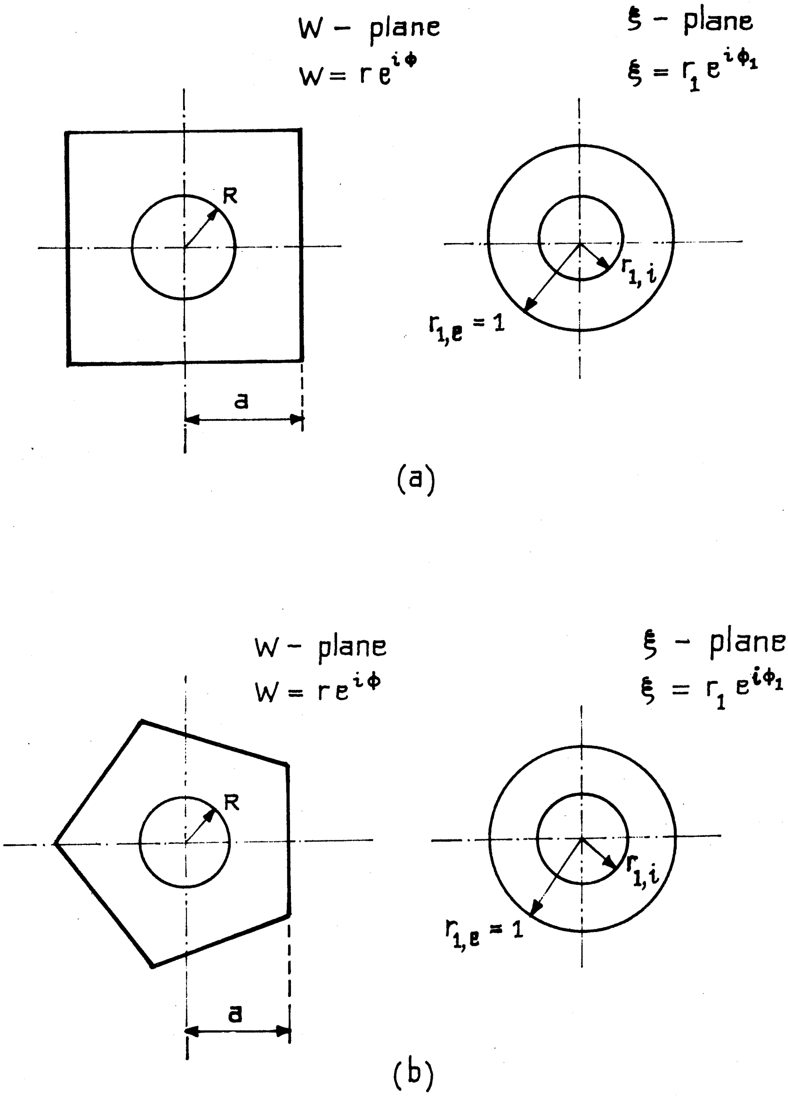

FIGURE 1 - Doubly Connected Cross Sections.-

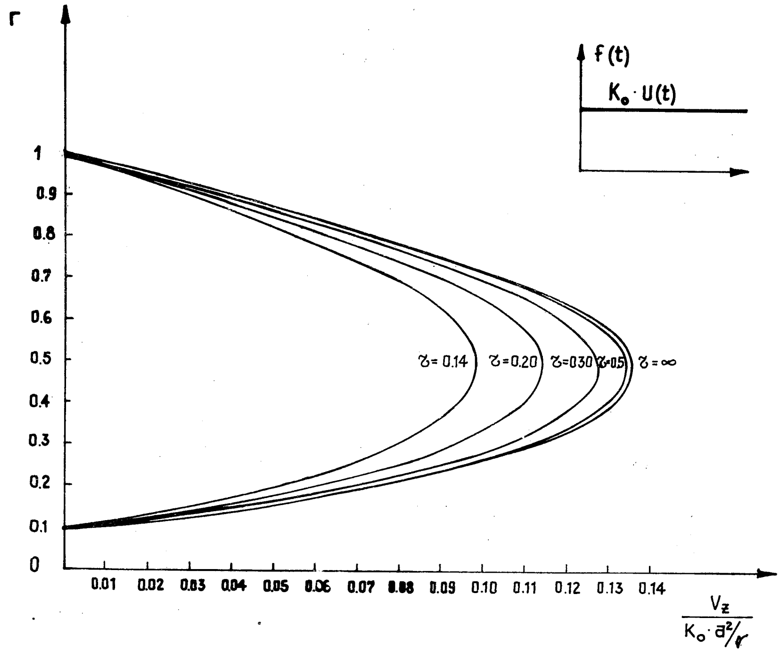

FIGURE 2 – Unsteady Flow in Duct of Doubly Connected Cross Section (CASE I)

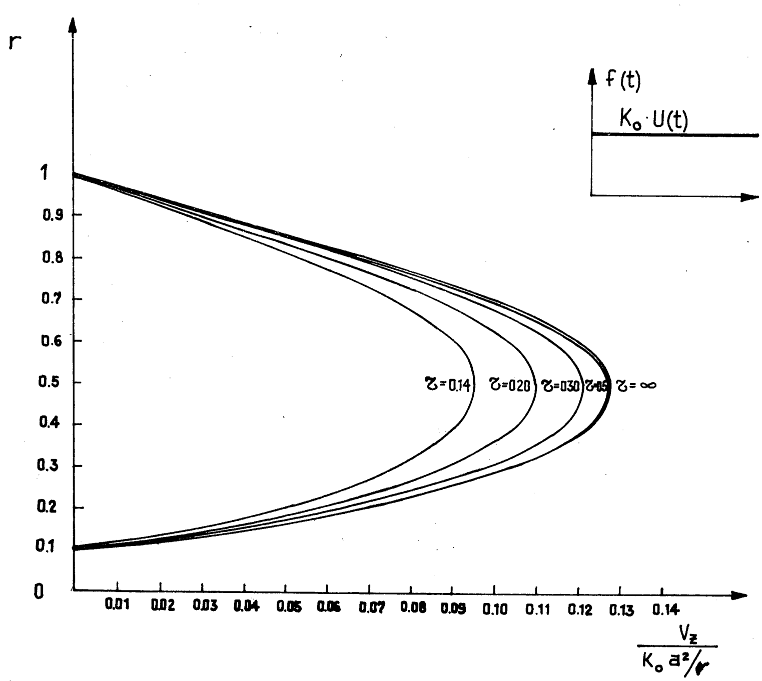

FIGURE 3 – Unsteady Flow in Duct of Doubly Connected Cross Section (CASE II)

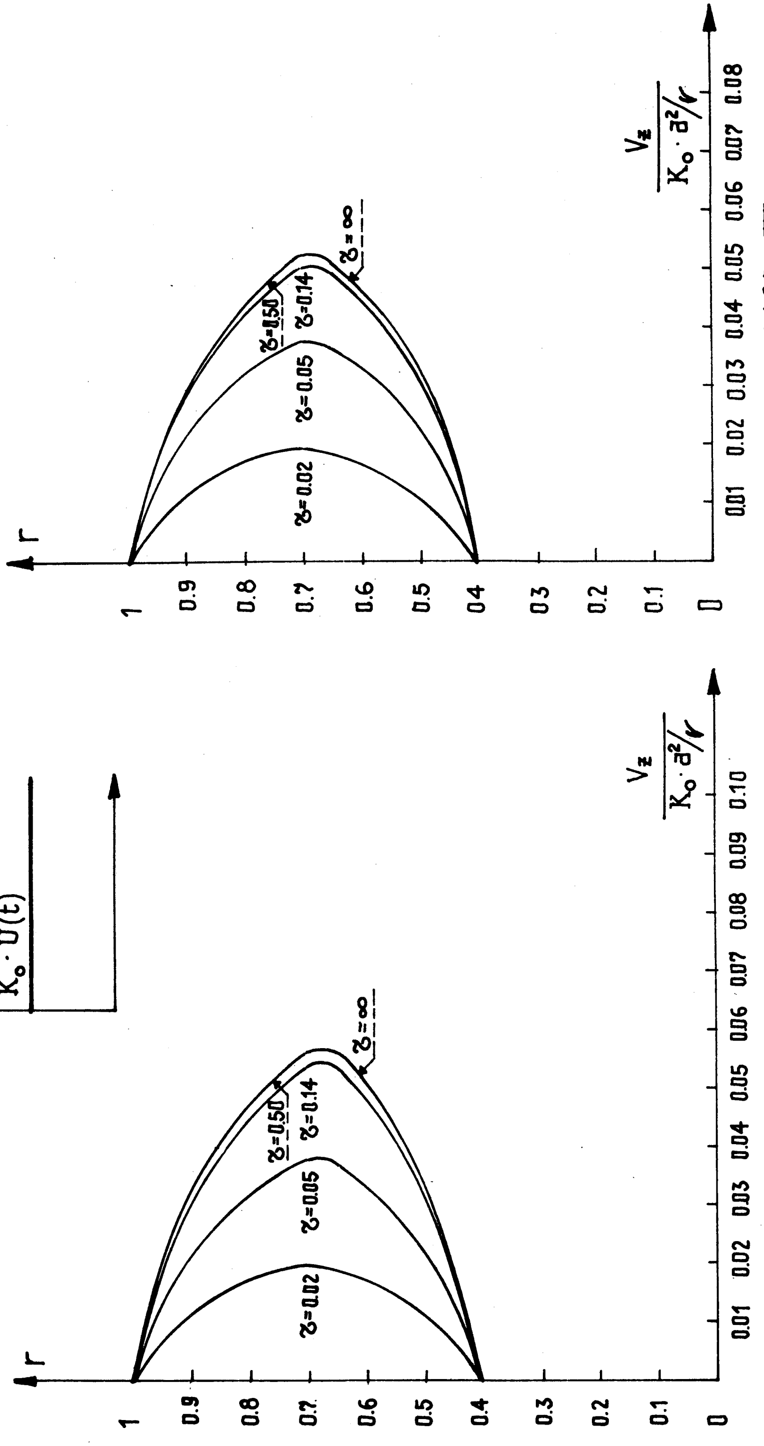

FIGURE 4 - Unsteady Flow in Conduits of Doubly Connected Cross Section.-

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