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FREE SYMMETRIC BOOLEAN ALGEBRAS

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Our purpose is to give a construction of the symmetric Boolean algebras with a finite set of free generators, different from that given by A. Monteiro ([5]).

1. INTRODUCTION.

We shall begin recalling some notions and results of the theory of symmetric Boolean algebras.

DEFINITION 1.1. A *symmetric Boolean algebra is a pair* (A,T), *where* A *is a Boolean algebra, and* T *is an automorphism of* A *of period two, that is, such that* $TTx = x$, *for all* $x \in A$. (Gr. C. Moisil, [2],[3]).

Briefly, we shall say that A is a symmetric algebra. A.Monteiro has independently studied these algebras under the name of "algèbres de Boole involutives" ([4],[5]).

DEFINITION 1.2. *A part* S *of a symmetric algebra* A *is said a symmetric subalgebra of* A *if:*

S1) $S \neq \emptyset$

S2) S *is closed under* $\wedge$, $\vee$, $-$ *and* T

It is clear that a symmetric subalgebra of A is a Boolean subalgebra of A.

Let G be a part of a symmetric algebra A; we shall represent by B(G) (respectively S(G)) the smallest Boolean (respectively symmetric) subalgebra of A containing G. B(G) (respectively S(G)) is called the Boolean (respectively symmetric) subalgebra generated by G. It is clear that $B(G) \subseteq S(G)$.

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We shall represent by $I = I(A)$ the set of all elements x of A such that $Tx = x$. It is clear that $I(A)$ is a Boolean subalgebra of A.

DEFINITION 1.3. *Let* A *and* A' *be symmetric algebras. A symmetric homomorphism from* A *into* A' *is a function* h *from* A *into* A' *such that:*

H1) $h(x \vee y) = h(x) \vee h(y)$

H2) $h(-x) = -h(x)$

H3) $h(Tx) = Th(x)$

for all x *and* $y \in A$.

If h is a surjective function, we say that A' is a homomorphic image of A. If h is a bijective function, we say that A is isomorphic to A', and we shall note $A \cong A'$.

From conditions H1) and H2) follows that h is a Boolean homomorphism. Therefore, a symmetric homomorphism is a Boolean homomorphism which verifies condition H3).

The kernel of a symmetric homomorphism h from A into A', that is, the set $\mathrm{Ker}\, h = h^{-1}(1)$, $1 \in A'$, is a filter which verifies the condition:

D) *If* $x \in \mathrm{Ker} h$ *then* $Tx \in \mathrm{Ker} h$.

DEFINITION 1.4. *A filter of a symmetric algebra* A *which verifies condition* D) *will be called a deductive system or a* T-*filter.*

LEMMA 1.5. *A principal filter* $F(x)$ *of* A *is a* T-*filter if and only if* $x \in I(A)$.

Proof. Necessary condition: as $x \in F(x)$ and $F(x)$ is a T-filter, then $Tx \in F(x)$, that is, $x \leq Tx$; hence $Tx \leq TTx = x$. Therefore $Tx = x$, that is, $x \in I(A)$.

Sufficient condition: Let us suppose $Tx = x$, and let y be such that $y \in F(x)$, that is, $x \leq y$; then $x = Tx \leq Ty$. Hence $Ty \in F(x)$.

If F is a T-filter of a symmetric algebra A, and we define: $a \equiv b \pmod F$ iff $(-a \vee b) \wedge (-b \vee a) \in F$, for all $a,b \in A$, then "$\equiv$" is a congruence relation on the algebra A. ([1]). Let us represent by $A' = A/\equiv$ or $A' = A/F$ the quotient set of A by the equivalence relation "$\equiv$", and let us note by $|x|$ the equivalence class containing the element $x \in A$. If we define:

$$|x| \wedge |y| = |x \wedge y| \quad ; \quad -|x| = |-x| \quad ; \quad T|x| = |Tx|$$

it is easy to prove that A' is a symmetric algebra.

The application h from A into A' defined by $h(x) = |x|$, for all $x \in A$, is a symmetric homomorphism from A onto A', that is, A' is a homomorphic image of A. h is called the natural homomorphism.

It is easy to prove that if A' is a homomorphic image of A, then there is a T-filter F of A, such that $A/F \cong A'$.

Therefore, we obtain all the homomorphic images doing the quotiens A/F , where F is a T-filter of A.

2. SIMPLE SYMMETRIC ALGEBRAS.

A symmetric algebra is called trivial if it has only one element.

DEFINITION 2.1. *A symmetric algebra is called simple if:*

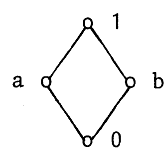
1) A *is non trivial.*
2) *All the homomorphic images of* A *are either trivial or isomorphic to* A.

A. Monteiro ([5]) proved that the only simple algebras are those whose Hasse diagrams and the corresponding automorphisms are shown in the next figure:

B_1

x	Tx
0	0
1	1

B_2



x	Tx
0	0
a	b
b	a
1	1

We shall next give another proof of this result. It is not difficult to prove the following theorem.

THEOREM 2.2. *In order that a non trivial symmetric algebra* A *be simple it is necessary and sufficient that* $I(A) = \{0,1\}$.

This result can be stated as follows: "In order that a symmetric algebra A be simple it is necessary and sufficient that the Boolean algebra I(A) be simple". Then, it is clear that the algebras B_1 and B_2 shown in the above figure are simple algebras. We shall now prove that they are the only simple symmetric algebras.

LEMMA 2.3. *Let* A *be a simple symmetric algebra with more than two elements. If* $x \in A - I(A)$, *then* $Tx = -x$.

Proof. Consider $y = x \wedge Tx$; then $y \in I(A)$. If $y=1$, then $x=1$ which is a contradiction. Therefore (1) $x \wedge Tx = 0$. Consider $z = x \vee Tx$; then $z \in I(A)$. If $z=0$, then $x=0$, which is a contradiction. Therefore (2) $x \vee Tx = 1$.
From (1) and (2) follows $Tx = -x$.

LEMMA 2.4. *Let* A *be a simple symmetric algebra with more than two elements. If* $x_1, x_2 \in A - I(A)$ *and* $x_1 \neq x_2$, *then* $x_1 = -x_2$.

Proof. By hypothesis, $x_1 \wedge x_2 \neq 1$. If $x_1 \wedge x_2 \neq 0$, then it follows from 2.3 $T(x_1 \wedge x_2) = -(x_1 \wedge x_2)$, that is, $Tx_1 \wedge Tx_2 = -x_1 \vee -x_2$. Besides $Tx_1 = -x_1$, $Tx_2 = -x_2$; hence $-x_1 \wedge -x_2 = -x_1 \vee -x_2$, and then $-x_1 = -x_2$, that is, $x_1 = x_2$, which is a contradiction. Therefore $x_1 \wedge x_2 = 0$. It can be likewise proved that $x_1 \vee x_2 = 1$.

It immediately follows from lemmas 2.3 and 2.4 that if A is a simple symmetric algebra which contains an element different from 0 and 1, then it contains exactly four elements $0,a,b,1$, with $a = -b$ and $T0 = 0$, $Ta = b$, $Tb = a$, $T1 = 1$.

3. T-FILTERS OF A SYMMETRIC ALGEBRA A AND FILTERS OF I(A).

Let $\mathcal{T} = \mathcal{T}(A)$ be the set of all the T-filters of a symmetric algebra A, and $\mathcal{F} = \mathcal{F}(I)$ the set of all the filters of the Boolean algebra I(A). It is clear that $\mathcal{T}$ and $\mathcal{F}$ are ordered sets if we order both by inclusion.

LEMMA 3.1. *The transformation* $\varphi: \mathcal{T} \longrightarrow \mathcal{F}$ *such that* $\varphi(D) = D \cap I(A)$, $D \in \mathcal{T}$, *is an order isomorphism.*

Proof. It is clear that if $D \in \mathcal{T}$, then $D \cap I(A) \in \mathcal{F}$. Given $F \in \mathcal{F}$, let $D = F_A(F)$ be the filter in A generated by the filter F of I(A). Let us prove that D is a T-filter. D is a filter by construction. If $x \in D$, then $Tx \in D$. It is well known that $F_A(F) = \{x \in A:$ there is $f \in F$ such that $f \leqslant x\}$. Hence, if $x \in A$, there is $f \in F$ such that $f \leqslant x$. Then $Tf \leqslant Tx$, and since $f \in I(A)$, $Tf = f$. Therefore, $f \leqslant Tx$, that is, $Tx \in D = F_A(F)$.

On the other hand, $\varphi(F_A(F)) = F_A(F) \cap I(A) = F$, that proves that φ is a surjective function.

It is clear that if $D_1, D_2 \in \mathcal{T}$ and $D_1 \subseteq D_2$, then $\varphi(D_1) \subseteq \varphi(D_2)$. Let us prove, if $D_1, D_2 \in \mathcal{T}$ and $\varphi(D_1) \subseteq \varphi(D_2)$, then $D_1 \subseteq D_2$.

Indeed, by hypothesis, $D_1 \cap I(A) \subseteq D_2 \cap I(A)$. Let x be an element of D_1, then $Tx \in D_1$ and $x \wedge Tx \in D_1$; moreover $x \wedge Tx \in I(A)$, then $x \wedge Tx \in D_1 \cap I(A)$. But $D_1 \cap I(A) \subseteq D_2 \cap I(A)$. Therefore $x \wedge Tx \in D_2 \cap I(A)$. In particular $x \wedge Tx \in D_2$ and then $x \in D_2$.

LEMMA 3.2. *If* $D \in \mathcal{T}$, $I(A/D) \cong I(A)/D \cap I(A)$.

Proof. Let us consider the natural homomorphism $h: A \longrightarrow A/D$ and h^* the restriction of h to I(A). It is clear that h^* is a Boolean homomorphism from I(A) into I(A/D), with kernel $I(A) \cap D$. Given

$y \in I(A/D)$, there is $x \in A$ such that $h(x) = y$. Then $h(x \wedge Tx) = h(x) \wedge Th(x) = y \wedge Ty = y$. But $x \wedge Tx \in I(A)$, hence $h^*(x \wedge Tx) = h(x \wedge Tx) = y$, that is, h^* is an epimorphism from $I(A)$ onto $I(A/D)$. Therefore $I(A/D) \cong I(A)/D \cap I(A)$.

DEFINITION 3.3. *A* T-*filter* D *of a symmetric algebra* A *is said a maximal* T-*filter if*

1) D *is a proper* T-*filter*.

2) *If* D' *is a* T-*filter such that* $D \subseteq D'$ *then* $D' = A$ *or* $D' = D$.

THEOREM 3.4. *If* D *is a maximal* T-*filter of a symmetric algebra* A, *then* A/D *is a simple symmetric algebra*.

Proof. If D is a maximal T-filter, then it follows from Lemma 1.1 that $D \cap I(A)$ is an ultrafilter (a maximal filter) of the Boolean algebra $I(A)$. Then $I(A)/D \cap I(A)$ is a simple Boolean algebra, hence, by Lemma 3.2, $I(A/D)$ is a simple Boolean algebra, that is, $I(A/D) = \{0,1\}$. Therefore, it follows from Theorem 2.2 that A/D is a simple symmetric algebra.

4. REPRESENTATION THEOREM.

Given a family $\{A_i\}_{i\in I}$ of symmetric algebras, the cartesian product $P = \prod_{i\in I} A_i$ is defined in the usual way.

Given a symmetric non trivial algebra A, let $M = \{M_i\}_{i\in I}$ be the family of all the maximal T-filters of A. A. Monteiro ([5]) proved that A is isomorphic to a subalgebra A* of the cartesian product $P = \prod_{i\in I} A/M_i$. The isomorphism is defined in the following way: let m_i be the natural homomorphism from A onto A/M_i. Then, if $f \in A$, $\varphi(f) = (m_i(f))_{i\in I} \in P$. The subalgebra A* of P is $\varphi(A)$. Moreover, if A is finite, then A is isomorphic to P.

5. FINITELY GENERATED SYMMETRIC BOOLEAN ALGEBRAS.

We shall prove that if a symmetric algebra A has a finite set of generators, then A is finite, that is, if G is a finite subset of A with n elements ($N(G) = n$) such that $S(G) = A$, then A is finite.

We know that A is isomorphic to a subalgebra A* of the symmetric algebra $P = \prod_{i\in I} A/M_i$, where $M = \{M_i\}_{i\in I}$ is the set of all the maximal T-filters of A. Moreover we know that the quotiens A/M,

$M \in \mathcal{M}$ are finite, more precisely, $N(A/M) = 2$ or $N(A/M) = 4$, because $A/M \cong B_1$ or $A/M \cong B_2$. It is sufficient then to prove that $\mathcal{M}$ is finite.

Let us consider
$$\mathcal{M}_1 = \{M \in \mathcal{M}: A/M \cong B_1\}$$
$$\mathcal{M}_2 = \{M \in \mathcal{M}: A/M \cong B_2\}$$

It is clear that $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$ and $\mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}$.

Let us note $Epi(A,B_1)$ the set of all the epimorphisms from A onto B_1, $F(G,B_1)$ the set of all the functions from G into B_1. We shall prove that:

I) $N(\mathcal{M}_1) = N(Epi(A,B_1)) \leqslant N(F(G,B_1)) = 2^n$

Consider the function $s: Epi(A,B_1) \longrightarrow \mathcal{M}_1$ defined by $s(h) = Ker\ h$, where $h \in Epi(A,B_1)$. It is clear that $Ker\ h \in \mathcal{M}_1$. Let be $M \in \mathcal{M}_1$; then $A/M \cong B_1$; if $\sigma_M: A/M \longrightarrow B_1$ is the isomorphism and $h_M: A \longrightarrow A/M$ is the natural homomorphism, then $h = \sigma_M . h_M$ is an epimorphism from A onto B_1, whose kernel is M, that is, $s(h) = M$. Hence s is a surjective function.

Let be $h_1, h_2 \in Epi(A,B_1)$, $M_1 = Ker\ h_1$, $M_2 = Ker\ h_2$, and suppose $M_1 = Ker\ h_1 = Ker\ h_2 = M_2$. Let be $x \in A$. If $x \in M_1 = M_2$, then $h_1(x) = h_2(x) = 1$; if $x \notin M_1 = M_2$, then $h_1(x) = h_2(x) = 0$. Hence $h_1 = h_2$, that is, s is an injective function. Hence, $N(\mathcal{M}_1) = N(Epi(A,B_1))$.

Consider now $r: Epi(A,B_1) \longrightarrow F(G,B_1)$ the application which maps each epimorphism $h: A \longrightarrow B_1$ into its restriction to G: $f = h/G$.

This is an injective application, because if $h/G = h'/G$, then $\{x \in A: h(x) = h'(x)\}$ is a symmetric subalgebra of A which contains G, and therefore $h = h'$.
Therefore $N(Epi(A,B_1)) \leqslant N(F(G,B_1)) = 2^n$.

We shall now prove that:

$$\text{II)} \quad N(\mathcal{M}_2) = \frac{N(Epi(A,B_2))}{N(Aut(B_2))} = \frac{N(Epi(A,B_2))}{2} \leqslant \frac{N(F^*(G,B_2))}{2} \leqslant$$
$$\leqslant \frac{N(F(G,B_2))}{2} = \frac{4^n}{2}$$

where $Epi(A,B_2)$ is the set of all the epimorphisms from A onto B_2, $Aut(B_2)$ is the set of all the automorphisms of B_2, $F^*(G,B_2)$ the set of all the functions f from G into B_2 such that $S(F(G)) = B_2$, and $F(G,B_2)$ the set of all the functions from G into B_2.

Consider $s: Epi(A,B_2) \longrightarrow \mathcal{M}_2$ the mapping defined by $s(h) = Ker\ h$, $h \in Epi(A,B_2)$. It can be proved as in I) that s is a surjective function. If $s(h) = M$, then it is easy to see that $s^{-1}(M) = \{\alpha \circ h: \alpha \in Aut(B_2)\}$. But there are only two automorphisms in B_2: the automorphism $Tx = x$, for all $x \in B_2$, and the automorphism T ($T0 = 0$, $T1 = 1$, $Ta = b$, $Tb = a$).

Then, for all $M \in M_2$, $s^{-1}(M)$ has exactly two elements. Hence

$$N(M_2) = \frac{N(Epi(A,B_2))}{N(Aut(B_2))} = \frac{N(Epi(A,B_2))}{2}$$

Consider now the mapping $r: Epi(A,B_2) \longrightarrow F^*(G,B_2)$, which maps each epimorphism $h: A \longrightarrow B_2$ into its restriction to G: $f = h/G$. As h is an epimorphism, then $S(h(G)) = B_2$, hence, $S(f(G)) = B_2$, that is, $f \in F^*(G,B_2)$.

It can be proved as in I) that r is an injective function. Then $N(Epi(A,B_2)) \leqslant N(F^*(G,B_2))$. It is clear that $N(F^*(G,B_2) \leqslant$ $\leqslant N(F(G,B_2)) = 4^n$. Then we have:

$$N(M_2) = \frac{N(Epi(A,B_2))}{N(Aut(B_2))} = \frac{N(Epi(A,B_2))}{2} \leqslant \frac{N(F^*(G,B_2))}{2} \leqslant$$

$$\leqslant \frac{N(F(G,B_2))}{2} = \frac{4^n}{2} < \infty$$

From I) and II) it follows that $M = M_1 \cup M_2$ is finite, and then A is finite, that is:

THEOREM 5.1. *Every finitely generated symmetric algebra is finite.*

It follows from the above, that if A is a finitely generated symmetric algebra, then $A \cong B_1^{N(M_1)} \times B_2^{N(M_2)}$.

6. SYMMETRIC ALGEBRAS WITH A FINITE SET OF FREE GENERATORS.

DEFINITION 6.1. *Given a cardinal number* $c > 0$, *we shall say that* $\mathcal{L}$ *is a symmetric algebra with* c *free generators if:*

L1) *There is a subset* G *of* $\mathcal{L}$, *of power* c, *such that* $S(G) = \mathcal{L}$.

L2) *Given a symmetric algebra* A *and an application* f *from* G *into* A, *there is a homomorphism* $\overline{f}$, *necessarily unique, from* $\mathcal{L}$ *into* A *such that* $\overline{f}$ *is an extension of* f.

If it is so, we shall say that G is a set of free generators of $\mathcal{L}$. A symmetric algebra is said to be free if it has a set of free generators. We shall note $\mathcal{L} = L(c)$. Since the symmetric algebras are defined by equations, we can state, by a theorem of universal algebra of G.Birkhoff ([1]), the existence and uniqueness, up to isomorphisms, of L(c).

In view of the preceding paragraph, we can state that L(n) is finite, for every natural number $n > 0$. Furthermore, $L(n) \cong B_1^{N(M_1)} \times B_2^{N(M_2)}$. We shall now compute $N(M_1)$ and $N(M_2)$.

LEMMA 6.2. *Let* G *be a set of free generators of* L(n), *and* B(n) = = B(G). *Then* G *is a set of free generators of the Boolean algebra* B(n).

Proof. We must prove that if A is a Boolean algebra, and f is an application from G into A, then f can be extended to a Boolean homomorphism from B(n) into A. Indeed, let A be a Boolean algebra and $f: G \longrightarrow A$. Consider the transformation $T: A \longrightarrow A$ defined by $Tx = x$ for all $x \in A$. Then (A,T) is a symmetric algebra. Then f can be extended to a symmetric homomorphism $\overline{h}: L(n) \longrightarrow A$. Consider $h = \overline{h}/B(n)$. It is clear that h is a Boolean homomorphism from B(n) into A and $h(g) = \overline{h}(g) = f(g)$, which proves that G is a set of free generators of the Boolean algebra B(n).

COMPUTATION OF $N(M_1)$. We know by paragraph 5, I) that

$N(M_1) = N(\mathrm{Epi}(L(n),B_1)) \leqslant N(F(G,B_1)) = 2^n$.

We now prove that $N(\mathrm{Epi}(L(n),B_1)) = N(F(G,B_1))$. The function $r: \mathrm{Epi}(L(n),B_1) \longrightarrow F(G,B_1)$ which maps each epimorphism $h: L(n) \longrightarrow B_1$ into its restriction to G, is injective. Let us see that it is surjective. If $f \in F(G,B_1)$, it is clear that $S(f(G) = = B_1$. Since L(n) is free, f can be extended to a homomorphism $\overline{f}: L(n) \longrightarrow B_1$; $\overline{f}$ is an epimorphism because $B_1 = S(f(G)) = = S(\overline{f}(G)) \subseteq S(\overline{f}(L(n))) = \overline{f}(L(n))$, that is, $\overline{f}(L(n)) = B_1$. Moreover, $r(\overline{f}) = \overline{f}/G = f$. Therefore, $N(M_1) = 2^n$.

COMPUTATION OF $N(M_2)$.

LEMMA 6.3. *If* $X \subset B_2$ *and* $S(X) = B_2$, *then* $B(X) = B_2$.

The application $r: \mathrm{Epi}(L(n),B_2) \longrightarrow F^*(G,B_2)$ such that $r(h) = h/G$, $h \in \mathrm{Epi}(L(n),B_2)$, is injective, and it is easy to prove that r is onto. Then $N(\mathrm{Epi}(L(n),B_2)) = N(F^*(G,B_2))$.

Consider B(n), the Boolean algebra generated by G. By Lemma 6.2, G is a set of n free generators of B(n). Consider $\mathcal{H}$ the set of all the Boolean epimorphisms from B(n) into B_2. Let us see that $N(F^*(G,B_2)) = N(\mathcal{H})$.

If $f \in F^*(G,B_2)$, we note $\overline{f}$ the extension epimorphism from L(n) onto B_2, and f' the restriction of $\overline{f}$ to B(n). We know that f' is a Boolean homomorphism. Moreover, $B_2 = S(f(G)) = S(f'(G)) = = B(f'(G)) \subset f'(B(n))$, that is, f' is a Boolean epimorphism. We define $\varphi(f) = f'$. φ is a mapping from $F^*(G,B_2)$ into $\mathcal{H}$. It is clear that φ is biyective. Since $B(n)$ has 2^n atoms, and B_2 has two atoms, it is well known that $N(\mathcal{H}) = V_{2^n,2} = \dfrac{2^n!}{(2^n-2)!}$ ([6], [7]).

Then $N(M_2) = \dfrac{N(Epi(L(n),B_2))}{N(Aut(B_2))} = \dfrac{N(F^*(G,B_2))}{2} = \dfrac{N(\mathcal{H})}{2} = \dfrac{V_{2^n,2}}{2} = \binom{2^n}{2}$

Therefore $L(n) = B_1^{2^n} \times B_2^{\binom{2^n}{2}} = B_1^{2^n} \times B_2^{\frac{4^n-2^n}{2}}$ and

$N(L(n)) = 2^{2^n} \times 4^{\frac{4^n-2^n}{2}} = 2^{4^n} = 2^{2^{2n}}$. Which coincides with the results obtained by A. Monteiro [5].

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