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# ON A FUZZY FUNCTIONAL INTEGRATION THEORY

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Once the bases for a probability theory on fuzzy sets or a generalization of the probability theory for an adequate class of functions, in a wide way have been given as for example by one of the authors of this note in [1], the next important theoretical step is to construct a corresponding functional integral theory.

On this direction some efforts have been made as that of Rankin in [4] obtaining important results, but the aim of it does not agree with a natural way of approaching an extension for fuzzy sets. For this reason we write this paper with the aim of developing a natural integral theory for fuzzy functional probability. Here, in the first part we introduce the concept of integral for simple functions and a new concept of convergence, which is a little bit stronger than the point wise convergence. In the second paragraph, we establish standard limit theorems for bounded functions and for non-bounded functions, later which play a similar role to measurable functions in the usual theory.

Finally, we present a general theorem which can be regarded as the dominance theorem for a wider class of functions.

The proof is based on the classical procedure that is quoted in [3] and [5]. We expect in the near future that such a theory could be of interest for many applications, perhaps in quantum mechanics, pattern recognitions, etc, as well as its relation with abstract studies. On this point one could study some extension of integrals related with normed spaces of fuzzy sets as introduced in [2].

1. Let $\mathcal{X}$ = (X,A,P) be a *fuzzy $\sigma$-algebra with a functional probability*, that is, X is a set, $A \subset [0,1]^X$ a family of functions closed under the usual countable infimum and supremum of functions and the quasi-complement operation. If $A \in A$, 1-A is its quasi-complement, where 1 indicates the characteristic function of X. Moreover 1 and $\phi$ are elements of A. By $\phi$ we mean the empty set and its characteristic function.

An element of A is called a *fuzzy set*. The *functional probability* P is a real function defined on A with values in [0,1] such that,

i) $P(A \vee B) + P(A \wedge B) = P(A) + P(B)$ , $P(1 - A) = P(1) - P(A)$

for any A,B in $A$, $P(1) = 1$. The symbols $\vee$ and $\wedge$ denote supremum and infimum.

ii) Given two sequences $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ , $B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$ of elements of $A$ with the property $\bigvee_n A_n \geqslant \bigwedge_n B_n$ then

$$\lim_n P(A_n) \geqslant \lim_n P(B_n)$$

This property is known as *monotone sequential continuity*.

The third author of this paper in [1] has studied many properties of functional probabilities. The main result is concerned with the existence and uniqueness of a functional probability defined on the fuzzy $\sigma$-algebra generated by a fuzzy algebra with a given functional probability. Among these properties we have the following: For a given sequence $\{A_n\}$ of fuzzy sets if $A_n \to A$ point wise then $P(A_n) \to P(A)$ when $n \to \infty$ . Moreover, the monotony property: that is $A \leqslant B$ implies $P(A) \leqslant P(B)$, is a simple consequence of monotone sequential continuity.

From now on, in this paragraph we are going to consider only positive functions. A *simple function* f defined on X is a linear combination of fuzzy sets of $A$ with non-negative coefficients. Such a class of functions is denoted by $S$. We always assume that $S$ is closed under finite infima and suprema.
We define the *fuzzy integral* or simple *integral* for the simple function $f \in S$ to be the real number

$$\int f = \operatorname{Sup} \{ \sum_i \lambda_i P(A_i) \; : \; \sum_i \lambda_i A_i \leqslant f \; , \; \lambda_i \geqslant 0 \; , \; A_i \in A \}$$

where the sum is always to be considered finite.

Under the following linear condition (L), which we always implicitly assume: For any two simple functions

$$\sum_i \mu_i A_i \leqslant \sum_i \nu_i B_i \quad \text{implies} \quad \sum_i \mu_i P(A_i) \leqslant \sum_i \nu_i P(B_i) \qquad \text{(L)}$$

where $A_i$ , $B_i \in A$ and $\mu_i, \nu_i \geqslant 0$, the number $P(f)$ defined by $\sum_i \lambda_i P(A_i)$ for a given simple function $f = \sum_i \lambda_i A_i$ is well defined.

Indeed, we have $\int f = P(f) = \operatorname{Sup}_{g \in D_f} P(g)$ where $D_f$ is the set of functions $g \in S$ such that $g \leqslant f$.

Immediately, from above we have for any two simple functions $f,g \in S$ and non-negative real numbers $\alpha,\beta$: $P(\alpha f + \beta g) = \alpha P(f) + \beta P(g)$. Furthermore, $f \leqslant g$ implies $P(f) \leqslant P(g)$.

Now we are going to define convergence. We say that a sequence $\{f_n\}$ of non-negative real functions *fuzzy-converges* to 0, and we write

$f_n \downarrow 0$ if there is a k such that for each $\varepsilon > 0$ there is a $g_\varepsilon^n \in A$ with the properties:

$$f_n \leqslant k\, g_\varepsilon^n + k\varepsilon \quad \text{and} \quad g_\varepsilon^n \to 0 \quad \text{when} \quad n \to \infty \quad \text{pointwise}. \tag{1}$$

Indeed, as can be seen, the second k in (1) is irrelevant. Similarly, $f_n \uparrow f$ if $f - f_n \downarrow 0$. We also call it *fuzzy limit*.

PROPOSITION 1.1. *If* $f_n$, $f$ *are simple functions in S such that* $f_n \uparrow f$, *then* $P(f_n) \to P(f)$.

*Proof*. Let $\{f_n\}$ be a sequence of simple functions fuzzy-converging to f, thus $f \leqslant k\, g_\varepsilon^n + k\varepsilon + f_n$ for the corresponding $g_\varepsilon^n \in A$ for a given $\varepsilon > 0$. Therefore $P(f) \leqslant kP(g_\varepsilon^n) + k\varepsilon + P(f_n)$ which implies for the superior limit $\overline{\lim\limits_n}\, \{P(f) - P(f_n)\} \leqslant 0$, but $P(f_n) \leqslant P(f)$.

PROPOSITION 1.2. *Let* $\{f_n\}$ *and* $\{f'_n\}$ *be two sequences in S such that* $f_n \uparrow f$ *and* $f'_n \uparrow f'$, *where* $f \leqslant f'$, *then* $\overline{\lim\limits_n} \int f_n \leqslant \underline{\lim\limits_n} \int f'_n$. *Moreover, if* $f=f'$, $\lim\limits_n \int f_n = \lim\limits_n \int f'_n$ *holds true.*

*Proof*. In order to prove the proposition we need the following secondary fact: if $f'_n \uparrow f'$ and $f_n \leqslant f'$ then $f_m \wedge f'_n \uparrow f_m$ for each m. Indeed, the inequality $f' \leqslant k\, g_\varepsilon^n + k\varepsilon + f'_n$ implies

$$f_m = f' \wedge f_m \leqslant k\, g_\varepsilon^n + k\varepsilon + f_m \wedge f'_n$$

Coming back to the proof of the proposition, we have for each m: $f_m \wedge f'_n \uparrow f_m$. By the previous proposition we have

$$\lim_n P(f_m \wedge f'_n) = P(f_m)$$

On the other hand, $P(f'_n) \geqslant P(f_m \wedge f'_n)$, hence

$$\underline{\lim_n}\, P(f'_n) \geqslant \lim_n P(f_m \wedge f'_n) = P(f_m) \quad \text{for each } m, \quad \text{then}$$

$$\underline{\lim_n}\, P(f'_n) \geqslant \overline{\lim_n}\, P(f_n)\ .$$

This result allows us to define the fuzzy integral for the class of functions which are obtained as the fuzzy limit of a sequence of simple functions. Thus, if $f_n \uparrow f$, $f_n$ in S then by definition we have

$$\int f = \lim_n \int f_n$$

Before we consider further aspects of the present study, we want to illustrate the fuzzy-limit in the usual case. In such a case the fuzzy-limit is the pointwise limit for measurable functions $f_n \leqslant k$. Since for these we have

$$f \leqslant k\, \chi_{[\varepsilon,\infty)} \circ f_n/k + k\varepsilon \qquad (2)$$

then $g^n_\varepsilon$ can be chosen as

$$\chi_{[\varepsilon,\infty)} \circ f_n/k = \chi_{\{f_n/k \geq \varepsilon\}}$$

where $\chi_A$ indicates the characteristic function of the set A. On the other hand, our version of fuzzy limit implies the usual point wise limit. Moreover, in the usual case they are equivalent.

2. Here we are going to extend the concept of integral for a wider class of functions. Let $J^+_k$ be the family of non-negative functions bounded by k, obtained as a fuzzy-limit of a sequence of simple functions in $S$, having the same k in the convergence formula (1). That is, if $f \in J^+_k$ then $f \leqslant k$ and there is a sequence $\{f_n\}$ in $S$ such that

$$0 \leqslant f - f_n \leqslant k\, g^n_\varepsilon + k\varepsilon$$

In such an instance we write $f_n \underset{k}{\uparrow} f$, then $f_n \leqslant k$.
Thus, we say that a sequence of functions $\{f_n\}$ in $J^+ = \bigcup_{k>0} J^+_k$ fuzzy-converges to some function f if for some k: $f_n \underset{k}{\uparrow} f$ and the sequence $\{f_n\}$ is in $J^+_k$.
Of course under such convergence $f \leqslant k$. It is clear that the fuzzy integral for any element of $J^+$ is well defined by virtue of proposition 1.2.

The next result is the key for studying an integration theory.

THEOREM 2.1. *For any two converging sequences* $f_n \uparrow f$ *and* $h_n \uparrow h$ *in* $J^+$ *and* $\alpha,\beta \geqslant 0$ *we have* $f,h \in J^+$ *and* $\alpha f_n + \beta h_n \uparrow \alpha f + \beta h$ , $\alpha f + \beta h \in J^+$.

*Proof*. Since $f_n \uparrow f$ and $h_n \uparrow h$, there are k and $\overline{k}$ such that $f_n \underset{k}{\uparrow} f$ and $h_n \underset{\overline{k}}{\uparrow} h$. Thus, adding these two inequalities

$$\alpha(f - f_n) \leqslant \alpha(k\, g^n_\varepsilon + k\varepsilon)$$

$$\beta(h - h_n) \leqslant \beta(\overline{k}\, \overline{g}^n_\varepsilon + \overline{k}\varepsilon)$$

we have

$$\alpha f + \beta h - (\alpha f_n + \beta h_n) \leqslant \alpha k\, g^n_\varepsilon + \beta\overline{k}\, \overline{g}^n_\varepsilon + (\alpha k + \beta\overline{k})\varepsilon \leqslant$$

$$\leqslant (\alpha k + \beta\overline{k})(g^n_\varepsilon \vee \overline{g}^n_\varepsilon) + (\alpha k + \beta\overline{k})\varepsilon \ , \quad \text{which says}$$

$$\alpha f_n + \beta h_n \underset{\alpha k+\beta\overline{k}}{\uparrow} \alpha f + \beta h \ .$$

Now, we prove the closeness. For a given sequence of functions $\{f_n\}$ in $J^+$ such that $f_n \underset{k}{\uparrow} f$ for some k, and $\{f_n\}$ in $J^+_k$, we always can choose for each n a sequence $f_{nm} \underset{k}{\uparrow} f_n$ of simple functions.

Thus, for $j \leqslant m$:

$$f - f_{jm} = (f - f_j) + (f_j - f_{jm}) \leqslant$$

$$\leqslant k(g^j_\varepsilon + g^{jm}_\varepsilon) + 2k\varepsilon \leqslant 2k(g^j_\varepsilon \vee g^{jm}_\varepsilon) + 2k\varepsilon$$

where $g^j_\varepsilon$ and $g^{jm}_\varepsilon$ are the corresponding elements in the convergence of $f_j$ and $f_{jm}$ respectively. From here, we derive the following inequality

$$f - \bigvee_{j\leq m} f_{jm} \leqslant 2k \bigwedge_{j\leq m} (g^j_\varepsilon \vee g^{jm}_\varepsilon) + 2k\varepsilon$$

which implies $\bar{f}_m = \bigvee_{j\leq m} f_{jm} \underset{2k}{\uparrow} f$

since $\bar{g}^m_\varepsilon = \bigwedge_{j\leq m} (g^j_\varepsilon \vee g^{jm}_\varepsilon) \to 0$ when $m \to \infty$

As we supposed $\bar{f}_m \in S$, then $f \in J^+_{2k}$.

Applying this result to the fuzzy integral we have the following

COROLLARY 2.2. *Let* $f_n \uparrow f$ *in* $J^+$, $h \in J^+$ *and* $\alpha,\beta \geqslant 0$, *then*

$$\int \alpha f + \beta h = \alpha \int f + \beta \int h \quad , \quad \int f = \lim_n \int f_n$$

*Moreover,* $f \leqslant h$ *implies* $\int f \leqslant \int h$ .

*Proof*. The linearity is clear. Let f,h be functions in $J^+$ such that $f \leqslant h$, then if $f_n \underset{k}{\uparrow} f$ and $h_n \underset{\bar{k}}{\uparrow} h$ for suitable $f_n$ and $h_n$ in $S$, we always have $f_n \vee h_n \underset{k\vee\bar{k}}{\uparrow} h$ where $f_n \vee h_n \in S$. This implies the monotonicity of the integral.

Applying this fact, if $f_n \underset{k}{\uparrow} f$ in $J^+$ we have $f \leqslant kg^n_\varepsilon + k\varepsilon + f_n$ which implies by using linearity and monotonicity

$$\int f \leqslant \underline{\lim}_n \int f_n$$

On the other hand, $f_n \leqslant f$ implies

$$\int f \geqslant \overline{\lim}_n \int f_n \ .$$

With this result, we now are allowed to extend the integral for non-negative unbounded functions. The space with a main role is

$$\mathscr{J}^+ = \{f:\ f \wedge n \in J^+ \ \text{ for all } \ n \in \mathbb{N}\}$$

where $\mathbb{N}$ is the set of non-negative integers.

As we will see in a moment, the following definition of fuzzy integral for an $f \in \mathcal{J}^+$ will become natural:

$$\int f = \sup\{P(h) : h \in S \text{ and } h \leqslant f\}$$

It is clear that on $J^+$ this and the integral previously considered are the same.

The next result relates a way of computing the integral.

PROPOSITION 2.3. *Let* f,g *be elements in* $\mathcal{J}^+$, *then*

$$\int (f+g) = \lim_n \int (f \wedge n + g \wedge n)$$

*Proof*. If I indicates the right hand of this equality, since for any n

$$f \wedge n + g \wedge n \leqslant f + g$$

we have $I \leqslant \int (f+g)$, by virtue of the monotonicity of the integral.

In order to prove the remaining inequality, we must consider two cases, namely: $\int (f+g)$ finite and infinite. In the first instance, given any $\varepsilon > 0$ there is a simple function $h \in S$ such that $h \leqslant f+g$ and $\int (f+g) - \varepsilon \leqslant P(h)$. Since there is an $n \geqslant h$ we have $h \leqslant f \wedge n + g \wedge n$, and therefore $P(h) \leqslant I$. When the integral $\int (f+g)$ is infinite, for each n there is an $h \in S$ such that $P(h) \geqslant n$ and $h \leqslant f+g$. Let $n_0$ be greater than h, therefore $h \leqslant f \wedge n_0 + g \wedge n_0$. This implies

$$n \leqslant P(h) \leqslant \int (f \wedge n_0 + g \wedge n_0) \leqslant I .$$

As an immediate consequence of this result we derive the linearity and a property like Beppo Levi's theorem. In order to present it, we now define the fuzzy convergence in $\mathcal{J}^+$ :

$$f_n \uparrow f \text{ in } \mathcal{J}^+ \text{ if } f_n \in \mathcal{J}^+ \text{ and } f_n \wedge m \uparrow f \wedge m \text{ in } J^+$$

for each $m \in \mathbb{N}$. It is clear that $\mathcal{J}^+$ is closed under the fuzzy-convergence. Of course, such a convergence coincides with the old one in $J^+$.

THEOREM 2.4. *Let* $f_n$,f *and* h *be elements of* $\mathcal{J}^+$, $f_n \uparrow f$, *and* $\alpha,\beta \geqslant 0$, *then* $\int (\alpha f + \beta h) = \alpha \int f + \beta \int h$ *and* $\int f = \lim_n \int f_n$.

*Proof*. It is clear that $f \in J^+$ implies $f \wedge \gamma \in J^+$ for an arbitrary $\gamma \geqslant 0$. Therefore, by using the equality:

$$\alpha f \wedge n = \alpha(f \wedge \tfrac{n}{\alpha}) \text{ with } \alpha \geqslant 0 \text{ for all } n,$$

it turns out that $f \in \mathcal{J}^+$ implies $\alpha f \in \mathcal{J}^+$.

Moreover, $\int \alpha f = \sup\{P(\alpha(\frac{h}{\alpha})): \frac{h}{\alpha} \leqslant f\} = \alpha \int f$ with the usual convention $0.\infty = 0$ when $\alpha = 0$.

For each n, the fuzzy-convergence in $J^+$ says:

$(\alpha f) \wedge n \leqslant k_n g_\varepsilon^{ni} + k_n \varepsilon + h_{ni}$ and $(\beta h) \wedge n \leqslant \overline{k}_n \overline{g}_\varepsilon^{ni} + \overline{k}_n \varepsilon + \overline{h}_{ni}$

From these two inequalites, one gets:

$$\int((\alpha f) \wedge n + (\beta h) \wedge n) \leqslant k_n P(g_\varepsilon^{ni}) + \overline{k}_n P(\overline{g}^{ni}) + (k_n + \overline{k}_n)\varepsilon + \\ + \int (\alpha f) \wedge n + \int (\beta h) \wedge n$$

since $h_{ni} \leqslant (\alpha f) \wedge n$ and $\overline{h}_{ni} \leqslant (\beta h) \wedge n$. Thus by proposition 2.3:

$$\int (\alpha f + \beta h) \leqslant \alpha \int f + \beta \int h .$$

The inverse inequality is obvious. In order to prove the second part, it is clear that $f_n \uparrow f$ in $\mathcal{J}^+$ assures

$$\int f \geqslant \overline{\lim_n} \int f_n$$

since $f_n \leqslant f$.

The reverse inequality can be seen as follows: if $\int f$ is finite, then for each $\varepsilon > 0$ there is an $n_0$ such that

$$\int f - \varepsilon \leqslant \int f \wedge n_0 = \lim_n \int f_n \wedge n_0$$

by virtue of proposition 2.3 and corollary 2.2. But there is a $\overline{n}_0$ such that for each $n \geqslant \overline{n}_0$

$$\int f \wedge n_0 - \varepsilon \leqslant \int f_n \wedge n_0 \leqslant \int f_n$$

Therefore $\int f \leqslant 2\varepsilon + \underline{\lim_n} \int f_n$ .

When $\int f$ is infinite we have that for each $\overline{n}$ there is an m such that $\int f \wedge m > \overline{n}$. But the equality

$$\int f \wedge m = \lim_n \int f_n \wedge m$$

implies that there is a $n_0$ such that for each $n \geqslant n_0$

$$\int f_n \wedge m \geqslant \overline{n} \quad , \quad \text{hence} \quad \lim_n \int f_n = \infty .$$

3. Now we will extend the theory for a wider class of functions. Let $\mathcal{J}$ be the set of real functions f defined on X such that $f = f_1 - f_2$ with $f_1, f_2 \in \mathcal{J}^+$, and the integrals of these functions

are finite. The last condition of boundness is assumed only for simplicity. Indeed, one could study a more general case: that only one of the integrals is finite. Here, even though the fuzzy convergence keeps its meaning, it involves both directions of approximation. Let $f_n$ be a sequence of elements in $\mathcal{J}$, we say that $f_n$ fuzzy converges to $f \in \mathcal{J}$ and write $f_n \uparrow f$ if there are two sequences $\{f_n^1\}, \{f_n^2\}$ and $f^1$, $f^2$ in $\mathcal{J}^+$ such that $f_n^1 \uparrow f^1$ and $f_n^2 \downarrow f^2$, in $\mathcal{J}^+$. The last reversed fuzzy-convergence means that there is an $h \in \mathcal{J}^+$, such that $h - f_n^2 \uparrow h - f^2$ in $\mathcal{J}^+$ and the integral $\int h$ is finite.

We note that $h - f_n^2$ and $h - f^2$ are in $\mathcal{J}^+$. In order to study properties of the fuzzy-convergence in $\mathcal{J}$ it becomes natural to extend the concepts of superior and inferior limits, which we are going to introduce.

Let $f_n$ be a sequence in $\mathcal{J}^+$ we say that an $\underline{f}$ in $\mathcal{J}^+$ is a *fuzzy inferior limit* if $\underline{f} \wedge i \leqslant k_i g_\varepsilon^{in} + k_i \varepsilon + f_m \wedge i$, for each $i,n \in N$ and $m \geqslant n$, where $g_\varepsilon^{in} \in A$ and for any $i \in N$ $g_\varepsilon^{in} \to 0$ when $n \to \infty$. Analogously, a *fuzzy superior limit* $\overline{f}$ in $\mathcal{J}^+$ is defined only if there is an $h \in \mathcal{J}^+$ with $\int h$ finite such that $\{h - f_n\}$ is in $\mathcal{J}^+$, then $h - \overline{f} = \underline{f_h}$ where $\underline{f_h}$ is a fuzzy inferior limit of $\{h - f_n\}$. We chose an integrable $h$ in this definition in order to prove the limit integral theorems.

Immediately from the definitions we always have that $\underline{f} \leqslant \underline{\lim}_n f_n \leqslant \overline{\lim}_n f_n \leqslant \overline{f}$ for any pair of fuzzy inferior $\underline{f}$ and superior $\overline{f}$ limits. Now, we extend the fuzzy inferior and superior limits to $\mathcal{J}$. Given a sequence $\{f_n\}$ in $\mathcal{J}$ we say that $\underline{f} = \underline{f}^1 - \overline{f}^2$ in $\mathcal{J}$ is a *fuzzy inferior limit* if $\underline{f}^1$ is a fuzzy inferior and $\overline{f}^2$ is a fuzzy superior limit for some $\{f_n^1\}$ and $\{f_n^2\}$ in $\mathcal{J}^+$, respectively, such that $f_n = f_n^1 - f_n^2$. Similarly, a fuzzy superior limit is an element $\overline{f}$ in $\mathcal{J}$ which equals $\overline{f}^1 - \underline{f}^2$ where $\overline{f}^1$ and $\underline{f}^2$ in $\mathcal{J}^+$ are fuzzy superior and inferior limits respectively for some $\{f_n^1\}$ and $\{f_n^2\}$ in $\mathcal{J}^+$.

Again, the above inequalities between inferior and superior limits in $\mathcal{J}$ take the following form:

$$\underline{f} \leqslant \underline{\lim}_n f_n^1 - \overline{\lim}_n f_n^2 \leqslant \overline{\lim}_n f_n^1 - \underline{\lim}_n f_n^2 \leqslant \overline{f}$$

By using these, it turns out that if $\underline{f} = \overline{f}$ for some fuzzy inferior and superior limits, implies

$$\overline{\lim}_n f_n^1 = \underline{\lim}_n f_n^1$$

and

$$\overline{\lim}_n f_n^2 = \underline{\lim}_n f_n^2$$

The integral is extended to $\mathcal{J}$ in a natural way, namely:

$\int f = \int f_1 - \int f_2$ with $f_1$ and $f_2 \in \mathcal{J}^+$ and $f = f_1 - f_2$. Of course, it is well defined and monotone in $\mathcal{J}$. As in theorem 2.4, we can obtain that $f_n \uparrow f$ in $\mathcal{J}$ implies

$$\int f = \lim_n \int f_n$$

On this way, we now present an important tool:

THEOREM 3.1. *Let* $\{f_n\}$ *be a sequence in* $\mathcal{J}$ *such that there exist* $\underline{f}$ *and* $\overline{f}$ *in* $\mathcal{J}$, *then* $\int \underline{f} \leqslant \underline{\lim}_n \int f_n \leqslant \overline{\lim}_n \int f_n \leqslant \int \overline{f}$.

*Proof*. Given $\{f_n\}$ in $\mathcal{J}$ with $f_n = f_n^1 - f_n^2$ where $f_n^1, f_n^2 \in \mathcal{J}^+$. By definition, we have

$$\underline{f}^1 \wedge i \leqslant k_i g_\varepsilon^{in} + k_i \varepsilon + f_m^1 \wedge i \leqslant k_i g_\varepsilon^{in} + k_i \varepsilon + f_m^1$$

for each $i, n \in N$ and $m \geqslant n$. Integrating and taking the infimum on $m \geqslant n$ in the right side of the inequality, and then limit on $n \in N$, we get

$$\int \underline{f}^1 \wedge i \leqslant k_i \varepsilon + \underline{\lim}_n \int f_n^1$$

since for each $i \in N$, $P(g_\varepsilon^{in}) \to 0$ as $n \to \infty$. Now, by taking $\varepsilon \to 0$ and after this $i \to \infty$, using proposition 2.3 it follows:

$$\int \underline{f}^1 \leqslant \underline{\lim}_n \int f_n^1$$

For the second fuzzy limit, we have $h - \overline{f}^2$ is a fuzzy inferior limit of $\{h - f_n^2\}$ for some integrable $h \in \mathcal{J}^+$.

Therefore $\int h - \int \overline{f}^2 = \int (h - \overline{f}^2) \leqslant \underline{\lim}_n \int (h - f_n^2) = \int h - \overline{\lim}_n \int f_n^2$

and since $\int h$ is finite we have $\int \overline{f}^2 \geqslant \overline{\lim}_n \int f_n^2$.

From here the result comes out immediately.

As a consequence of this result of the type of Fatou's lemma, we get the following obvious theorem.

THEOREM 3.2. *Let* $\{f_n\}$ *be a sequence in* $\mathcal{J}$ *such that there is an* $f \in \mathcal{J}$ *which is an inferior and superior limit of* $\{f_n\}$, *then*

$$\int f = \lim_n \int f_n .$$

We note that the required domination in theorem 3.1 and 3.2 is hidden in the definition of fuzzy inferior and superior limits.

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