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STRICTLY CYCLIC WEIGHTED SHIFTS

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ABSTRACT. A method is given to construct a strictly cyclic bilateral weighted shift on ℓ^2 from a strictly cyclic unilateral shift with non-increasing sequence of weights. It is also shown that a unilateral weighted shift in ℓ^p whose weights either decrease to 0, or decrease to 1 and satisfy the boundedness condition $\sum_0^\infty (a_1 a_2 \ldots a_n)^{-\varepsilon} < \infty$ for all $\varepsilon > 0$ is not necessarily stricly cyclic for any p, $1 < p < \infty$.

1. INTRODUCTION.

A *weighted shift* T in the complex Banach space $\ell^p(K)$, $1 \leq p < \infty$, is a (bounded linear) operator defined by the equations $Te_n = a_n e_{n+1}$, where $\{e_n\}$ is the canonical basis (i.e., e_n is the sequence $\{\delta_{nk}\}_{k \in K}$, where δ_{nk} denotes the Kronecker's delta function) and $\{a_n\}$ is a (necessarily bounded) sequence of positive reals. If n runs over the set $K = \mathbb{N}$ of all non-negative integers ($K = \mathbb{Z}$ of all integers), then T is called a *unilateral* (*bilateral*, resp.) weghted shift.

Given $A \in L(\ell^2)$ (= the algebra of all operators acting on ℓ^2), let $A(A)$ ($A^a(A)$) and $A'(A)$ denote the weak closure of the polynomials (the rational functions with poles off the spectrum $\Lambda(A)$ of A, resp.) in A and the commutant of A in $L(\ell^2)$, respectively. Assume that there exists a vector x in ℓ^2 such that $A(A)x = \{Lx: L \in A(A)\}$ ($A^a(A)x$) coincides with ℓ^2; then A is said to be a *strictly cyclic* (*analytically strictly cyclic*, resp.) operator. It is well known (see [7]) that a strictly cyclic unilateral weighted shift (analytically strictly cyclic bilateral weighted shift B) always satisfies the following: $A(T) = A^a(T) = A'(T)$ (B is invertible and $A(B) \neq A^a(B) = A'(B)$; moreover, $A'(B)$ is the norm closure of the polynomials in B and B^{-1}) and the Gelfand spectrum of the Banach algebra $A(T)$ ($A'(B)$) can be naturally identified with $\Lambda(T)$ ($\Lambda(B)$, respectively).

Strictly cyclic unilateral weighted shifts (SCUWS) have been analyzed by several authors (see, e.g., [1];[2];[4];[7];[8];[9];[10];[11];[12];[13]), but very little is known about analytically strictly cyclic bilateral weighted shifts (ASCBWS). In *section* 2 a method will be given to construct an ASCBWS in $\ell^2(\mathbb{Z})$ by using a SCUWS in $\ell^2(\mathbb{N})$ whose weights are bounded below from zero and satisfy a certain boundedness condition (This result applies, in particular, if the weights of the SCUWS form a non-increasing sequence converging to some positive number).

In *section* 3 a different kind of problem is analyzed: It is shown that there exist UWS in $\ell^p(N)$ whose weights either decrease to 0, or decrease to 1 and satisfy the condition $\sum_0^\infty (a_0a_1\ldots a_n)^{-\varepsilon} < \infty$, which are not strictly cyclic for any p, thus answering in the negative a question of A.L.Shields ([13]).

2. A CLASS OF SYMMETRIC ASCBWS.

Let B be a BWS with weight sequence $\{a_n\}_{n\varepsilon Z}$ and define $w_0 = 1$, $w_n = a_0a_1\ldots a_{n-1}$, $w_{-n} = (a_{-1}a_{-2}\ldots a_{-n})^{-1}$ for $n > 0$. For T a UWS, the sequence $\{w_n\}_{n\varepsilon N}$ is similarly defined.

THEOREM 1. *Let* T *be a SCUWS with weights* $\{a_n\}_{n\varepsilon N}$ *such that*

$$\text{spectral radius } (T) = \lim_{n\to\infty} (\sup_k w_{n+k}/w_k)^{1/n} = 1 \tag{1}$$

and

$$w_n/w_{n+k} = (a_na_{n+1}\ldots a_{n+k-1})^{-1} \leqslant C \tag{2}$$

for some constant $C \geqslant 1$, *and for all* $n,k \in N$.

Define the BWS B *by*

$$Be_n = \begin{cases} a_ne_{n+1}, & \text{if } n \in N \\ (1/a_{-n})e_{n+1}, & \text{if } n \in Z\setminus N \end{cases}$$

Then B *is analytically strictly cyclic and* $\Lambda(B)$ *is the boundary* ∂D *of the unit disc* D.

Proof. Clearly, $w_n(B) = w_{-n}(B) = w_n(T)$ for all $n \in N$. Let R be the unitary map defined by $Re_n = e_{1-n}$; then $R = R^{-1}$ and $B^{-1} = RBR$; in particular, B is invertible.

Let $A \in A'(B)$; then A is the strong limit of the sequence of Cesàro averages of a formal Laurent series $\sum_{n\varepsilon Z} c_nB^n$ (see [5];[6]). B is ASC if and only if, given $x = \sum_{n\varepsilon Z} b_ne_n$, there exists an A_x in $A'(B)$ such that $A_xe_0 = x$ (see [7]). This is equivalent to say that the central column of the matrix of A_x (with respect to the canonical basis) coincides with the column vector x. By using the fact that $w_n = w_{-n}$, the matrix of A = (formally) $\sum_{n\varepsilon Z} c_nB^n$ is equal to :

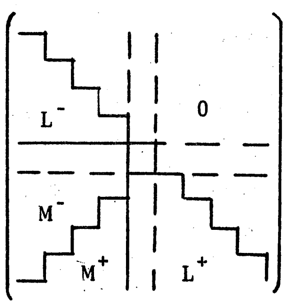
(i.e., the (j,k)-entry is equal to $c_{j-k}w_j/w_k$ for all $j,k \in Z$). The dotted lines remark the central column and the central row.

$$A = \left(\begin{array}{ccc|c|ccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & c_{-1}w_3/w_2 & c_{-2}w_3/w_1 & c_{-3}w_3 & c_{-4}w_3/w_1 & c_{-5}w_3/w_2 & \cdots \\
\cdots & c_0 & c_{-1}w_2/w_1 & c_{-2}w_2 & c_{-3}w_2/w_1 & c_{-4} & \cdots \\
\cdots & c_1w_1/w_2 & c_0 & c_{-1}w_1 & c_{-2} & c_{-3}w_1/w_2 & \cdots \\
\hline
\cdots & c_2/w_2 & c_1/w_1 & c_0 & c_{-1}/w_1 & c_{-2}/w_2 & \cdots \\
\hline
\cdots & c_3w_1/w_2 & c_2 & c_1w_1 & c_0 & c_{-1}w_1/w_2 & \cdots \\
\cdots & c_4 & c_3w_2/w_1 & c_2w_2 & c_1w_2/w_1 & c_0 & \cdots \\
\cdots & c_5w_3/w_2 & c_4w_3/w_1 & c_3w_3 & c_2w_3/w_1 & c_1w_3/w_2 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right). \quad (3)$$

Let $A_+ = \sum_{n\in N} c_n B^n$, $A_- = \sum_{n\in Z\setminus N} c_n B^n$, $x_+ = \sum_{n\in N} b_n e_n$ and $x_- = \sum_{n\in Z\setminus N} b_n e_n$. Then $A =$ (formally) $A_+ + A_-$ and $x = x_+ + x_-$. Assume that A_+ and A_- actually define bounded linear maps; then it is clear that $A_+e_0 \in \ell^2(N)$, while $A_-e_0 \in \ell^2(Z\setminus N)$. Thus, in order to complete the proof, it suffices to show that if $c_n w_n = b_n$ for all $n \in Z$, then A_+ is a bounded linear map in $A(B)$ and A_- is a bounded linear map in $A(B^{-1})$, whence it readily follows that $A = A_+ + A_-$ and $Ae_0 = A_+e_0 + A_-e_0 = x_+ + x_- = x$.

The matrix of A_+ is obtained from (3) by replacing the c_n's by 0's for all negative n. Decompose this matrix according to the heavy lines of (3). Then $A_+ =$ (formally) $L^- + L^+ + M^- + M^-$, where $L^-(L^+)$ is the up-er left (lower right, resp.) quarter of A_+ and $M^-(M^+)$ is the upper (lower, resp.) part of the lower left quarter of A_+ (and the remaining entries of L^-, L^+, M^- and M^+ are 0's).

Schematically, we have

$A_+ =$ 

$L^+ : \ell^2(N) \to \ell^2(N)$, $L^- : \ell^2(Z\setminus N) \to \ell^2(Z\setminus N)$ and $M^-(M^+) : \ell^2(Z\setminus N) \to \ell^2(N)$.

$L^+ = A_+|\ell^2(N) \in A'(T)$ (The vertical bar denotes "restriction") and therefore, since T is a SCUWS and the first column of L^+ belongs to $\ell^2(N)$, L^+ is actually bounded ([9]).

Condition (1) implies that $\Lambda(T) = D^- = \{z: |z| \leqslant 1\}$ = point spectrum of T^*, the adjoint of T (see [3];[7];[9]). Thus, in particular, $\{1/w_n\}_{n\in N} \in \ell^2(N)$ and $|c_n| \leqslant \|x\|/w_n \leqslant C'$ for a suitable constant $C' \geqslant 1$ and for all $n \in Z$. Consider L^-; clearly, $\|L^-\| \leqslant \sum_{n\in N} \|L^-(n)\|$, where $L^-(n)$ is obtained from L^- by replacing c_m by 0 for every $m \neq n$, $n = 0,1,2,\ldots$. Hence, by (2),

$$\|L^-\| \leqslant \sum_{n\in N} |c_n| \max\{w_j/w_{n+j};\ j \in N\} \leqslant C \sum_{n\in N} |c_n| \leqslant$$

$$\leqslant C\{\sum_{n\in N} (|c_n| w_n)^2\}^{1/2} \{\sum_{n\in N} w_n^{-2}\}^{1/2} < \infty$$

Similarly, by considering the formal decomposition $M^- = \sum_{n\in N\setminus\{0\}} M^-(n)$, where $M^-(n)$ is that part of the matrix M^- corresponding to a fixed coefficient c_n, it is not difficult to conclude that M^- is also bounded.

It is easy to see that M^+ has the same norm as the operator $M:\ell^2(N) \to \ell^2(N)$ defined by the matrix

$$M = \begin{pmatrix} c_2 & c_3w_2/w_1 & c_4w_3/w_1 & c_5w_4/w_1 & \cdots \\ 0 & c_4 & c_5w_3/w_2 & c_6w_4/w_2 & \cdots \\ 0 & 0 & c_6 & c_7w_4/w_3 & \cdots \\ 0 & 0 & 0 & c_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Let $y = \sum_{n\in N} d_n e_n \in \ell^2(N)$; then $My = \sum_{n\in N} \{\sum_{k=n+1}^{\infty} c_{k+n}\, y_{k-1}\, w_k/w_n\}\, e_n$. Hence

$$\|M^+\|^2 = \|M\|^2 = \sup_{\|y\|=1} \sum_{n\in N} |\sum_{k=n+1}^{\infty} c_{k+n}\, y_{k-1} w_k/w_n|^2 \leqslant$$

$$\leqslant \sup_{\|y\|=1} \sum_{n\in N} \{\sum_{k=n+1}^{\infty} (|c_{k+n}|\, w_k/w_n)^2\} \{\sum_{k=n}^{\infty} |y_k|^2\} \leqslant$$

$$\leqslant \sum_{n\in N} \sum_{m\in N} (|c_m|\, w_{m-n}/w_n)^2 \leqslant C^2\{\sum_{n\in N} w_n^{-2}\} \{\sum_{m\in N} (|c_m|\, w_m)^2\} < \infty$$

The boundedness of A_- follows by a completely symmetric argument. The details are left to the reader.

3. NON-SCUWS WITH NON-INCREASING SEQUENCES OF WEIGHTS.

In ([13], Question 15), A.L.Shields asked the following: If $a_n \searrow 1$ and $\sum w_n^{-2} < \infty$, must T be a SCUWS (in $\ell^2(N)$)?.If $a_n \searrow 0$, must T be strictly cyclic?. It is worth to recall that both questions have an affirmative answer in $\ell^1(N)$ ([3];[5]; see also [1]). Nevertheless, the answer is NO in $\ell^p(N)$ for $1 < p < \infty$. We shall need the following result:

THEOREM 2. (E.Kerlin and A.L.Lambert, [8], Theorem 3.2). *If* $\{a_n\}$ *is monotonically non-increasing and* $Te_n = a_n e_{n+1}$, *then* T *is strictly cyclic in* $\ell^p(N)$ *if and only if*

$$\sup_{n\in N} \sum_{k=0}^{n} (w_n/w_k w_{n-k})^q < \infty \quad , \quad \textit{where} \quad q = p/(p-1).$$

COROLLARY. (i) *There exists a sequence* $a_n \searrow 1$ *such that* $\sum w_n^{-\varepsilon} < \infty$ *for every* $\varepsilon > 0$, *but* $Te_n = a_n e_{n+1}$ *does not define a SCUWS in* $\ell^p(N)$ *for any* p, $1 < p < \infty$.

(ii) *There exists a sequence* $a_n \searrow 0$ *such that* $Te_n = a_n e_{n+1}$ *does not define a SCUWS in* $\ell^p(N)$ *for any* p, $1 < p < \infty$.

Proof. (i) Set $a_0 = 2$, $n_0 = 0$, $n_1 = 1$, $n_{j+1} > 3n_j$ (n_j to be defined) for all $j = 1,2,\ldots$ and $a_n = 1 + 2^{-j}$ for $n_j \leqslant n < n_{j+1}$; then $a_n \searrow 1$. If $2n_j < n \leqslant n_{j+1} - n_j$ and $n_j < k \leqslant n - n_j$, we have

$$w_n/w_k w_{n-k} = (1 + 2^{-j})^{n_j}/w_{n_j} = c_j > 0$$

so that we can inductively define the n_j's in such a way that

$$\sum_{k=0}^{n_{j+1}} (w_{n_{j+1}}/w_k w_{n_{j+1}-k})^j \geqslant (n_{j+1} - 2n_j)(c_j)^j > j \text{ for all } j=1,2,\ldots$$

Clearly, the above inequalities remain true if the n_j's are replaced by m_j's so that $m_j > n_j$ and $m_{j+1}/m_j > n_{j+1}/n_j$, $j = 1,2,\ldots$. Therefore, without loss of generality we can assume that the n_j's tend to ∞ fast enough so that $a_n > [(n+2)/(n+1)]^{n^{1/2}}$, whence it follows that $\sum w_n^{-\varepsilon} < \infty$ for every $\varepsilon > 0$.

(ii) Take n_j as above, except that now we take $a_n = 2^{-1}$ for $n_j \leqslant n < n_{j+1}$; then $a_n \searrow 0$. For n and k as above, $w_n/w_k w_{n-k} = 2^{-jn_j}/w_{n_j} = d_j > 0$, so that we can inductively define the n_j's in such a way that

$$\sum_{k=0}^{n_{j+1}} (w_{n_{j+1}}/w_k w_{n_{j+1}-k})^j \geqslant (n_{j+1}-2n_j)(d_j)^j > j, \quad j = 2,3,\ldots .$$

The conclusion is the same in both cases: By THEOREM 2, T cannot be a SCUWS in $\ell^p(N)$, for $1 < p < \infty$.

THEOREM 2 and its COROLLARY are actually true for the space $c_0(N)$ of all sequences converging to 0, under the maximum norm, if the exponent q is replaced by 1. (The details for these changes are left to the reader).

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