Revista de la
Unión Matemática Argentina
Volumen 28 , 1976 .

MULTIPLIERS WITH RESPECT TO ABEL-BOUNDED SPECTRAL MEASURES IN LOCALLY CONVEX SPACES

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1. INTRODUCTION.

In this note, which can be regarded as Part II of [7], we would like to continue our previous investigations on multipliers in abstract spaces. Whereas ([1], III) was concerned with Abel-bounded, discrete expansions in Banach spaces, ([2], I) with Riesz-bounded, continuous spectral measures in Banach spaces, and ([5];[7]) with Cesàro-bounded, discrete expansions in locally convex spaces, it is the purpose of this note to consider the more general situation of Abel-bounded spectral measures in locally convex spaces. To this end, Section 2 presents the general theory. Section 3 deals with some examples of weight spaces in connection with the (continuous) Fourier spectral measure on $L^2(\mathbf{R})$ as well as with (discrete) expansions into Hermite and ultraspherical polynomials. This would also enable one to discuss certain fundamental problems in approximation theory such as the comparison of processes, saturation, Bernstein-type inequalities etc. in connection with concrete examples of multiplier operators such as the Weierstrass or Picard means. However, in carrying out these applications one would proceed as in the Banach space frame so that one can refer the reader to [1], [2], [5], [7] and the literature cited there.

The authors would like to express their sincere gratitude to Professor P.L. Butzer for a critical reading of the manuscript.

2. GENERAL THEORY.

Let $(X,\{p_r\}_{r\in J})$, J being an arbitrary index set, denote a complete, locally convex Hausdorff space whose topology T is generated by a family of filtrating seminorms p_r. Let $[X]$ be the class of all continuous linear operators of X into itself. As in the Banach space frame (cf. [1],III; [2],I) it is appropriate to look for some auxiliary Hilbert space H with $H \cap X$ nonempty and dense in X. The operators are then assumed to be generated by some spectral measure E on H via the following procedure:

* Work supported in part by DFG grant Ne 171/1 which is gratefully acknowledged.

Let Σ be the family of Borel sets σ in $\mathbf{R}$, the set of real numbers, and let E be a spectral measure for H on $\mathbf{R}$, i.e. $E(\sigma) \in [H]$ for each $\sigma \in \Sigma$ and (ϕ being the void set, id the identity mapping)

$$E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2) \quad \text{for all } \sigma_1,\sigma_2 \in \Sigma ,$$

$$(2.1) \qquad E(\phi) = 0 , \qquad E(\mathbf{R}) = \mathrm{id} ,$$

$$E\Big(\bigcup_{j=1}^{\infty} \sigma_j\Big) = \sum_{j=1}^{\infty} E(\sigma_j) \text{ with } \sigma_j \in \Sigma,\ \sigma_j \cap \sigma_k = \phi \text{ for } j \neq k .$$

Let $L^\infty(\mathbf{R};E)$ denote the set of E-essentially bounded Borel measurable functions τ on $\mathbf{R}$. Then the integral $\int_{\mathbf{R}} \tau(u)dE(u)$ is well defined in the strong operator topology as an element of $[H]$ (cf. [3],II, p.900).

A function $\tau \in L^\infty(\mathbf{R};E)$ is called a multiplier on $(X,\{p_r\})$ if to each $f \in H \cap X$ there exists $f^\tau \in H \cap X$ such that

$$f^\tau = \int_{-\infty}^{\infty} \tau(u)dE(u)f$$

and to each $r \in J$ there exist $t \in J$ such that

$$p_r(f^\tau) \leq B(r,t;\tau)p_t(f) \qquad (f \in H \cap X)$$

with constant $B(r,t;\tau) > 0$ independent of f. Then the operator $T:(H \cap X,\{p_r\}) \to (H \cap X,\{p_r\})$, defined via $Tf:=f^\tau$, has continuous linear extension to all of X, thus $T \in [X]$. The set of all multipliers τ on $(X,\{p_r\})$ is denoted by M. In the same way, following the approach in Banach spaces (see [1],III;[2],I), one may construct certain closed linear operators B^ψ defined on some subspace $X^\psi \subset X$ corresponding to some function $\psi \in L^\infty_{loc}(\mathbf{R};E)$ which is allowed to have e.g. polynomial growth at infinity.

To derive a multiplier criterion based on Abel summability let us consider for $f \in H \cap X$, $y > 0$

$$(2.2) \qquad P(f;y) := \int_{-\infty}^{\infty} \exp\{-y|u|\}dE(u)f$$

The spectral measure E is called Abel-bounded on $(X,\{p_r\})$ if $P(f;y)$ belongs to $H \cap X$ for all $f \in H \cap X, y > 0$, and if to each $r \in J$ there exists $t \in J$ such that

$$(2.3) \qquad p_r(P(f;y)) \leq A(r,t)p_t(f) \qquad (f \in H \cap X \ ; \ y > 0)$$

with constant $A(r,t) > 0$ independent of f and y.

In this connection let us consider the following class of functions (cf. [1],III):

$$(2.4) \qquad CBV := \{\lambda \in C[0,\infty] , \ \lambda(x)-\lambda(\infty) = \int_0^{\infty} \exp\{-xy\}db(y) \text{ for some } b \in BV[0,\infty]\}$$

$$\|\lambda\|_{CBV} := \int_0^\infty |db(y)| + |\lambda(\infty)|$$

where $\lambda(\infty) := \lim_{x\to\infty} \lambda(x)$, and $C[0,\infty]$ and $BV[0,\infty]$ are the sets of functions which are continuous and of bounded variation on $[0,\infty]$, respectively.

THEOREM. *If* E *is an Abel-bounded spectral measure for* $(X,\{p_r\})$ *and if* $\lambda \in CBV$, *then* $\lambda(|u|) \in M$.

Proof. For each $f \in H \cap X$ set (cf. [1],III; [2],I)

$$f^\lambda := \int_0^\infty P(f;y)db(y) + \lambda(\infty)f$$

As $\{p_r\}_{r\varepsilon J}$ is filtrating, it follows by (2.3) that to each $r \in J$ there exist $t \in J$ and $s=s(r,t) \in J$ such that

$$p_r(f^\lambda) \leqslant \int_0^\infty p_r(P(f;y))|db(y)| + |\lambda(\infty)|p_r(f) \leqslant$$

$$\leqslant A(r,t)p_t(f) \int_0^\infty |db(y)| + |\lambda(\infty)|p_r(f) \leqslant$$

$$\leqslant A(r,t)\|\lambda\|_{CBV}\, p_s(f)$$

Furthermore, we have for $f \in H \cap X$ by (2.1), (2.2) and (2.4) that

$$f^\lambda = \int_0^\infty\int_{-\infty}^\infty \exp\{-y|u|\}dE(u)f\; db(y) + \lambda(\infty)f =$$

$$= \int_{-\infty}^\infty dE(u)f\left(\int_0^\infty \exp\{-y|u|\}db(y) + \lambda(\infty)\right) = \int_{-\infty}^\infty \lambda(|u|)dE(u)f$$

which completes the proof.

3. APPLICATIONS.

As already indicated in the introduction, we shall here confine ourselves to a short description of how to construct suitable locally convex spaces X in connection with classical orthogonal expansions. Let us commence with some general remarks (see also [6],[7]).

The complete locally convex Hausdorff spaces $(X,\{p_r\}_{r\varepsilon J})$ to be considered are representable as projective or inductive limits of Banach spaces. Thus let

$$X_r := L^p(a,b;U_r(x)), \quad 1 \leqslant p < \infty, \quad -\infty \leqslant a < b \leqslant \infty,$$

$$\|f\|_{p,U_r} := p_r(f) := \left\{\int_a^b |f(x)|^p U_r(x)dx\right\}^{1/p}$$

denote the usual Banach space of measurable functions, pth power integrable with respect to the weight $U_r(x) \geqslant 0$, $r \in J$. If e.g. J is some open set in R, one may define complete locally convex Hausdorff spaces via

$$X_{P,J} := \bigcap_{r\varepsilon J} X_r \text{ and } X_{U,J} := \bigcup_{r\varepsilon J} X_r$$

If, furthermore, J can be replaced by a countable set without affecting the topology, $X_{P,J}$ is a countably normed space if the norms $\{p_r\}_{r\varepsilon J}$ are in concordance (cf. [4], p.5). Concerning $X_{U,J}$, this would lead to an analogous study of countable union spaces (of distributions) (cf. [4], p.20). However, we shall confine ourselves in the following to product spaces $X_{P,J}$.

3.1. CONTINUOUS SPECTRA.

Let $H := L^2(-\infty,\infty;1) := L^2(R)$ and define the Fourier transform $F[f]$ of $f \in H$ by

$$\lim_{N\to\infty} \| F[f](v) - \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(u)e^{-ivu}du \|_H = 0$$

Let F^{-1} be its inverse and P_σ, $\sigma \in \Sigma$, be the multiplication projection

$$P_\sigma f(u) := p_\sigma(u)f(u), \; p_\sigma(u) := \begin{cases} 1, & u \in \sigma \\ 0, & u \notin \sigma \end{cases}$$

Setting, for arbitrary $\sigma \in \Sigma$,

$$E(\sigma) := F^{-1}P_\sigma F \tag{3.1}$$

it is a familiar fact (cf. [3],III, p.1989) that E is a spectral measure on H.

In order to construct suitable locally convex spaces X, let us consider spaces $L^p(-\infty,\infty;U_r(x))$ in connection with the Poisson integral.

$$f(x,y) := \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x-u)^2+y^2}\,du \qquad (y > 0)$$

Then a result of Muckenhoupt [10] asserts:

Let μ and ν be Borel measures and $1 \leqslant p < \infty$, $y > 0$. There exists a constant $C(\mu,\nu) > 0$ independent of f such that

$$\int_{-\infty}^{\infty} |f(x,y)|^p d\mu(x) \leqslant C(\mu,\nu) \int_{-\infty}^{\infty} |f(x)|^p d\nu(x) \tag{3.2}$$

if and only if for each interval $I \subset R$ with length $|I|$ there exists a constant B, independent of I, such that

$$A_1 := \left(\frac{\mu(I)}{|I|}\right) \operatorname*{ess\,sup}_{x\in I} \left(\frac{d\nu_a(x)}{dx}\right)^{-1} \leqslant B \qquad (p = 1)$$

(3.3)

$$A_p := \left(\frac{\mu(I)}{|I|}\right)\left(\frac{1}{|I|}\int_I \left[\frac{d\nu_a(x)}{dx}\right]^{-1/(p-1)} dx\right)^{p-1} \leqslant B \qquad (1 < p < \infty),$$

ν_a *denoting the absolute continuous part of* ν. A similar version holds for 2π-periodic functions.

To give an example, (3.3) is true with $d\mu(x) = U_r(x)dx$, $d\nu(x) = U_t(x)dx$, and $U_r(x) = (1+|x|)^r$ for $r \leqslant t$ and $r,t \in J = J(p) := (-1,p-1)$ if $1 \leqslant p < \infty$, and additionally for $r=0$ if $p=1$. Obviously, the Hilbert space $H = L^2(R)$ is dense in $L^p(-\infty,\infty;(1+|x|)^r)$, $1 \leqslant p < \infty$ and $r \in J(p)$, and therefore in $X := \bigcap_{r\in J} L^p(-\infty,\infty;(1+|x|)^r)$. Moreover, for the operator $P(f;y)$ in (2.2) we have for all $f \in H \cap X$

$$P(f;y)(x) = \int_{-\infty}^{\infty} \exp\{-y|v|\}F[f](v)e^{ixv}dv = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{y^2+(x-u)^2}\,du$$

so that the Fourier spectral measure (3.1) is Abel-bounded on X by (3.3).

3.2. DISCRETE SPECTRA.

Let H be an arbitrary Hilbert space and $\{P_k\}_{k\in \mathbf{P}} \subset [H]$, $\mathbf{P}$ being the set of all non-negative integers, a complete system of mutually orthogonal projections of H into itself, so that each $f \in H$ admits an expansion

$$f = \sum_{k=0}^{\infty} P_k f \qquad (f \in H)$$

Then a spectral measure E on H may be defined by

$$E(\sigma) := \sum_{k\in\sigma} P_k \tag{3.4}$$

σ being an arbitrary Borel set of $\mathbf{R}$. Given a locally convex space $(X,\{p_r\})$ such that $H \cap X$ is dense in X, the operators T to be considered are then generated via expansions of type

$$Tf = \sum_{k=0}^{\infty} \tau(k)P_k f \qquad (f \in H \cap X)$$

To treat concrete examples, let $H_k(x)$ denote the kth Hermite polynomial given via

$$\sum_{k=0}^{\infty} \frac{H_k(x)}{k!} s^k = \exp\{2xs-s^2\}$$

On $H := L^2(-\infty,\infty;\exp\{-x^2\}) := L^2_w$ with $w(x) := \exp\{-x^2\}$ we define projections P_k via

$$(3.5)\quad (P_k f)(x) := \left(\frac{1}{\sqrt{\pi}2^k k!}\int_{-\infty}^{\infty} f(u)H_k(u)\exp\{-u^2\}du\right)H_k(x) \qquad (k \in P)$$

Note that P_k is in general not defined on L_w^p, $p \neq 2$. However, the projections $\{P_k\}_{k\varepsilon P}$ belong to $[L_w^2]$ and are mutually orthogonal. Indeed, a well known result of Pollard [11] states that for each $f \in H$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\left|\sum_{k=0}^{n}(P_k f)(x) - f(x)\right|^2\exp\{-x^2\}dx = 0$$

and this fails to hold for $p\neq 2$. Furthermore, $P_k(H) \subset L_w^p$ and the set Π of all finite linear combinations of the Hermite polynomials $H_k(x)$ is dense in L_w^p, $1 \leq p < \infty$ (cf. [1],III and the literature cited there). With

$$f(x,v) := \int_{-\infty}^{\infty} P(v,x,u)f(u)\exp\{-u^2\}du$$

$$P(v,x,u) := \sum_{k=0}^{\infty}\frac{v^k H_k(x)H_k(u)}{\sqrt{\pi}\ 2^k k!}$$

it follows that for $f \in \Pi$ (cf. [8])

$$\|f(x,v)\|_{p,w} = \left\|\sum_{k=0}^{\infty} v^k (P_k f)(x)\right\|_{p,w} \leq \|f\|_{p,w} \qquad (1 \leq p < \infty)$$

Thus the spectral measure (3.4) corresponding to (3.5) is Abel-bounded on the complete locally convex space $X = \bigcap_{p\varepsilon J} L_w^p$ for some open $J \subset [1,\infty)$.

Analogous results are true for expansions into Laguerre polynomials.

Finally we examine expansions into ultraspherical polynomials $C_n^\lambda(x)$, $\lambda > 0$, where

$$\sum_{n=0}^{\infty} s^n C_n^\lambda(x) = (1-2xs+s^2)^{-\lambda} \qquad (x \in (-1,1))$$

Given the Hilbert space $H := L^2(0,\pi;\sin^{2\lambda}\theta) := L^2_{m_\lambda}$ with $m_\lambda(\theta) := \sin^{2\lambda}\theta$, each $f \in H$ has an ultraspherical expansion (in H)

$$(3.6)\qquad f(\theta) = \sum_{k=0}^{\infty} a_k C_k^\lambda(\cos\theta) := \sum_{k=0}^{\infty}(P_k^\lambda f)(\theta)$$

$$a_k := \gamma_k\int_0^\pi f(\theta)C_k^\lambda(\cos\theta)m_\lambda(\theta)d\theta,\quad \gamma_k := \frac{[\Gamma(\lambda)]^2(k+\lambda)k!}{2^{1-2\lambda}\pi\Gamma(k+2\lambda)}$$

By a condition similar to (3.3) it follows for the Abel means

$$f(\theta,v) := \sum_{k=0}^{\infty} v^k (P_k^\lambda f)(\theta) = \int_0^\pi P(v,\theta,\phi)f(\phi)m_\lambda(\phi)d\phi$$

$$P(v,\theta,\phi) := \sum_{k=0}^{\infty} v^k \gamma_k C_k^\lambda(\cos\theta) C_k^\lambda(\cos\phi)$$

that for $f(\theta)\sin^{2\lambda}\theta$ integrable on $[0,\pi]$ one has (cf. [9])

$$\int_0^\pi |f(\theta,v)|^p U_r(\theta)d\theta \leqslant C(r) \int_0^\pi |f(\theta)|^p U_r(\theta)d\theta \qquad (1 \leqslant p < \infty)$$

for e.g. the weights

$$(3.7) \qquad U_r(\theta) = \theta^r \quad \text{for} \quad \begin{cases} r \in J(p) := (-1,p-1), & 1 < p < \infty \\ r \in J(1) := (-1,0], & p=1 \end{cases}$$

$$(3.8) \qquad U_r(\theta) = (\tan\theta/2)^r \quad \text{for} \quad \begin{cases} r = 0 & p=1 \\ r \in J(p) := (-1,p-1) \cap (1-p,1) & 1<p<\infty \end{cases}$$

Defining the Banach spaces

X_r^λ := $\{f; f(\theta)\sin^{2\lambda}\theta$ integrable on $[0,\pi]$ such that

$$p_r(f) := \left(\int_0^\pi |f(\theta)|^p U_r(\theta)d\theta\right)^{1/p} < \infty\},$$

$P_k^\lambda(H) \subset X_r^\lambda$ and the set Π of all finite linear combinations of the ultraspherical polynomials C_k^λ (cos θ) is dense in X_r^λ, $1 \leqslant p < \infty$. Therefore the spectral measure (3.4) corresponding to (3.6) is Abel-bounded (with r=t) on the complete locally convex Hausdorff space $X_{p,J} := \bigcap_{r\in J} X_r^\lambda$ for each open set $J \subset J(p)$ according to (3.7) or (3.8), respectively. Thus again our theorem may be applied.

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Recibido en octubre de 1975.