Revista de la
Unión Matemática Argentina
Volumen 28, 1976.

INTERPOLATION BETWEEN TWO PUTNAM'S INEQUALITIES

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1. Let $T = H + iJ$ be the cartesian decomposition ($H = \text{Re } T$, $J = \text{Im } T$) of the (bounded linear) operator T acting on a complex separable Hilbert space $\mathcal{H}$. T is called *hyponormal* if its self-commutator

$$D = T^*T - TT^* = 2i(HJ - JH)$$

is a positive semi-definite hermitian operator.

Let $\Lambda(T)$ denote the spectrum of T and let m_1 and m_2 denote the linear Lebesgue measure (on a given line) and planar Lebesgue measure on the complex plane C, respectively. C.R.Putnam ([5]) proved the following

$$(1) \qquad \pi\|D\| \leqslant m_2[\Lambda(T)] \qquad \text{(Putnam's inequality)}.$$

If H has simple spectrum (in the sense of [3]), then (1) can be sharpened to (see [6])

$$(2) \qquad \pi|D|_1 \leqslant m_2[\Lambda(T)]$$

where $|D|_p$, $1 \leqslant p \leqslant \infty$, denotes the p-norm of an element of the Schatten's bilateral ideal C_p of compact operators on $\mathcal{H}$ (C_1 denotes *trace class* operators, C_2 denotes the class of *Hilbert-Schmidt* operators, C_∞ is the ideal of all *compact* operators, etc., The reader is referred to [8] for the definition and properties of these ideals).

Actually, Putnam has also obtained the following improvement of (2): Let $H = \int \lambda \, dE_\lambda = \bigoplus_{n=1}^{\infty} \int_{B_n} \lambda \, dE_\lambda$ be the spectral decomposition of H, where $\Lambda(H) = \cup \{B_n : 1 \leqslant n \leqslant \aleph_o\}$ is the (essentially unique) decomposition of the spectrum of H into pairwise disjoint Borel subsets such that $H_n = \int_{B_n} \lambda \, dE$ is an hermitian operator of *uniform spectral multiplicity* n on the space $E(B_n)\mathcal{H}$ (see [3]) and let $F(t) = m_1[\Lambda(T) \cap \{z: \text{Re } z = t\}]$; then

$$(3) \qquad \pi|D|_1 \leqslant \sum(1 \leqslant n \leqslant \aleph_o)\; n\int_{B_n} F(t)\, dt.$$

Clearly, we can restrict our attention to the case when $B_{\aleph_0} = \phi$ (see [7]).

In this note, the following two results will be proven:

THEOREM 1. *Let* $T = H + iJ$ *be a hyponormal operator such that* $B_{\aleph_0} = \phi$. *Then* $D \in C_\infty$.

THEOREM 2. (Interpolation theorem) *Let* T *be as above. Then*

(i) $\pi |D|_p \leq \sum_n n^{1/p} \int_{B_n} F(t)dt \leq \sum_n [n \int_{B_n} F(t)^p dt]^{1/p} \{m_1[\Lambda(H)]\}^{1/q}$

for all p, $1 < p < \infty$, *where* $q = p/(p-1)$.

(ii) *Let* $0 < p < 1$. *If* $|D|_p^{\,p}$ *denotes the invariant metric of the ideal* C_p, *then*

$$\pi^p |D|_p^{\,p} \leq \sum_n n^p [\int_{B_n} F(t)\, dt]^p \leq \sum_n n^p \int_{B_n} F(t)^p\, dt.$$

COROLLARY. *Let* T *be as above and assume that the left spectrum coincides with the right one. Then there exist a normal operator* N *and a sequence* $\{F_n\}_{n=1}^{\infty}$ *of finite rank operators such that* $\Lambda(N) = \Lambda(T)$, $-1 \notin \Lambda(F_n)$ $(n = 1,2,\dots)$ *and*

$$\lim_{n\to\infty} \|T - (I+F_n)N(I+F_n)^{-1}\| = 0 .$$

The proof follows from *Theorem 1* and [4, §2].

REMARKS. (a) The condition $B_{\aleph_0} = \phi$ is sufficient, but not necessary. Indeed, a concrete example of a *completely hyponormal* operator T (i.e., there is no non-zero reductive subspace M such that the restriction $T|M$ is normal in M) such that $B_{\aleph_0} = \Lambda(H)$ and $B_n = \phi$ $(n = 1,2,\dots)$, but $D \in C_1$ can be found in [1]. *Theorem 1* affirmatively answers the author's *Conjecture* (*b*) of [4].

(b) *Example.* Let $\mathcal{J}$ be an arbitrary bilateral ideal properly contained in C_∞. Then there exists a sequence $\{r_n\}_{n=1}^{\infty}$ decreasing to 0 such that no (necessarily compact!) hermitian operator of the form $L = \sum_{n=1}^{\infty} r_n \varphi_n \otimes \varphi_n$ (where $\{\varphi_n\}$ is a suitable orthonormal system of $\mathcal{H}$ and the operator $\varphi \otimes \varphi$ is defined by $\varphi \otimes \varphi(\psi) = (\psi,\varphi)\varphi$; see [8]) belongs to $\mathcal{J}$. Let c_n be the positive square root of r_n and set $T = \bigoplus_{n=1}^{\infty} c_n S$, where S denotes the unilateral shift; then

$$D = T^*T - TT^* = \oplus_n c_n^2 [S^*S - SS^*] = \sum_n r_n \varphi_n \otimes \varphi_n$$

does not belong to J. In this example, $B_n = [-c_n, c_n] \setminus [-c_{n+1}, c_{n+1}]$ $(n = 1,2,\ldots)$.

2. PROOF OF THEOREM 1.

Let $E_N = E(\bigcup_{n=1}^{N} B_n)$. Following [7], we have

$$D = D^{1/2} I D^{1/2} = D^{1/2} E_N D^{1/2} + D^{1/2}(I - E_N) D^{1/2}$$

and

$$|D^{1/2} E_N D^{1/2}|_1 = |\sum_{n=1}^{N} D^{1/2} E(B_n) D^{1/2}|_1 = |\sum_{n=1}^{N} E(B_n) D E(B_n)|_1 =$$

$$= \sum_{n=1}^{N} |E(B_n) D E(B_n)|_1 \leqslant \sum_{n=1}^{N} (n/\pi) \int_{B_n} F(t)\, dt < \infty$$

and therefore $D^{1/2} E_N D^{1/2} \in C_1 \subset C_\infty$.

On the other hand (by [5;6;7])

$$\|D^{1/2}(I - E_N) D^{1/2}\| = \|(I - E_N) D (I - E_N)\| \leqslant (1/\pi) m_2[\Lambda(T) \setminus \bigcup_{n=1}^{N} B_n] \to 0$$

as $N \to \infty$, because $m_2[\Lambda(T)] < \infty$ and $m_2(B_{\aleph_0}) = 0$.

We conclude that D is the norm limit of a sequence $\{D_n\}$ of compact operators and therefore, D itself is compact.

3. PROOF OF THEOREM 2.

(i) As in [7], $D = \sum_{n=1}^{\infty} D^{1/2} E(B_n) D^{1/2} = \sum_n D_n$ where

$$|D_n|_1 = |E(B_n) D E(B_n)|_1 \leqslant (n/\pi) \int_{B_n} F(t)\, dt \,. \tag{4}$$

On the other hand, Putnam's inequality (1) implies that

$$\|D_n\| = |D_n|_\infty = |E(B_n) D E(B_n)|_\infty \leqslant (1/\pi) \int_{B_n} F(t)\, dt \tag{5}$$

so that we can interpolate the C_p-norm between (4) and (5) in order to obtain

$$\pi |D_n|_p \leqslant \pi |D_n|_1^{1/p} |D_n|_\infty^{1/q} \leqslant n^{1/p} \int_{B_n} F(t)\, dt \leqslant n^{1/p} [\int_{B_n} F(t)^p dt] .$$

$$.[m_1(B_n)]^{1/q}, \quad \text{for all } p,\ 1 < p < \infty \text{ (see [8]).}$$

Therefore

$$\pi |D|_p \leqslant \pi \sum_n |D_n|_p \leqslant \sum_n n^{1/p} \int_{B_n} F(t)\, dt \leqslant$$

$$\leqslant \sum_n [\int_{B_n} n\, F(t)^p\, dt]^{1/p} . [m_1(B_n)]^{1/q} \leqslant$$

$$\leqslant \{m_1[\Lambda(H)]\}^{1/q} \sum_n [\int_{B_n} n\, F(t)^p\, dt]^{1/p} .$$

(ii) For $0 < p < 1$, we have

$$\pi^p |D|_p^p \leqslant \pi^p \sum_n |D_n|_p^p \leqslant \pi^p \sum_n |D_n|_1^p \leqslant \sum_n n^p \, [\int_{B_n} F(t)\, dt]^p \leqslant$$

$$\leqslant \sum_n n^p \int_{B_n} F(t)^p\, dt \ .$$

The author suggests that analogous results should be true for the trace estimates of C.A.Berger and B.I.Shaw ([1;2]).

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Recibido en abril de 1975.