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SOME CLASSES OF RINGS DEFINED BY PROPERTIES OF MODULES

B. S. Chwe and J. Neggers

INTRODUCTION. Suppose that A is a ring with 1 and suppose all modules are right unitary. In [2], the authors identified the class of local right perfect rings (right Steinitz rings) via the following property:
P_0: every linearly independent subset of a free module can be extended to a basis by adjoining elements of a given basis.

In [2], the authors also proved that a ring A is a Steinitz ring if and only if the maximal ideal R is left vanishing or left T-nilpotent in the sense that for any infinite sequence $\{x_i\}$ of elements of R there is n such that $x_n.x_{n-1}\ldots x_1 = 0$. In [1], Bass showed that right perfect rings need not be left perfect and his example, which actually involved a right Steinitz ring, shows that property P_0 is not symmetric either. Some properties below will not be symmetric because of this example. For convenience of discussion however, we shall drop the prefix "right" and refer to perfect rings, Steinitz rings, T-nilpotent sets, etcetera, with the understanding that every property or class of rings under discussion possesses such a prefix. One of the properties of Steinitz rings, which is characteristic in the class of local rings, is the following property:
P_1: Every module has a minimal generating set.

It is the purpose of this paper to discuss several properties listed below and to identify in each case the class of rings satisfying this property.

P_2: Every minimal generating set of a finite free module is a basis (local rings);

P_3: Every minimal generating set of a free module is a basis (local rings);

P_4: (In the class of local rings) Every maximal linearly independent subset of a finite free module is a basis (every finite set of non-units has a non-zero right annihilator);

P_5: Every maximal linearly independent subset of a free module is a basis (Steinitz rings).

Although properties P_2, P_3, P_4 and P_5 are related to property P_1

the precise relations connecting these properties are not yet clear. In another paper the second author discusses a class of rings which is at least conjectured to be the class of rings having property P_1.

PROPERTIES P_2 AND P_3.

We shall establish that property P_2 implies that A is local, and that if A is local then it satisfies property P_3.

Suppose A has property P_2. Let R be a maximal right ideal and suppose $x \in R$. We assert that x is a unit.

Since $x \notin R$, $1 = m + xa$ for some $m \in R$, and $\{m,x\}$ is a generating set of the free module A. Hence, either $\{m,x\}$ or $\{x\}$ is a minimal generating set, since $m \in R$.

If $\{m,x\}$ is a minimal generating set, then $mx + x(1-ax) = 0$ implies $x = 0$ and $1-ax = 0$, a contradiction. Thus, $\{x\}$ is itself a minimal generating set and $xy = 1$ for some y. Therefore, $x\ (1-yx) = 0$ implies $1 = yx$, i.e., x is a unit. We now show that if A is local, then it satisfies P_3.

Suppose that A is a local ring with maximal ideal R.

PROPOSITION. *If* S *is a minimal generating set of the module* M *and if* $\varphi\colon M \to M/MR$ *is the canonical map, then the restriction of* φ *to* S *is an injection and* $\varphi(S)$ *is a basis of* M/MR *as an* A/R*-espace.*

Proof. Suppose $\varphi(S)$ is not linearly independent. Then, say $\sum \varphi(s_i)\, a_i = 0$ with $a_1 \notin R$. Since $\sum s_i a_i \in MR$, $\sum s_i a_i = \sum s_j r_j$, where $r_j \in R$.

Hence $s_1 = \sum_{j\geq 2} s_j(r_j-a_j).(a_1-r_1)^{-1}$, contradicting the minimality of S.

Now suppose M, φ and S are as in the proposition and suppose M is free. We claim that S is linearly independent and hence a basis. Let $\{s_1,\dots,s_t\}$ be any finite subset of S and let $\{u_i | i \in I\}$ be a basis of M. Set $s_\ell = \sum u_i a_{i\ell}$.

Since $\varphi(s_\ell) \neq 0$, some $a_{i\ell}$ is not in R. Say $a_{1\ell} = 1$. Then $u_1 = s_1 - \sum_{i>1} u_i a_{i1}$. Whence, $\{s_1\} \cup \{u_i | i \neq 1\}$ is a basis of M. By induction say $s_n = s_1 a_1 + \dots + s_{n-1}a_{n-1} + \sum_{i\geq n} u_i a_{in}$, and $\{s_1,\dots,s_{n-1}\} \cup \{u_j | i \in (1,\dots,n-1)\}$ is a basis. Then, some a_{in} is not in R, say $a_{nn} \notin R$, since otherwise $s_n \equiv s_1a_1+\dots+s_{n-1}a_{n-1}$ (mod MR), a contradiction. Thus, $\{s_1,\dots,s_n\}$ is part of a basis, and, in particular, the set is linearly independent. Hence $\{s_1,\dots,s_t\}$ is also linearly independent and the assertion follows.

PROPERTY P_4.

We begin the discussion with a lemma.

LEMMA. *Let* A *be a ring with* 1. *Then, each maximal linearly independent subset of the module* A *generates* A *if and only if each left non-zero-divisor of* A *has a right inverse. Thus, in such rings every non-unit is a left zero-divisor.*

Proof. If each left non-zero-divisor is a unit, then it is clear that if $\{a\}$ is a linearly independent subset of the module A, then a is a unit and $aA = A$.

Conversely, suppose each maximal linearly independent subset of A generates A. Then, each maximal linearly independent subset of A is finite. Indeed, if $u_1, u_2, \ldots$ is a basis of A then $1 = \sum_{i=1}^{n} u_i x_i$ for some n and $u_{n+1} = \sum_{i=1}^{n} u_i(x_i u_{n+1})$, contradicting the linear independence of the basis $\{u_1, u_2, \ldots\}$.

If $\{a,b\}$ is a linearly independent subset of A, then $\{b, ab, a^2b, \ldots, a^nb, \ldots\}$ is an infinite linearly independent subset of A. Indeed, if $\sum_{i=s}^{t} a^i bx_i = 0$, for some $x_i \in A$, then $\sum_{i=s}^{t} a^{i-s} bs_i = 0$ and $bx_s + a \sum_{i=s+1}^{t} a^{i-s-1} bx_i = 0$. Hence $x_s = 0$ and $\sum_{i=s+1}^{t} a^{i-s-1} bx_i = 0$. By induction on t-s it follows that $x_s = x_{s+1} = \ldots = x_t = 0$. Now this contradicts the fact that any linearly independent set is finite. Hence, if $a \in A$ is not a left zero-divisor, then $aA = A$, whence $ax = 1$ for some $x \in A$. Since x is not a left zero-divisor either, we have $xy = 1$ for some $y \in A$ and thus $y = a$, i.e., a is a unit. The lemma follows.

Now suppose A has property P_4. Then, by the lemma, every left non-zero divisor is a unit, and this is the case of a free module generated by one element. In general we have the following theorem:

THEOREM 1.*Let* A *be a local ring. Then, each maximal linearly independent subset of a free module with a basis of at most* (n-1) *elements is a basis if and only if, for each set* $\{x_1, \ldots, x_t\}$ *of non-units, where* $1 \leqslant t \leqslant n-1$, *there is an element* $y \in A$ *such that* $x_1 y = \ldots = x_t y = 0$, $y \neq 0$.

Proof. We proceed by induction on n. Suppose the theorem holds for $n \leqslant r$. Let $\{u_1, \ldots, u_r\}$ be a basis of the free module M, and suppose $\{v_1, \ldots, v_s\}$ is a maximal linearly independent subset of M. Set $v_i = \sum_{\ell=1}^{r} u_\ell T_{\ell i}$, where $T_{\ell i} \in A$. To prove the "if" part, assume A is a local ring such that any r non-units of A have a common annihilator. Then, v_1 cannot be a free unless one of the $T_{\ell 1}$ is a unit. Say T_{11} is a unit. Then, $\{v_1, u_2, \ldots, u_r\}$ is a basis and we may take $\{v_2, \ldots, v_s\}$ as a maximal linearly independent subset of the free module with basis $\{u_2, \ldots, u_r\}$. Hence, by the induction hypothesis $\{v_2, \ldots, v_s\}$ is a basis of this module and $\{v_1, \ldots, v_s\}$ is a basis of M. Conversely, to prove the "only if" part, suppose A is a local ring such that each maximal linearly independent subset of a free module with a basis of less than r+1 elements, is a basis.

Let M be a free module with basis $\{u_1, \ldots, u_r\}$ and a maximal linearly independent subset $\{v_1, \ldots, v_s\}$ as above. Furthermore, let

$$v_i = \sum_{\ell=1}^{r} u_\ell T_{\ell i}.$$

If none of the $T_{\ell i}$ is a unit then clearly $\{v_1, \ldots v_s\}$ does not generate M. Hence, by renumbering of u's and v's if necessary, suppose that T_{11} is a unit. Then the usual computations show that $\{v_1, u_2, \ldots, u_r\}$ is a basis of M and $\{v_2, \ldots, v_s\}$ generates a maximal linearly independent subset of a free module with a basis of r-1 elements. Indeed, in $M/[v_1]$, where $[v_1]$ is the free module generated by v_1, we have a basis $\{u_2+[v_1], \ldots, u_r+[v_1]\}$ and a maximal linearly independent set $\{u_2+[v_1], \ldots, u_s+[v_1]\}$. By the inductive hypothesis both are basis and thus $v_i + [v_1] = \sum_{\ell=2}^{r} u_\ell S_{\ell i} + [v_1]$, $i = 2, \ldots, s$ with some $S_{\ell i}$ a unit.

If we let $v_i = \sum_{\ell=2}^{r} u_\ell S_{\ell i} + v_1 a_i$, then

$$v_i = u_1 T_{11} a_i + \sum_{\ell=2}^{r} u_\ell (S_{\ell i} + T_{\ell 1} a_i).$$

Thus, if a_i is not a unit, then $T_{\ell 1} a_i$ is not a unit. But then $S_{\ell i} + T_{\ell 1} a_i$ is a unit whenever $S_{\ell i}$ is a unit. On the other hand, if a_i is a unit then $T_{11} a_i$ is a unit. Since $v_i = \sum_{\ell=1}^{r} u_\ell T_{\ell i}$, a comparison of coefficients shows that for each i at least one of the $T_{\ell i}$ is a unit.

Now suppose $\{x_1, \ldots, x_r\}$ does not have a common annihilator. Let $v_1 = u_1 x_1 + \ldots + u_r x_r$. Then $v_1 a = 0$ implies $a = 0$, whence v_1 can be included in some maximal linearly independent subset $\{v_1, \ldots, v_s\}$ of M. Since by construction $T_{\ell 1} = x_\ell$, it follows that some x_ℓ is a unit. Thus, if each x_i of $\{x_1, \ldots, x_r\}$ is a non-unit, then $\{x_1, \ldots, x_r\}$ has a non-zero annihilator.

Since the theorem now follows for $n \leqslant r+1$, the induction step is complete. Also, for r=1 the assertion is trivial and thus the theorem follows in general.

From theorem 1, A has property P_4 if and only if each finite set of non-units has a common annihilator. The restriction that the rings be local is only apparent, i.e., actually property P_4 is characterized in the class of all rings, although theorem 1 fails in that class. We note that in the next situation, i.e., Steinitz rings, the set of non-units has a common annihilator.

PROPERTY P_5.

Clearly, if A is a Steinitz ring, then A has property P_5. For the converse we prove the following theorem

THEOREM 2. *Let* A *be a ring with* 1. *Then each maximal linearly independent subset of a free module is a basis if and only if* A *is a Steinitz ring.*

Proof. The proof is exactly the same as in [2, theorem 2], except that we now make use of the lemma. Let M be a free module with a countable basis $\{u_i | i \in \omega\} = U$ and let $\{x_i | i \in \omega\}$ be a sequence of elements of A which do not have a left inverse. Set $v_i = u_i - u_{i+1}x_i$, for $i \in \omega$, $V = \{v_i | i \in \omega\}$ and let [V] be the submodule of M generated by V.

Suppose each maximal linearly independent subset of M generates M.

V is clearly a linearly independent set. To show V is a maximal linearly independent subset of M, let $m \in M$ be such that $\{m\} \cup V$ is linearly independent. Since $u_i \equiv u_{i+1}x_i \pmod{[V]}$, for each i, there is a u_j and an $x \in A$, such that $u_j x \equiv m \pmod{[V]}$, and $\{u_j x\} \cup V$ is linearly independent. Then, x is not a left zero-divisor of A and by the lemma, x is a unit, whence $\{u_j\} \cup V$ must be linearly independent. Since $u_i \equiv u_{i+1}x_i \pmod{[V]}$, for each i, it follows that $\{u_j\} \cup V = V'$ is a maximal linearly independent set, hence a basis. Therefore $u_{j+1} \equiv u_j y \pmod{[V]}$ for $y \in A$ and $u_j \equiv u_{j+1}x_j \equiv u_j y x_j \pmod{[V]}$, and $yx_j = 1$, since V' is a basis of M. This contradicts the fact that x_j does not have a left inverse. Thus, $[V] = M$ and $u_1 = \sum_{i=1}^{n} (u_i - u_{i+1}x_i)a_i$, whence by comparison of coefficients, we obtain $x_n \ldots x_1 = 0$.

Hence, the set of elements without left inverse is T-nilpotent (left vanishing). Thus, by [3], A is a Steinitz ring. Conversely, if A is a Steinitz ring, then from its definition [2], any maximal linearly independent subset of a free module is a basis. The theorem follows.

PROPERTY P_1.

Of all properties P_1 appears to be the most elusive, although it is a natural generalization of the following property:

P: Every module has a basis.

To construct a module without a minimal generating set, one takes a non-Steinitz local ring and a sequence of non-units $\{x_i | i \in \omega\}$ which is not T-nilpotent. Let M, U, V and [V] be as in the proof of theorem 2, then from property P_3 it follows that M/[V] does not have a minimal generating set. Because if M/[V] has a minimal generating set, then (M/[V])R = M/[V], when R is the maximal ideal, implies M/[V] = 0 and hence $\{x_i | i \in \omega\}$ is T-nilpotent from a similar argument as above. What we hope to show is that this situation is somehow typical, i.e., if there are modules without minimal generating sets, then there are modules of the type M/[V] without minimal generating sets by starting off a sequence of non-units which is not T-nilpotent. In order to do this one must determine the structure of modules of the type M/[V] in more general situations than those treated above.

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Department of Mathematics
The University of Alabama
University, Alabama 35486
U.S.A.

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