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INNER FUNCTIONS UNDER UNIFORM TOPOLOGY. II.

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The set of all inner functions in the unit disc is a topological semigroup under pointwise multiplication in the open disc and the metric topology induced by the H^∞-norm. In this second article it is shown that, given any non-empty closed subset Γ of the unit circle ∂D, the set of components containing nothing but Blaschke products whose zeroes cluster exactly on Γ (i.e., $Sp(b) \cap \partial D = \Gamma$, in the notation of [6], [7]) has the power of the continuum; moreover, the same result is true in $(F^{\sim},\tau)$, thus answering in the strongest negative form a question raised by R.G. Douglas in [1].

The results also include a matricial version of the classical theorem of Otto Frostman ([2]) about the density of Blaschke products, which is used to extend some of the previous results to matrix-valued inner functions (i.e., the case when $\dim K < \infty$).

This paper is a sequel of [6] and [7], and we shall continue using the notation introduced there without further reference.

1. COMPONENTS OF $(F^{\sim}_{\Gamma},\tau)$.

The aim of this section is to prove the following

THEOREM 1.1. *Let Γ be any non-empty closed subset of ∂D and let $\{\theta_k\}_{k=1}^{\infty}$ be any arbitrary sequence of arguments such that*

$$\Gamma = \bigcap_{N=1}^{\infty} \text{clos}\,\{e^{i\theta_k}: k \geq N\}$$

If the sequence $0 = r_1 < r_2 < \dots < r_k < \dots < 1$ converges to one rapidly enough, then there exist c *sub-products $\{b_\theta: 0 \leq \theta \leq \pi/4\}$ of $b(z) = \prod_{k=1}^{\infty} b(z,r_k e^{i\theta_k})$ such that $b_\theta H^2 \in F^{\sim}_{\Gamma} \cap (\det)^{\sim}_{B}$; moreover, $b_\theta H^2$ and $b_{\theta'} H^2$ belong to different components of $(F^{\sim},\tau)$ if $\theta \neq \theta'$.*

The idea of the proof is very simple and it is based on the following result of O. Frostman:

THEOREM 1.2. ([2, §58, thm. 2, p. 107]). *Let* $q \in F$ *and assume that* q *does not admit zero as a radial limit; then* q *is a Blaschke product.*

We are going to choose the sequence $\{r_k\}_{k=1}^{\infty}$ in such a way that, for each k, there is an R_k, $r_k < R_k < r_{k+1}$ such that $|b(z)| \simeq 1$ for $|z| = R_k$ and, moreover, for every element qH^2 of the component of bH^2 in $(F^{\sim},\tau)$, $|q(z)| \simeq |b(z)|$ whenever $R < |z| < 1$, for some R depending on q. Then, by *thm. 1.2*, q is also a Blaschke product.

For each pair of functions, f , $g \in H^{\infty}$, define

$$[f,g] = \sup\{||f(z)| - |g(z)||: z \in D\}$$

Similarly , we shall write $\{[f,g]: z \in \Lambda\}$ when that supremum is taken on the subset $\Lambda \subset D$.

It is apparent that for each non-denumerable $\Lambda \subset D$, $(pH^2,qH^2)_{\Lambda} = \{[p,q]: z \in \Lambda\}$ $(p,q \in F)$ defines a metric in $F^{\sim}$.

Let $p,q \in F$ satisfy $\tau_0(pH^2,qH^2) < \varepsilon < 1$, then there exists $f \in (N)_{1-\varepsilon}$ such that $\|p-qf\|_{\infty} < \varepsilon$; hence

$$[p,q] \leqslant [p,qf] + \|q\|_{\infty}[f,1] \leqslant \|p-qf\|_{\infty} + [f,1] < \varepsilon + [(1-\varepsilon)^{-1}-1] = \varepsilon(2-\varepsilon)/(1-\varepsilon).$$

Since $0 \leqslant [p,q] \leqslant 1$, a simple computation shows that

LEMMA 1.3. *Let* $p,q \in F$; *then*

$$[p,q] \leqslant (27/10)\tau_0(pH^2,qH^2).$$

We shall need some auxiliary results, which are contained in the following three lemmas:

LEMMA 1.4. *For each* $a \in D$, *define*

$$A_1(a) = \{a' \in D: [b(z,a),b(z,a')] < 1/2\}$$

and (by induction)

$$A_n(a) = \{a'' \in D: [b(z,a'),b(z,a'')] < 1/2 , \text{ for some } a' \in A_{n-1}(a)\} , n=2,3,\ldots .$$

Then:

i) $A_n(a)$ *is an increasing family of open neighborhoods of* a *with a smooth boundary, such that* $\bigcup_{n=1}^{\infty} A_n(a) = D$ *and* $\text{clos}[A_n(a)] \subset D$, *for each* n;

ii) *For fixed* n, *if* $|a| \to 1$, *then diameter* $[A_n(a)] \to 0$.

LEMMA 1.5. *The results of lemma 1.4 are also true for finite Blaschke products up to a few minor modifications: instead of elementary Blaschke products, consider the elements of* $(\det)_m$ *(for each fixed* m*) and, instead of* $A_n(a)$*, define* $A_n(a_1,\dots,a_m)$ *(where* $a_1,\dots,a_m$ *are the zeroes of a given* $p \in (\det)_m$*), in the obvious way.*

LEMMA 1.6. *There exists an increasing sequence* $0 < R_1 < R_2 < \dots < < R_n < R_{n+1} < \dots < 1$, *such that for any Blaschke product* $b(z) = = \prod_{k=n+1}^{\infty} b(z,a_k)$, *where* $R_k \leqslant a_k < R_{k+1}$,

$$|b(z)| > 9/10$$

on the closed disc $\{z: |z| \leqslant R_n\}$, *for* $n=1,2,3,\dots$.

The proof of *lemmas 1.4* and *1.5* follow from some elementary arguments based on the continuity of finite Blaschke products in the closed unit disc and the properties of the inner functions in $F\backslash F_\phi$. *Lemma 1.6* follows from the fact that $\lim b(z,a) = 1 (|a|\to 1)$, uniformly on compact subsets of D.

The definition of the sequence $\{r_k\}$ proceeds as follows: By induction, we shall construct a sequence of 4-tuples $\{r_n,s_n,t_n,u_n\}_{n=1}^{\infty}$ so that

1) $r_1 < s_1 < t_1 < u_1 < r_2 < \dots < r_n < s_n < t_n < u_n < r_{n+1} < \dots < 1.$

2) if $q_o(z) = \prod_{k=1}^{n} b(z,a_k)$, where $|a_k| \leqslant r_n$ $(k=1,\dots,n)$, and $q_1,\dots,q_n \in (\det)_n$ satisfy $\max\{[q_{j-1},q_j] < 1/2: j:1,\dots,n\}$, then the zeroes of q_n lie on $\{z:|z| < s_n\}$.

3) let q_n be as above; then $|q_n(z)| > 9/10$ for $|z| \geqslant t_n$.

4) for all $p \in (\det)_B$ having no zeroes in $\{z:|z| < u_n\}$ and no more than one zero in the annulus $\{z: u_m \leqslant |z| < u_{m+1}\}$, for $m=n,n+1,\dots,$

$$|p(z)| > 9/10 \quad \text{for} \quad |z| \leqslant t_n \ .$$

5) $A_{n+1}(r_{n+1}) \cap \{z:|z| < u_n\} = \phi$, for all $n=1,2,\dots$.

Now, the 4-tuples are chosen as follows:

Set $r_1 = 0$ and $s_1 = 1/2$. Let t_1 be the greatest real zero of $|b(t,1/2)| = 9/10$. If k_1 is the first index (in *lemma 1.6*) such that $t_1 \leqslant R_{k_1}$, then set $u_1 = R_{k_1+1}$; clearly, $r_1 < s_1 < t_1 < u_1$ and 2), 3), 4) can be easily checked for n=1.

Assume that $r_1 < s_1 < \dots < r_n < s_n < t_n < u_n < 1$ have been chosen so that they satisfy 2), 3), 4) and 5). Now use *lemma 1.4* to find an $r_{n+1} < 1$ satisfying 5); $r_{n+1} > u_n$. Next, use *lemma 1.5* to find an $s_{n+1} < 1$ so that 2) is verified; then $s_{n+1} > r_{n+1}$. Let t_{n+1} be the greatest real zero of $|b(t,s_{n+1})|^{n+1} = 9/10$. Finally, if k_{n+1} is the first index (in *lemma 1.6*) such that $t_{n+1} \leqslant R_{k_{n+1}}$, choose $u_{n+1} = R_{k_{n+1}+1}$.

It is not difficult to check that the sequence of 4-tuples thus constructed satisfies 1) to 5).

CLAIM. If $b(z) = \prod_{k=1}^{\infty} b(z, r_k e^{i\theta_k}) = \prod_{k=1}^{\infty} b_k(z)$ and $q \in F$ satisfies $\tau_o(bH^2, qH^2) < 1/54$, then

$$q(z) = \lambda \prod_{k=1}^{\infty} b(z,a_k)$$

where $\lambda \in \partial D$ and $a_k \in A_1(r_k e^{i\theta_k})$, $k=1,2,\ldots$.

By 3) and 4), $|\prod_{k=1}^{n-1} b_k| = |b_n'| > 9/10$ for $|z| \geqslant t_{n-1}$ (*a fortiori* the same result is true for any subproduct) and $|\prod_{k=n+1}^{\infty} b_k| = |b_n''| > 9/10$ for $|z| \leqslant t_n$; hence $|b| > (9/10)^2 > 4/5$ for $|z| = t_n$ and $|b/b_n| > 4/5$ on the annulus $\Lambda_n = \{z : t_{n-1} \leqslant |z| \leqslant t_n\}$, for all $n=1,2,\ldots$. $(t_o = -1)$.

By *lemma 1.3*, $[b,q] < 1/20$ and therefore $|q| > 4/5 - 1/20 = 3/4$ for $|z| = t_n$ and $|q(re^{i\theta_n})| < 1/20 < 3/4$, $n=1,2,\ldots$. Using *thm. 1.2* we infer that q is a Blaschke product; moreover, $q(z) \neq 0$ for $|z| = t_n$ (for all n) and q has at least one zero in Λ_n, for each n.

It follows from the inequality $\tau_o(bH^2, qH^2) < 1/54$ that $\|b - qf\|_\infty < 1/54$ for some suitable function $f \in (N)_{53/54}$. The last inequality shows, in particular, that $|b - qf| < |b|$ for $|z| = t_n$ $(n=1,2,\ldots)$. Thus, applying *Rouché's theorem* and using the fact that f is invertible in H^∞, we conclude that q has *exactly one zero* in Λ_n, for each $n=1,2,\ldots$. Hence,

$$q(z) = \lambda \prod_{k=1}^{\infty} b(z,a_k) = \lambda \prod_{k=1}^{\infty} q_k(z) = \lambda q_n' \, q_n \, q_n''$$

where $\lambda \in \partial D$, $a_n \in \Lambda_n$ $(n=1,2,\ldots)$ and q_n', q_n and q_n'' denote the subproducts corresponding to those zeroes lying in the regions $|z| < t_{n-1}$, $t_{n-1} < |z| < t_n$ and $|z| > t_n$, respectively.

Then the previous estimates and the *modulus maximum theorem* imply that $|q_n' \, q_n''| > 3/4$ for $z \in \Lambda_n$ and

$$\{[b_n,q_n]:z \in D\backslash\Lambda_n\} \leqslant \{[b_n,1]:z \in D\backslash\Lambda_n\} + \{[1,q_n]:z \in D\backslash\Lambda_n\} <$$
$$< 1/5 + 1/5 + 1/20 < 1/2$$

(to see this, observe that $|b| > 4/5$ for $|z| = t_n$ and $|z| = t_{n-1}$ implies that $|q_n| > 4/5-1/20$ for all $z \in D\backslash\Lambda_n$).

Assume that $\{[b_n,q_n]:z \in \Lambda_n\} \geqslant 1/2$; then

$$\begin{aligned}[b,q] &\geqslant \{[b,q]:\ z \in \Lambda_n\} \geqslant \\ &\geqslant \{[b,b_n'q_nb_n'']:\ z \in \Lambda_n\} - \{[b_n'q_nb_n'',q]:\ z \in \Lambda_n\} \geqslant \\ &\geqslant (\min\{|b_n'b_n''|:\ z \in \Lambda_n\})\{[b_n,q_n]:\ z \in \Lambda_n\} - \\ &- \{[b_n'b_n'',q_n'q_n'']:\ z \in \Lambda_n\} > (4/5)(1/2)-1/4 = 3/20 > [b,q]\end{aligned}$$

a contradiction.

Therefore $[b_n,q_n] < 1/2$ for all values of n and it follows that $q_n \in A_1(r_ne^{i\theta_n})$ for $n=1,2,\dots$. Similarly, it can be proven that $[b_n',q_n'] < 1/2$. Thus, by 2), the zeroes of q_n' lie in $|z| < s_{n-1}$, $n=2,3,\dots$.

Let $pH^2 (p \in F)$ belong to the component of bH^2 in $(F^{\sim},\tau)$; then, by [6, *Thms. 5.5* and *6.1*] there exists a finite family $p_o = b,p_1,p_2,\dots,p_m = p$ of inner functions such that

$$\max\ \{\tau_o(p_{j-1}H^2,p_jH^2):\ 1 \leqslant j \leqslant m\} < 1/54\ .$$

By *lemma 1.3*, $\max\{[p_{j-1},p_j]:\ 1 \leqslant j \leqslant m\} < 1/20$. Now by induction on j, it follows that p is a Blaschke product; moreover

$$p(z) = \lambda \prod_{k=1}^{\infty} b(z,a_k)$$

where $\lambda \in \partial D$, $|a_h| \leqslant t_n$ for $h=1,2,\dots,n$, and $a_n \in A_m(r_ne^{i\theta_n})$, for all $n \geqslant m$.

Finally, choose a subsequence $\{\theta_{k_j}\}_{j=1}^{\infty}$ of $\{\theta_k\}$ such that $e^{i\theta_{k_j}} \longrightarrow \lambda_o \in \Gamma$ as $j \longrightarrow \infty$, and write

$$\begin{aligned}b(z) &= \{\prod_{j=1}^{\infty} b(z,e^{i\theta_{k_j}})\}.\Pi\{b(z,e^{i\theta_k}):\ k\neq k_j,\ \text{for all } j\} = \\ &= b^+(z)b^-(z).\end{aligned}$$

Then, if $b_\theta(z) = b_\theta^+(z).b^-(z)$, where $b_\theta^+(z)$ is defined as in [9] for $0 \leqslant \theta \leqslant \pi/4$, then the b_θ's inherit all the properties that we already proved for b. In particular:

The components of $(F^{\sim},\tau)$ containing the $b_\theta H^2$'s are disjoint for different values of θ, $0 \leqslant \theta \leqslant \pi/4$ and, moreover, all these components are contained in $(\det)^{\sim}_B \cap F^{\sim}_\Gamma$.

The proof of *theorem 1.1* is complete now.

REMARKS. a) The only criterion that we know to decide whether or not pH^2 and qH^2 belong to the same component of $(F^{\sim},d)$ (and the one that we already used in [7] and here is this: the distance from the component of pH^2 to the component of qH^2 is larger than some positive number. According to [6, *Thm.6.1*] this is enough for $(F^{\sim},\tau)$; however, we do not even know whether or not all the components of $(F^{\sim},d)$ are arcwise connected or open subsets. We conjecture that the results of [6, *Thm.6.1*] are also true for $(F^{\sim},d)$, but we were unable to prove or disprove it.

b) The topology induced on $F^{\sim}$ by the metric $(.,.)_D$ is strictly weaker than the τ-topology. Moreover, as it follows from *Remark a)* to [7, *Thm.2.3*], if $d(z) = d(z;0,1)$ and $0 < t < t'$, then

$$\theta(d^tH^2,d^{t'}H^2) = \theta(H^2,d^{t'-t}H^2) = 1.$$

On the other hand, the map $t \longrightarrow d^tH^2$ from the positive reals into $(F^{\sim},(.,.)_D)$ is continuous!.

2. EXTENSIONS OF THE RESULTS TO THE CASE WHEN K IS FINITE DIMENSIONAL.

Throughout this section we shall assume that $2 \leqslant \dim K = N < \infty$. As for $\dim K = 1$, we have $(\det)_R = F_R$ (where R denotes the subindex of a given subclass). If $U \in F$, then $U(z)$ can be represented by an N x N matrix with entries in H^∞ such that the non-tangential limits $U(\lambda)$ are unitary matrices for almost every $\lambda \in \partial D$ and $\det U(z)$ is an inner function with the same kind of singularities as $U(z)$ (see [4;5;10]); in particular

$$(\det U)H^2_K \subset UH^2_K \quad \text{and} \quad \mathrm{Sp}\ (\det U) = \mathrm{Sp}\ (U).$$

The canonical form of an element of F or FN can be found in [3;8].

The mappings $U \longrightarrow \det U$ (from $H^\infty(K)$ onto H^∞) and $UH^2_K \longrightarrow (\det U)H^2$ (from $F^{\sim}(K)$ onto $F^{\sim}(C)$ with d-topologies or τ-topologies in both sides) are continuous.

Partof the results of *section 1* and [7] can be extended to the case when $2 \leq \dim K = N < \infty$ with minor changes; for example, if $\{b_\theta : 0 \leq \theta \leq \pi/4\}$ is defined as in the *proof* of *thm. 1.1*, and

$$(*) \qquad B_\theta = \begin{pmatrix} b_\theta & 0 \\ 0 & I_{N-1} \end{pmatrix}$$

then the component of B_θ in $(F^\sim, \tau)$ is contained in $F^\sim_\Gamma \cap (\det)^\sim_B$ and B_θ, $B_{\theta'}$ belong to different components of $(F^\sim, \tau)$ for $\theta \neq \theta'$. The proof follows from [6, *Thms. 5.5, 6.1*], *Thm. 1.1* and the following

LEMMA 2.1. *If* $U, V \in F$ *and* $\tau_o(UH^2_K, VH^2_K) < \varepsilon < 1/2$, *then* $\tau_o[(\det U)H^2, (\det V)H^2] < c(N)\varepsilon$, *where* $N = \dim K$ *and* $c(N)$ *is a positive constant depending on* N.

Proof. If $\tau_o(UH^2_K, VH^2_K) < \varepsilon$, then there exists a $C \in (N)_{1-\varepsilon}$ such that $\|U-VC\| < \varepsilon$; i.e.,

$$\text{ess sup } \{\|I - V(\lambda)C(\lambda)U^*(\lambda)\|_K : \lambda \in \partial D\} < \varepsilon < 1/2$$

and therefore $\det C \in (N)_\delta$, where $\delta = (1-\varepsilon)^{c'(N)}$.

$$\|\det U - \det V \det C\| = \|1 - \det(VCU^*)\|_\infty < c'(N)\varepsilon$$

for a suitable constant $c'(N) > 0$.

Hence

$$\tau_o[(\det U)H^2, (\det V)H^2] \leq \tau_\delta[(\det U)H^2, (\det V)H^2] <$$
$$< c'(N)2^{c'(N)}\varepsilon = c(N)\varepsilon \quad . \quad \text{qed.}$$

Our first partial extension of the results of [7] is a generalization of *Frostman's theorem*. The following theorem is contained in the author's thesis (see [5, p. 85]); its proof is based on a result due to W. Rudin ([8; 9]).

Since the proofs are the same for one or for several complex variables, we shall state the results for the most general setting. Some definitions are needed:

DEFINITION. Let $f(z_1, \ldots, z_n)$ be an analytic function defined in the unit polydisc

$$D^n = \{z = (z_1, \ldots, z_n) : z_j \in D, \ j = 1, \ldots, n\}$$

the n-dimensional torus $(\partial D)^n$ is the *distinguished boundary* of D^n and dm_n is the normalized Lebesgue measure on $(\partial D)^n$. f belongs to

$N_*(D^n)$ (the *Universal Hardy Class*) if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $m_n(A) < \delta$ (A a measurable subset of $(\partial D)^n$) then

$$\int_A \log^+|f(rw)|\,dm_n < \varepsilon\ , \quad w \in (\partial D)^n$$

uniformly with respect to r, $0 \leqslant r < 1$.

For $f \in N_*(D^n)$, the radial limits $f(w)$ do exist a.e. (dm_n) on $(\partial D)^n$. The inequalities

$$\log^+|a+b| \leqslant \log^+|a| + \log^+|b| + \log 2$$

$$\log^+|ab| \leqslant \log^+|a| + \log^+|b| \quad (a,b \in C)$$

show that $N_*(D^n)$ is actually an algebra under pointwise operations ([8]).

DEFINITION. ([8]). $f(z) \in H^\infty(D^n)$ is an *inner function* if $|f(w)| = 1$ a.e. (dm_n) on $(\partial D)^n$. If, in addition, the minimal n-harmonic function $u[f]$ such that $\log|f(z)| \leqslant u[f](z)$ is identically zero on D^n, f is said to be a *good inner function.*

According to this definition, we shall say that the analytic $N \times N$ matrix of functions $F(z) = (F_{jk}(z))_{j,k=1}^N$ is an inner (good inner, resp.) matrix if $\|F(z)\|_K \leqslant 1$, for $z \in D^n$ and $\det F(z)$ is an inner (good inner, resp.) function.

Observe that if $F(z)$ is an inner matrix, then $F(w)$ is unitary a.e.

For n=1, the good inner functions are just the Blaschke products (see [2], [8], [9]).

THEOREM 2.2. *Let* $f(z,\alpha) = \alpha^N + \sum_{k=o}^{N-1} f_k(z)\alpha^k$, $f_k(z) \in N_*(D^n)$, $k = 0,.,\ldots,N-1$, *and let* $\Lambda \subset C$ *be a compact set such that* cont log cap Λ *is positive (see definition in* [8], [9]). *Then there exists* $\alpha \in \Lambda$ *such that*

$$u[f(z,\alpha)] = \int_{(\partial D)^n} f(e^{i\phi_1},\ldots,e^{i\phi_n},\alpha) \prod_{j=1}^n P(r_j,\theta_j-\phi_j)\,dm_n(\phi_1,\ldots,\phi_n)$$

where $z = (r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})$, $0 \leqslant r_j < 1$, *and*

$P(r,\theta-\phi) = (1-r^2)/[1+r^2-2r(\cos(\theta-\phi)]$ *is the Poisson kernel for* D.

THEOREM 2.3. *Let* $F(z) = (F_{jk}(z))_{j,k=1}^{N}$, $z \in D^N$, *be an* $N \times N$ *inner matrix. Then for all* $\alpha \in D$, *except a subset of continuous logarithmic capacity zero,*

$$F_\alpha(z) = [F(z)-\alpha I][I-\bar{\alpha}F(z)]^{-1}$$

is a good inner matrix.

For N=n=1, this is *Frostman's theorem*. For N=1 and arbitrary finite n, both results are due to W. Rudin.

LEMMA 2.4. *Let* Λ *be a compact set of positive continuous logarithmic capacity and let* μ *be a positive Borel measure with* $\operatorname{supp}(\mu) \subset \Lambda$, *such that*

$$g(u) = \int_\Lambda \log^+(|\alpha-u|^{-1})d\mu(\alpha)$$

is continuous on C. *Then*

$$G(s_o,s_1,\dots,s_{N-1}) = \int_\Lambda \log^+(|\alpha^N + \sum_{k=o}^{N-1} s_k\alpha^k|^{-1})d\mu(\alpha)$$

is a bounded uniformly continuous function on C^N.

Proof. Given $(s_o,s_1,\dots,s_{N-1}) \in C^n$ we can find $u_1,u_2,\dots,u_N \in C$ (the roots of the polynomial) such that

$$\alpha^N + \sum_{k=o}^{N-1} s_k\alpha^k = \prod_{j=1}^{N} (\alpha-u_j).$$

The function g(u) has compact support in C. Therefore g(u) attains its maximum in C. Now, the continuity of G follows from the continuity of g and the inequality

$$\log^+|\prod_{j=1}^{N} c_j| \leqslant \sum_{j=1}^{N} \log^+|c_j| \ , \quad c_j \in C.$$

This also shows that

$$\max\{G(s_o,s_1,\dots,s_{N-1}):(s_o,\dots,S_{N-1}) \in C^N\} = N.\max\{g(u):u \in C\}.$$

The proof will be completed by showing that

$$\lim G(s_o,\dots,s_{N-1}) = 0, \quad \text{as} \quad \sum_{k=o}^{N-1} |s_k| \to \infty .$$

Actually, if the sum of the modulus of the s_k's tends to infinity then (by an elementary algebraic fact) at least one of the roots u_j must tend to infinity. Let $R > 0$, large enough so that $|\alpha| < R$, for all $\alpha \in \Lambda$; then

$$\lim \{G(s_o,\dots,s_{N-1}): \sum_{k=o}^{N-1} |s_k| \longrightarrow \infty\} =$$

$$= \lim \{\int_\Lambda \log^+(|\prod_{j=1}^{N}(\alpha-u_j)|^{-1})d\mu(\alpha): \sum_{j=1}^{N}|u_j| \longrightarrow \infty\} \leqslant$$

$$\leqslant \lim_{|u_1|\to\infty} \int_\Lambda \log^+(|(|u_1|-R)\prod_{j=2}^{N}(\alpha-u_j)|^{-1})d\mu(\alpha) = 0$$

qed.

Let $f(z,\alpha)$ be as in *thm. 2.2:* then defining

$$B_r(\alpha) = \int_{(\partial D)^n} \log^+(|f(rw,\alpha)|^{-1})dm_n(w); \quad B(\alpha) = \liminf B_r(\alpha), (r\to 1)$$

we have (following Rudin's proof)

$$\int_\Lambda B(\alpha)d\mu(\alpha) \overset{(1)}{\leqslant} \liminf_{r\to 1} \int_\Lambda B_r(\alpha)d\mu(\alpha) \overset{(2)}{=}$$

$$= \liminf_{r\to 1} \int_{(\partial D)^n} G(f_o(rw),\dots,f_{N-1}(rw))dm_n(w) \overset{(3)}{=}$$

$$= \int_{(\partial D)^n} G(f_o(w),\dots,f_{N-1}(w))\, dm_n(w) \overset{(4)}{=}$$

$$= \int_\Lambda \int_{(\partial D)^n} \log^+(|f(w,\alpha)|^{-1})dm_n(w)\, d\mu(\alpha) \overset{(5)}{\leqslant} \int_\Lambda B(\alpha)d\mu(\alpha)$$

(1) and (5) are justified by Fatou's lemma; (2) and (4) are consequences of Fubini's theorem; (3) follows from Lebesgue's theorem on majorized convergence and *lemma 2.4.*

We conclude that

$$\liminf_{r\to 1} \int_{(\partial D)^n} \log^+(|f(rw,\alpha)|^{-1})dm_n = \int_{(\partial D)^n} \log^+(|f(w,\alpha)|^{-1})dm_n$$

for almost every $\alpha \in \Lambda$, with respect to $d\mu$.

Now the proof of *thm. 2.2* follows as the proof of *thm. 3.6.2* in [8].

If $U(z)$ is an inner function-operator and $\alpha \in D$, then it is easy to see that

$$U_\alpha(z) = (U(z)-\alpha I)(I-\bar{\alpha}U(z))^{-1}$$

is also an inner function-operator and

$$\|U(z)-U_\alpha(z)\|_K \leq 2|\alpha|/(1-|\alpha|).$$

Let $F(z)$ be as in *thm. 2.3*. Then $(-1)^N \det(F(z)-\alpha I)$ has the same form as $f(z,\alpha)$ in *thm. 2.2*, with $f_o,\ldots,f_{N-1}$ bounded functions. On the other hand, $\log|\det(I-\bar{\alpha}F(z))|$ is bounded and n-harmonic in D^n. Hence *thm. 2.3* follows from *thm. 2.2* as *thm. 5.3.3* in [8]. The above acotation gives the following

COROLLARY 2.5. *The family of all good (Blaschke, for* $n=1$*) matrices forms a dense subset in the family of all inner matrices, in the norm-topology of* $H^\infty(K)$.

Now we are in a position to "translate" the results of [7] to the case when $2 \leq \dim K = N < \infty$. All these results are included in the following theorem; the details of the proof are left to the reader.

THEOREM 2.6. i) *Let* $U_r \in F$ *be defined by* (*) *with* $d^r(z;0,1)$ *instead of* b_θ. *Then the component of* U_1 *is isometric to the component of* U_r, *even though* U_1 *and* U_r *belong to different components for every* $r \neq 1$. *If* $0 < r < 1$, *the subsets* $\{QU_{1-r} \in$ component of $U_1\}$ *and* $\{U_{1-r}Q \in$ component of $U_1\}$ *are isometric to the whole component of* U_1.

ii) *For each* $r > 0$, *and for each* $Q \in (\det)_s$ *in the component of* U_r, $\det Q(z) = \lambda\, d^r(z;0,1)$, *for some* $\lambda \in \partial D$. *Moreover, if* $V \in (\det)_s$ *and* $Sp(V) \cap \partial D$ *is countable, then* $\det Q = \lambda \det V$, *for all* $Q \in (\det)_s$ *in the component of* V.

iii) *There exists a component of* F *containing* c *elements of* $(\det)_s$ *whose determinants are pairwise coprime. If for some* $\alpha \neq 0$, $Q = PS$ *and* $Q_\alpha = PR$, *where* $P,S,R \in F$, *then* $P \in (\det)_o$.

iv) $(\det)_s$ *is closed and nowhere dense in* F. $(\det)_B$ *is dense, but not open in* F.

v) *The* "τ_o-diameter" *of each component of* $(F^\sim,\tau)\setminus\{H^2_K\}$ *is* 2 *and no component of* $(F^\sim,\tau)\setminus\{H^2_K\}$ *(or* $F\setminus(\det)_o$*) is compact.*

vi) *If* B_t (B_θ, resp.) $\in (\det)_B$ *and* $\det B_t = b_t$, *as defined in* [7, *thm. 2.1*] ($\det B_\theta = b_\theta$, *as defined in* [7, *thm. 2.2*], *resp.*), *then the component of* B_t (B_θ, *resp.*) *in* F *is contained in* $(\det)_B$. *Moreover, if* $0 \leq t < t' < \infty$ ($0 \leq \theta < \theta' \leq \pi/4$, *resp.*), *then* B_t *and* $B_{t'}$ (B_θ *and* $B_{\theta'}$, *resp.*) *belong to different components.*

However, the direct analog of [7, *thm. 1.3*] is false; in fact we have:

PROPOSITION 2.7. *If* $U \in (\det)_s$ *is non-constant, then the component of* UH^2_K *in* $(F^{\sim},d)$ *contains* c *elements of* $(\det)^{\sim}_s$.

Proof. If U does not have the form qX for some inner function q and $X \in (\det)_o$, then for each $Y \in (\det)_o$ we write $U_Y(z) = YU(z)Y^*$. Clearly, U_Y is arcwise connected with U in F and therefore $U_Y H^2_K$ is arcwise connected with UH^2_K in $(F^{\sim},d)$; moreover $\det U = \det U_Y$. Thus, it only remains to show that $Y \to U_Y H^2_K$ is not a constant mapping from $(\det)_o$ into $F^{\sim}$.

The set of all $\lambda \in \partial D$ such that the non-tangential limits $U(\lambda)$ are well-defined and unitary has Lebesgue measure (dm) one; let $\Gamma = \Gamma(U)$ denote such set. Clearly, $\Gamma(U_Y) = \Gamma(U)$, for all $Y \in (\det)_o$.

Assume that $U_Y H^2_K = UH^2_K$ for all $Y \in (\det)_o$. Since $UH^2_K = UXH^2_K$ for any $X \in (\det)_o$, without loss of generality we can assume that $U(\lambda_o) = I$, for some $\lambda_o \in \Gamma$. Then $U_Y H^2_K = UH^2_K$ implies that, for all $\lambda \in \Gamma$, $YU(\lambda) = U(\lambda)X$, for some $X = X(Y) \in (\det)_o$, independent of $\lambda \in \Gamma$. For $\lambda = \lambda_o$, the above equality implies: $X(Y) = Y^* = Y^{-1}$, i.e., $U(\lambda)$ commutes with Y for all $Y \in (\det)_o$. Therefore, $U(\lambda) = u(\lambda)I$ for some inner function u and for $\lambda \in \Gamma$, contradicting our assumption about U. We conclude that $Y \to U_Y H^2_K$ is a non-constant map.

Finally, if $U = qX$, then $U = U_1 U_2 X$, where

$$U_1 = \begin{pmatrix} q & 0 \\ 0 & I_{N-1} \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} I_1 & 0 \\ 0 & qI_{N-1} \end{pmatrix} \qquad (N = \dim K > 1)$$

and the result follows by applying the previous argument to $U'_Y = (YU_1Y^*)U_2X$, $Y \in (\det)_o$. qed.

Replacing "Blaschke products" by "good inner functions", many of our results are still true for inner functions (or inner matrices) of several complex variables in the unit polydisc of C^n.

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