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SPHERICAL FUNCTIONS

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INTRODUCTION.

The fundamental properties of spherical functions have been established by R. Godement in a well known paper [1] in 1952. There he defines in a general manner the notion of spherical function associated to an irreducible representation of a locally compact unimodular group G. Moreover, he gives a characterization of such functions as characters of certain subalgebras of the algebra of all continuous functions on G with compact support. For certain purposes it is best not to work with the characters of such subalgebras but rather directly with their finite dimensional representations. This leads one to consider spherical functions with values in the endomorphism ring of a finite dimensional vector space and not just complex valued functions.

Despite the importance of the close connection between the spherical functions and the representations of G, and being of interest in their own right, it is desirable to have an intrinsic definition for the important notion of spherical function. Such definition is given and explored in §1.

In fact, it is possible to start from two different points which leads to the same concept. The reason of our choice is the existence of the general notion of μ-spherical function, where $\mu = (\mu_1,\mu_2)$ is a double representation of a compact subgroup K of G on a finite dimensional vector space E. By this one understands a continuous function ϕ from G to E such that

$$\phi(k_1gk_2) = \mu_1(k_1)\phi(g)\mu_2(k_2) \qquad (k_1,k_2 \in K;\ g \in G).$$

In §2 we establish the close connection between the spherical functions and the representations of certain algebras of functions on G, from which the most important properties of spherical functions follow. In §3 we discuss thoroughly the relation between the two different view points we mentioned above. In §4 we study the differential properties of spherical functions on Lie groups.

Since we have dropped every irreducibility assumption, some interesting questions naturally arise. For example, we don't know if any spherical function is associated to a representation of G. If G is a compact

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group, then we know that any spherical function is a direct sum of irreducibles, because if ϕ: G $\longrightarrow$ End(V) is spherical there is an inner product (,) on V such that

$$(\phi(g)v_1,v_2) = (v_1,\phi(g^{-1})v_2) \qquad (v_1,v_2 \in V\ ;\ g \in G)$$

In some other place we shall be concern with local spherical functions and complete reducibility of spherical functions on semi-simple Lie groups.

1. Throughout this paper we shall denote by G a locally compact unimodular group and by K a compact subgroup of G. We shall often use the following notation: if X denotes a group, then x will denote a generic element of X and e will denote the identity element of X.

Let $\hat{K}$ denote the set of all equivalence classes of finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite dimensional vector space over the field C of all complex numbers and by End(V) the space of all endomorphisms of V. Whenever we shall refer to a topology on such vector spaces we shall be talking about the unique Hausdorff linear topology on them.

By definition, a *zonal spherical function* φ on G is a continuous, complex valued function which satisfies $\varphi(e) = 1$ and

$$\text{(1)} \qquad \int_K \varphi(xky)\,dk = \varphi(x)\varphi(y) \qquad x,y \in G$$

A fruitful generalization of the functional equation above is the equation

$$\text{(2)} \qquad \int_K \chi_\delta(k^{-1})\phi(xky)\,dk = \phi(x)\phi(y) \qquad x,y \in G$$

whose End(V)-valued solutions will be called spherical functions on G.

The purpose of this paper, then, is to present in a systematic fashion the generalities which lie at the basis of the theory of spherical functions on those pairs (G,K) where G is a locally compact unimodular group, K a compact subgroup of G.

DEFINITION 1.1. *Let* $\delta \in \hat{K}$. *A spherical function* ϕ *(on* G*) of type* δ *is a continuous function on* G *with values in* End(V) *such that:*

(i) $\phi(e) = I$ (I = identity) ;

(ii) $\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)\,dk$ for all $x,y \in G$.

PROPOSITION 1.2. *If* ϕ: G $\longrightarrow$ End(V) *is a spherical function of type* δ *then:*

(i) $\phi(kgk') = \phi(k)\phi(g)\phi(k')$ *for all* $k,k' \in K$, $g \in G$;

(ii) $k \mapsto \phi(k)$ *is a representation of* K *such that any irreducible subrepresentation belongs to* δ.

Proof. (i) Let $k' \in K$ and $g \in G$. Then we have from the definition

$$\phi(k'g) = \phi(e)\phi(k'g) = \int_K \chi_\delta(k^{-1})\phi(kk'g)dk$$

by the symmetry of χ_δ we can interchange k and k'

$$\phi(k'g) = \int_K \chi_\delta(k^{-1})\phi(k'kg)dk = \phi(k')\phi(g) .$$

In a similar way it follows that $\phi(gk') = \phi(g)\phi(k')$, which completes the proof of (i).

(ii) Since $\phi(e) = I$, (i) implies that $\phi(kk') = \phi(k)\phi(k')$; that ϕ is continuous is obvious, therefore, $k \mapsto \phi(k)$ is a representation of K. Now,

$$I = \phi(e)\phi(e) = \int_K \chi_\delta(k^{-1})\phi(k)dk$$

but, it is well-known that the right hand side is a projection of V onto the space of all vectors which under $k \mapsto \phi(k)$ transform according to δ. This proves (ii).

Concerning the definition let us point out that the spherical function ϕ determines its type univocaly and let us say that the number of times that δ occurs in the representation $k \mapsto \phi(k)$ is called the *height* of ϕ.

Whenever K is a central subgroup of G (i.e. K is contained in the center of G) and ϕ is a spherical function, we have

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk = \int_K \chi_\delta(k^{-1})\phi(k)\phi(xy)dk = \phi(xy) \quad , \ x,y \in G$$

in other words, ϕ is a representation of G. Therefore, if we take K reduced to the identity, the spherical functions are precisely the finite dimensional representations of G, and if G is abelian the spherical functions are the finite dimensional representations of G such that 1.2 (ii) is satisfied.

Another extreme case occurs when G is compact and K = G. In this case the spherical functions are also the finite dimensional representations of G, with all their irreducible subrepresentations equivalent.

The function 0: $G \longrightarrow End(V)$ identically zero satisfies the functional equation 1.1 (ii) for any $\delta \in \hat{K}$. If $K = \{e\}$ the functional equation reduces to $\phi(x,y) = \phi(x)\phi(y)$ which implies that $\phi(e)$ is a projection commuting with all $\phi(G)$. Let V_1 and V_2 be respectively the kernel and the image of $\phi(e)$. Then $V = V_1 \oplus V_2$; if we write $\phi = \phi_1 \oplus \phi_2$ accordingly, we have that ϕ_2 is spherical while ϕ_1 is identically zero. For a moment one may think that something of this sort happens in general with the solutions of 1.1 (ii). But, the following example will show that this is not the case. Let $G = R^*$ be the multiplicative group of all non-zero real numbers and let $K = \{1,-1\}$. The two possible irreducible characters $\chi_\pm$ of K are given by $\chi_\pm(-1) = \pm 1$. Let ϕ: $R^* \longrightarrow M_2(C)$ be of the form

$$\phi(g) = \begin{pmatrix} 0 & f(g) \\ 0 & 0 \end{pmatrix}$$

where f: $R^* \longrightarrow C$ is continuous. Then ϕ satisfies the functional equation with the character χ_+(resp. χ_-) if and only if f is an odd (resp. even) function.

Later on we shall prove (see Lemma 4.1) that, when G is a Lie group, every spherical function is C^∞ (moreover analytic). Therefore one cannot expect to "build up" the solutions of 1.1 (ii) out of spherical functions and "elementary functions".

Let φ be a complex valued continuous solution of the equation (1). If φ is not identically zero then $\varphi(e) = 1$ (cf. Helgason [1,p.399]). This result generalizes in the following way: we shall say that a function ϕ: $G \longrightarrow \mathrm{End}(V)$ is *irreducible* whenever $\phi(G)$ is a non-trivial irreducible family of endomorphisms of V; then, we have

PROPOSITION 1.3. *Let* ϕ *be an* End(V)*-valued continuous solution of the equation* (2). *If* ϕ *is irreducible then* $\phi(e) = I$.

Proof. Let W_v denote the vector space spanned by $\{\phi(g)v\colon g \in G\}$. Now

$$\phi(x)\phi(y)v = \int_K \chi_\delta(k^{-1})\phi(xky)v\,dk \in W_v$$

which shows that W_v is $\phi(G)$-invariant, therefore W_v is either $\{0\}$ or V. We also have

$$\phi(x)\phi(e)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xk)\phi(y)\,dk = \iint_{K\times K} \chi_\delta(k^{-1})\chi_\delta(k_1^{-1})\phi(xkk_1y)\,dk\,dk_1 =$$

$$= \int_K \Big(\int_K \chi_\delta(k^{-1})\chi_\delta(k_1^{-1}k)\,dk\Big)\phi(xk_1y)\,dk_1 =$$

$$= \int_K \chi_\delta(k_1^{-1})\phi(xk_1y)\,dk_1 = \phi(x)\phi(y)$$

where we have used that $\chi_\delta * \chi_\delta = \chi_\delta$ (orthogonality relations). From this and what we observed before it follows that $\phi(g)\phi(e) = \phi(g) = \phi(e)\phi(g)$, all $g \in G$. Hence, $\phi(e)$ is a projection which commutes with every $\phi(g)$, therefore $\phi(e) = I$.

Spherical functions of type δ arise in a natural way upon consideration of representations of G. We recall that a continuous representation of G on a locally convex, Hausdorff, topological vector space E over C is a homomorphism $g \mapsto U(g)$ of G into the group of topological automorphisms of E, such that, the map $(g,a) \mapsto U(g)a$ of $G \times E$ into E is continuous. We also want to be able to lift U in the well-known way to a homomorphism $\mu \mapsto U(\mu)$ of the algebra $M_c(G)$ of Radon measures on G with compact support, into the algebra of continuous linear operators on E. Thus we want that the integral

$$U(\mu)a = \int_G U(g)a \, d\mu(g)$$

defines an element in E for every $a \in E$. This will be the case if we assume for example that E is complete.

Let $P(\delta)$ be defined by

$$P(\delta) = U(\overline{\chi}_\delta) = \int_K \overline{\chi}_\delta(k)U(k)dk$$

$P(\delta)$ is a continuous projection of E onto $P(\delta)E = E(\delta)$; $E(\delta)$ consists of those vectors in E, the linear span of whose K-orbit is finite dimensional and splits into irreducible K-submodules of type δ. Whenever $E(\delta)$ is finite dimensional, the function ϕ: $G \longrightarrow \mathrm{End}(E(\delta))$ defined by $\phi(g)a = P(\delta)U(g)a$, $g \in G$, $a \in E(\delta)$, is spherical of type δ. In fact, if $a \in E(\delta)$ we have

$$\phi(x)\phi(y)a = P(\delta)U(x)P(\delta)U(y)a = \int \overline{\chi}_\delta(k)P(\delta)U(x)U(k)U(y)a \, dk =$$

$$= \left(\int_K \chi_\delta(k^{-1})\phi(xky)dk\right)a$$

($\overline{\chi}_\delta(k) = \chi_\delta(k^{-1})$ for all $k \in K$).

In the next paragraph we shall consider the question of seeing when a spherical function is obtained in this way.

There is an important class of pairs (G,K), namely those where K is a large compact subgroup of G, where the above construction works. A compact subgroup K of G is said to be large (in G) if for each $\delta \in K$ there exists an integer $m(\delta) \geq 1$ such that $\dim E(\delta) \leq m(\delta)$ in every topologically completely irreducible Banach representation (E,U) of G. Examples of groups which admit large compact subgroups include the connected semisimple Lie groups with finite center and the motion groups (cf. Warner [1], §4.5).

If the representation $g \mapsto U(g)$ is topologically irreducible (i.e. U admits no non-trivial closed G-invariant subspace) then the associated spherical function ϕ is also irreducible. In fact, let W be a non-zero $\phi(G)$-invariant subspace of $E(\delta)$ and let Q: $E(\delta) \longrightarrow W$ be a projection of $E(\delta)$ onto W. Then

$$0 = P(\delta)U(g)QP(\delta) - QP(\delta)U(g)QP(\delta) = (I-Q)P(\delta)U(g)QP(\delta)$$

(I = identity transformation of $E(\delta)$). Since the vectors $U(g)a$, $g \in G$, $a \in W$, span a dense subspace of E, it follows that $I = Q$ which proves our assertion.

2. THE ALGEBRAS $C_{c,\delta}(G)$ AND THEIR REPRESENTATIONS.

We consider the given group G, its compact subgroup K and the function χ_δ, $\delta \in \hat{K}$, introduced before.

We shall denote by $M_c(G)$ (resp. $C_c(G)$) the algebra, with respect to convolution "*", of Radon measures (resp. continuous functions) on G with compact support, and by $M_\omega(G)$ (resp. $C_\omega(G)$) the space of Radon measures (resp. continuous functions) on G with support contained in the compact subset ω of G. We shall equip $M_c(G)$ (resp. $C_c(G)$) with the inductive limit of the topologies defined by the norm on the spaces $M_\omega(G)$ (resp. $C_\omega(G)$). We shall always identify a measure $\alpha \in M(K)$ on K with the measure $\alpha \in M_c(G)$ on G given by $f \mapsto \int_K f(k)d\alpha(k)$; in this way we get an isomorphism of the algebra M(K) into the algebra $M_c(G)$. We shall choose once and for all a left Haar measure on G, and we shall always identify every continuous function f(g) with the corresponding measure f(g)dg. In the same way, every continuous function on K will be identified with a measure on K, hence with a measure on G.

It is well-known that $C_c(G)$ is a two-sided ideal in $M_c(G)$, and it is clear that

$$(\alpha * f)(e) = (f * \alpha)(e)$$

for all $\alpha \in M_c(G)$ and all $f \in C_c(G)$. We shall also use for measures the operation $\alpha \to \check{\alpha}$; $\check{\alpha}$ is the transform of α under $g \mapsto g^{-1}$. In particular, if $f \in C_c(G)$, $\check{f}(g) = f(g^{-1})$ and $\check{\alpha}(f) = \alpha(\check{f})$ for all $\alpha \in M_c(G)$, $f \in C_c(G)$. Of course we have

$$(\alpha * \beta)\check{} = \check{\beta} * \check{\alpha}$$

for all $\alpha, \beta \in M_c(G)$.

Now, we may consider the set $C_{c,\delta}(G)$ of those $f \in C_c(G)$ which satisfy $\overline{\chi}_\delta * f = f = f * \overline{\chi}_\delta$; since $\chi_\delta * \chi_\delta = \chi_\delta$ (orthogonality relations), it is clear that $C_{c,\delta}(G)$ is a subalgebra of $C_c(G)$ and that $f \mapsto \overline{\chi}_\delta * f * \overline{\chi}_\delta$ is a continuous projection of $C_c(G)$ onto $C_{c,\delta}(G)$. We shall consider $C_{c,\delta}(G)$ as a topological subspace of $C_c(G)$.

We are in a position to take up a very important result, which establishes a close connection between spherical functions of type δ and representations of the algebra $C_{c,\delta}(G)$.

THEOREM 2.1. *If Φ is a spherical function on G of type δ, then the mapping*

$$\Phi : f \mapsto \int_G f(g)\Phi(g)dg$$

is a continuous finite dimensional representation of $C_{c,\delta}(G)$ such that $I \in \Phi(C_{c,\delta}(G))$. Conversely, if L is a continuous finite dimensional representation of $C_{c,\delta}(G)$ such that $I \in L(C_{c,\delta}(G))$ then L is represen-

ted as above by a spherical function of type δ.

Needless to say that if L is an irreducible finite dimensional representation of $C_{c,\delta}(G)$ then $I \in L(C_{c,\delta}(G))$ (Burnside's theorem).

The proof of this theorem requires the following proposition.

PROPOSITION 2.2. *Let* ϕ: G $\to$ End(V) *be a continuous function such that* $\chi_\delta * \phi = \phi = \phi * \chi_\delta$. *Then* ϕ *satisfies the functional equation* 1. (2) *if and only if the mapping*

$$\phi : f \mapsto \int_G f(g)\phi(g)dg$$

is a representation of $C_{c,\delta}(G)$.

Proof. Let f,h be two functions in $C_c(G)$, then

$$\phi(f) = \int_G f(g)\phi(g)dg = (\phi * \check{f})(e)$$

Therefore

$$(1) \qquad \phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta) = (\phi * (\overline{\chi}_\delta * f * \overline{\chi}_\delta)^\vee)(e) = (\phi * \chi_\delta * \check{f} * \chi_\delta)(e) =$$
$$= (\phi * \check{f} * \chi_\delta)(e) = (\chi_\delta * \phi * \check{f})(e) = (\phi * \check{f})(e) = \phi(f)$$

we have used that $\overline{\chi}_\delta = \check{\chi}_\delta$, which is well-known. Now

$$(2) \qquad \phi((\overline{\chi}_\delta * f * \overline{\chi}_\delta) * (\overline{\chi}_\delta * h * \overline{\chi}_\delta)) = \phi(f * \overline{\chi}_\delta * h) = \int_G (f * \overline{\chi}_\delta * h)(y)\phi(y)dy =$$
$$= \int_G \int_G (f * \overline{\chi}_\delta)(x)h(x^{-1}y)\phi(y)dx\, dy =$$
$$= \int_G \int_G \int_K f(xk^{-1})\overline{\chi}_\delta(k)h(y)\phi(xy)dk\, dx\, dy =$$
$$= \iint_{G\times G} f(x)h(y)\left(\int_K \chi_\delta(k^{-1})\phi(xky)dk\right)dx\, dy \; .$$

On the other hand

$$(3) \qquad \phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta)\phi(\overline{\chi}_\delta * h * \overline{\chi}_\delta) = \phi(f)\phi(h) = \iint_{G\times G} f(x)h(y)\phi(x)\phi(y)dx\, dy \; .$$

Considering (2) and (3), the proposition follows immediately.

Proof of Theorem 2.1. Let ϕ: G $\to$ End(V) be a spherical function on G of type δ. Then, by Propositions 1.2 and 2.2 the mapping ϕ: $C_{c,\delta}(G) \to$ End(V) is a representation of $C_{c,\delta}(G)$, which is obviously continuous.

In order to prove that $I \in \phi(C_{c,\delta}(G))$ we first notice that $\phi(C_{c,\delta}(G)) = \phi(C_c(G))$. The neighborhoods O of $g \in G$ form a directed system under inclusion, and if $f_O \in C_c(G)$ is a nonnegative function with spt $f_O \subset O$

and satisfying $\int_G f_\mathcal{O}(g)dg = 1$, then $f_\mathcal{O}*f \to \delta_g*f$ in $C_c(G)$ (δ_g is the Dirac measure at g). Then

$$\phi(f_\mathcal{O}) = (f_\mathcal{O}*\check{\phi})(e) \to (\delta_g*\check{\phi})(e) = \phi(g) \tag{4}$$

hence the linear span of $\{\phi(g)\colon g \in G\}$ is contained in $\phi(C_c(G))$. Since the other inclusion is obvious we get

$$\phi(C_{c,\delta}(G)) = \{\phi(g)\colon g \in G\}_C \tag{5}$$

Now it is clear that $I \in \phi(C_{c,\delta}(G))$.

Conversely, let $L\colon C_{c,\delta}(G) \to \mathrm{End}(V)$ be a continuous representation of $C_{c,\delta}(G)$ such that $I \in L(C_{c,\delta}(G))$. The mapping $\phi\colon f \mapsto L(\overline{\chi}_\delta*f*\overline{\chi}_\delta)$ defines an End(V)-valued Radon measure on G. Let $h \in C_{c,\delta}(G)$ be an element such that $L(h) = I$, then

$$\phi(f) = L(\overline{\chi}_\delta*f*\overline{\chi}_\delta)L(h) = L(\overline{\chi}_\delta*f*h) = \phi(f*h) = (\phi*\check{h}*\check{f})(e) = (\phi*\check{h})(f)$$

for all $f \in C_c(G)$. Therefore $\phi = \phi*\check{h}$ is a continuous function on G which represents L. But we also have

$$(\chi_\delta*\phi*\chi_\delta)(f) = ((\chi_\delta*\phi*\chi_\delta)*\check{f})(e) = (\phi*(\overline{\chi}_\delta*f*\overline{\chi}_\delta)^\vee)(e) = (\phi*\check{f})(e) = \phi(f)$$

if $f \in C_c(G)$, which implies that $\phi = \chi_\delta*\phi*\chi_\delta$. Hence by Proposition 2.2

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk .$$

In particular

$$\phi(e)\phi(g) = (\chi_\delta*\phi)(g) = \phi(g) = (\phi*\chi_\delta)(g) = \phi(g)\phi(e) \tag{6}$$

hence $\phi(e)$ is an identity of $L(C_{c,\delta}(G))$ and therefore $\phi(e) = I$. This completes the proof of Theorem 2.1.

REMARK 2.3. Under the hypothesis of Proposition 2.2 the function ϕ is spherical of type δ if and only if the representation ϕ of the algebra $C_{c,\delta}(G)$ cannot be decomposed as a direct sum of two representations, one of which is the trivial zero representation. This follows at once from (6).

Let $\phi\colon G \to \mathrm{End}(V)$ be a spherical function of type δ. Then a direct consequence of (5) is that a subspace W of V is $\phi(G)$-invariant if and only if it is $\phi(C_{c,\delta}(G))$-invariant. In particular we have the following corollary:

COROLLARY 2.4. *A spherical function* $\phi\colon G \to \mathrm{End}(V)$ *is irreducible if and only if the linear span of* $\phi(G)$ *coincides with* End(V).

We shall say that the spherical functions $\phi\colon G \to \mathrm{End}(V)$ and

$\phi_1: G \to \mathrm{End}(V_1)$ are *equivalent* if there exists a linear isomorphism T of V onto V_1 such that $\phi_1(G) = T\phi(g)T^{-1}$ for all $g \in G$. It is clear that this equivalence relation preserves the type and the height of the spherical functions. Moreover we have

PROPOSITION 2.5. *The spherical functions* $\phi: G \to \mathrm{End}(V)$ *and* $\phi_1: G \to \mathrm{End}(V_1)$ *of type* δ *are equivalent, if and only if the corresponding representations* $\phi: C_{c,\delta}(G) \to \mathrm{End}(V)$ *and* $\phi_1: C_{c,\delta}(G) \to \mathrm{End}(V_1)$ *are equivalent.*

Proof. Let T be an isomorphism of V onto V_1 such that $\phi_1(f) = T\phi(f)T^{-1}$ for all $f \in C_{c,\delta}(G)$. Then

$$\phi_1(f) = \phi_1(\overline{X}_\delta * f * \overline{X}_\delta) = T\phi(\overline{X}_\delta * f * \overline{X}_\delta)T^{-1} = T\phi(f)T^{-1}$$

for any $f \in C_c(G)$. Therefore $\phi_1(g) = T\phi(g)T^{-1}$, all $g \in G$.

The other assertion is obvious.

As a corollary of Theorem 2.1 and Proposition 2.5 we obtain the following result

THEOREM 2.6. *The irreducible spherical functions* ϕ *and* ϕ_1 *are equivalent if and only if* $\mathrm{tr}\,\phi(g) = \mathrm{tr}\,\phi_1(g)$ *for all* $g \in G$.

Proof. It is obvious that if ϕ and ϕ_1 are equivalent they have the same trace. Conversely, $\mathrm{tr}\,\phi(g) = \mathrm{tr}\,\phi_1(g)$, all $g \in G$, implies $\mathrm{tr}\,\phi(f) = \mathrm{tr}\,\phi_1(f)$ for all $f \in C_{c,\delta}(G)$. Since ϕ and ϕ_1 are two irreducible finite dimensional representations of an associative algebra over C with the same trace, they are equivalent. Hence, ϕ and ϕ_1 are equivalent.

REMARK 2.7. Theorem 2.6 does not hold in general if we drop the irreducibility hypothesis, because, it is not even true for finite dimensional representations. For example, the functions $\phi, \phi_1: R \to M_2(C)$ defined by

$$\phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \phi_1(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x \in R$$

are two spherical functions of the pair $(R,\{0\})$ with the same trace which are not equivalent. But, as one can expect, when G is compact it is not necessary to assume that the spherical functions are irreducible for Theorem 2.6 to be true.

The possible heights of the various irreducible spherical functions ϕ on (G,K) are not entirely arbitrary. In order to clarify this, it is convenient to recall the following algebraic fact due to Kaplansky: let A be an associative algebra over C, and let n be a fixed integer; if there are enough representations of A of dimensions $\leq n$ to separate the points of A, then, every irreducible finite dimensional representation of A has dimension $\leq n$ (cf. Godement

[1], p.503). Some interesting examples of pairs (G,K) which have the property that $C_{c,\delta}(G)$ has a separating family of representations of dimensions $\leqslant n$ are:

(1) G is a motion group, i.e. G is the semi-direct product of a closed normal abelian subgroup H and a compact subgroup K;

(2) G is a connected semi-simple Lie group which admits a faithful finite dimensional representation and K is a maximal compact subgroup. In both cases it can be proved (cf. Godement [1], §1) that the integer n can be taken equal to $d(\delta)$. Therefore, the height of an irreducible spherical function ϕ on (G,K) ((G,K) as in (1) or (2)) of type δ is $\leqslant d(\delta)$.

Let us turn now to consider when a spherical function on (G,K) is obtained from a representation of G as described in §1. To avoid some technical troubles we shall now assume that our locally compact group G is, furthermore, countable at infinity. We shall also presuppose that the reader is familiar with inductive limits and strict inductive limits. A good reference is Horváth [1]. A space (E,τ) is called a *strict* LF-*space* if (E,τ) is the strict inductive limit of Fréchet spaces (τ being the topology on E); for example $C_\omega(G)$ (ω a compact subset of G) is a Fréchet space, while $C_c(G)$ is a strict LF-space. Thus, a strict LF-space is a locally convex, complete, Hausdorff, topological vector space. We shall be concerned with continuous representations of G on a strict LF-space E, and with the corresponding quotient representations of G on E/J, J being a closed G-stable subspace of E. Even if E/J is not complete, we can lift by integration the representation of G on E/J to a representation of $M_c(G)$.

Let ϕ: G $\longrightarrow$ End(V) be an irreducible spherical function of type δ, and let L be a maximal left ideal in End(V). If I is the set of all $f \in C_{c,\delta}(G)$ such that $\phi(f) \in L$, then I is a closed regular maximal left ideal in $C_{c,\delta}(G)$. Now let J be the set of all $f \in C_c(G)$ such that

$$\overline{X}_\delta * h * f * \overline{X}_\delta \in I \quad \text{for every} \quad h \in C_c(G)$$

then J is a closed regular maximal left ideal in $C_c(G)$, $I = J \cap C_{c,\delta}(G)$, and we have $f * \overline{X}_\delta \equiv f \pmod J$ for all $f \in C_c(G)$ (for the proof see Godement [1], p.513).

Since J is a closed left ideal in $C_c(G)$ it is invariant under left translation by elements of G, otherwise said there is induced on the space $C_c(G)/J$ a natural representation U of G. The corresponding lift of U to $M_c(G)$ associates with each $\mu \in M_c(G)$ the operator which transforms the class of $f \in C_c(G)$ (mod J) into the class of $\mu * f$ (mod J); thus, its restriction to the ideal $C_c(G)$ is an algebraically irreducible (J is maximal) representation of $C_c(G)$. That is to say that U is an algebraically irreducible representation of G.

The projection operator $P(\delta) = U(\overline{\chi}_\delta)$ is given by $P(\delta)(f + J) = \overline{\chi}_\delta * f + J$; on the other hand $f * \overline{\chi}_\delta \equiv f \pmod{J}$ for all $f \in C_c(G)$, hence $f \mapsto f + J$ is a mapping of $C_{c,\delta}(G)$ onto $E(\delta)$ (the range of $P(\delta)$). Since $I = J \cap C_{c,\delta}(G)$ it is clear that $\dim E(\delta) = \dim C_{c,\delta}(G)/I = \dim \mathrm{End}(V)/L = \dim V$. The associated spherical function $\phi_1 \colon G \to \mathrm{End}(E(\delta))$ is equivalent to ϕ. To see this it is sufficient to show that the representations $\phi \colon C_{c,\delta}(G) \to \mathrm{End}(V)$ and $\phi_1 \colon C_{c,\delta}(G) \to \mathrm{End}(E(\delta))$ of $C_{c,\delta}(G)$ are equivalent (Proposition 2.5). If $f \in C_{c,\delta}(G)$ is such that $\phi(f) = 0$ then $\phi(f*h) = 0$ for all $h \in C_{c,\delta}(G)$, and $\phi_1(f)(h+J) = f*h + J = 0$ ($f*h \in I \subset J$); therefore $\phi_1(f) = 0$. Consequently, since it is a question of finite dimensional irreducible representations of an associative algebra, it follows that ϕ and ϕ_1 are equivalent.

The preceding discussion serves to prove that any irreducible spherical function on G can be obtained from an algebraically irreducible representation U of G, U being a quotient of a representation of G on a strict LF-space. To complete this circle of ideas, it remains to show that if the irreducible spherical function ϕ, comes from a representation U of G as above, the construction of the representation U_ϕ of G out of ϕ, gets us back to U.

Let E and E_ϕ be the representation spaces of U and U_ϕ respectively, let $E(\delta)$ and $E_\phi(\delta)$ be the corresponding K-isotypic subspaces, $P(\delta)$ and $P_\phi(\delta)$ the corresponding projections. If ϕ_1 is the spherical function of type δ associated to U_ϕ, there exist non-zero vectors $v \in E(\delta)$, $v_1 \in E_\phi(\delta)$ such that $\phi(f)v = 0$ if and only if $\phi_1(f)v_1 = 0$, $f \in C_c(G)$ (ϕ and ϕ_1 are equivalent). Let $S \colon C_c(G) \to E$ and $S_1 . C_c(G) \to E_\phi$ be the linear maps defined by $S(f) = U(f)v$, $S_1(f) = U_\phi(f)v_1$. Then $\mathrm{Ker}(S) = \mathrm{Ker}(S_1)$. In fact, if $f \in \mathrm{Ker}(S)$ we have $\phi(h*f)v = U(\overline{\chi}_\delta)U(h)U(f)v = 0$ therefore $0 = \phi_1(h*f)v_1 = U_\phi(\overline{\chi}_\delta)U_\phi(h)U_\phi(f)v_1$, $h \in C_c(G)$, which implies that $f \in \mathrm{Ker}(S_1)$ (algebraic irreducibility). In the same way one proves that $\mathrm{Ker}(S_1) \subset \mathrm{Ker}(S)$, and therefore they are equal. The maps S and S_1 are clearly continuous surjective linear maps, hence they are strict morphisms (cf. Horváth [1], Prop. 11, p.306). From this it follows that the continuous representations U and U_ϕ of G are equivalent, i.e. there exists a linear bicontinuous bijection $Q \colon E \to E_\phi$ such that $QU(g) = U_\phi(g)Q$ for all $g \in G$.

One can play exactly the same game as before, but with Fréchet representations of G, and prove that any irreducible spherical function $\phi \colon G \to \mathrm{End}(V)$ can be obtained from a topologically irreducible representation of G on a Fréchet space E. For this, one writes $G = \cup K_n$ as a countable union of an increasing sequence of compact subsets K_n of G, and such that every compact subset of G is contained in some

K_n. Then, $\quad \|f\|_n = \sup_{x\in K_n} \int_G \|\phi(xy)f(y)\|dy \qquad n = 1,2,\ldots$.

($\|\ \|$ a norm on End (V)) are semi-norms on $C_c(G)$, and $\|f\|_n = 0$ for every n is equivalent to f=0. Then the Fréchet space L(G) which is the completion of $C_c(G)$ by these semi-norms, plays the role of $C_c(G)$ in the construction of the representation of G (for the details see Shin'ya [1]).

If a spherical function ϕ is associated to a Banach representation U of G then

$$\|\phi(g)\| \leqslant \|U(g)\| \quad \text{for all } g \in G$$

($\|\ \|$ is the usual operator norm). The function

$$\rho(g) = \|U(g)\|$$

is a positive real valued lower semi-continuous function which is bounded on compact subsets of G and satisfies

$$\rho(xy) \leqslant \rho(x)\rho(y)$$

for all $x,y \in G$; such a function is called a *semi-norm* on G.

A Banach space valued function f on G is said to be *quasi-bounded* if there exists a semi-norm ρ on G such that $\sup_{g\in G} \|f(g)\|/\rho(g) < \infty$.
Thus, if a spherical function comes from a Banach representation of G it is quasi-bounded. Conversely, if ϕ is an irreducible quasi-bounded spherical function on G, then it is associated to an algebraically irreducible Banach representation of G. Let ρ be a semi-norm on G such that $\sup_{g\in G} \|\phi(g)\|/\rho(g) < \infty$. One constructs the Banach representation as before, but replacing the space $C_c(G)$ by the Banach algebra obtained by completing $C_c(G)$, with respect to the ρ-norm

$$\|f\|_\rho = \int_G |f(g)|\rho(g)dg \qquad (f \in C_c(G))$$

(cf. Godement [1]).

3. THE ALGEBRAS $I_{c,\delta}(G)$ AND THEIR REPRESENTATIONS.

In what follows we shall denote by $I_c(G)$ the set of functions $f \in C_c(G)$ which are K-central, i.e. invariant under $g \mapsto kgk^{-1}$; thus $I_c(G)$ is a subalgebra of $C_c(G)$ and the operator

$$f \mapsto f^0(g) = \int_K f(kgk^{-1})dk$$

is a continuous projection (in the inductive limit topology) of $C_c(G)$ onto $I_c(G)$. We shall pût $I_{c,\delta}(G) = I_c(G) \cap C_{c,\delta}(G)$, this is also a subalgebra of $C_c(G)$ and $f \mapsto f^0$ maps $C_{c,\delta}(G)$ onto $I_{c,\delta}(G)$. If $f \in I_c(G)$ and if $\overline{\chi}_\delta * f = f$, then also $f = f * \overline{\chi}_\delta$; this means that the map $f \mapsto \overline{\chi}_\delta * f$ is a continuous projection of $I_c(G)$ onto $I_{c,\delta}(G)$.

The topology induced by $C_c(G)$ is the one we shall consider on $I_{c,\delta}(G)$. If α is a Radon measure on G then α^0 can be defined by the following "weak" integral: $\alpha^0 = \int_K (\delta_k * \alpha * \delta_{k^{-1}})dk$ (δ_x denotes the Dirac measure at x). Observe that

$$(1) \qquad (\alpha^0 * \beta)^0 = (\alpha * \beta^0)^0 = \alpha^0 * \beta^0$$

whenever α or β has compact support.

Let ϕ: G $\longrightarrow$ End(V) be a spherical function of type δ and height p. Then V is a K-module under π: $k \longmapsto \phi(k)$. Let $\mathrm{End}_K(V)$ be the commutator of the representation π. Now, since the representation π decomposes into p equivalent irreducible representations, it is clear that its commutator is isomorphic with the algebra $M_p(C)$ of all p x p matrices, and such isomorphism is unique up to an inner automorphism of $M_p(C)$. If $f \in I_c(G)$ then

$$\phi(f) = \int_G f(g)\phi(g)dg \in \mathrm{End}_K(V) \quad , \quad \text{in fact}$$

$$\pi(k)\phi(f) = \int_G f(g)\phi(kg)dg = \int_G f(k^{-1}g)\phi(g)dg = \int_G f(gk^{-1})\phi(g)dg =$$

$$= \int_G f(g)\phi(gk)dg = \phi(f)\pi(k).$$

Therefore, we may view ϕ: $I_{c,\delta}(G) \longrightarrow \mathrm{End}_K(V)$ as p-dimensional representation of $I_{c,\delta}(G)$. Also note that if $\phi(f) = I$ then $\phi(f^0) = I$ ($f \in C_c(G)$). Hence, we have proved the first part of the following theorem:

THEOREM 3.1. *If* ϕ: G $\longrightarrow$ End(V) *is a spherical function of type* δ *then* ϕ: $I_{c,\delta}(G) \longrightarrow \mathrm{End}_K(V)$ *gives a continuous representation of* $I_{c,\delta}(G)$ *such that* $I \in \phi(I_{c,\delta}(G))$. *Conversely, any continuous finite dimensional representation* L *of* $I_{c,\delta}(G)$ *such that* $I \in L(I_{c,\delta}(G))$ *is equivalent to one given by a spherical function* ϕ *of type* δ.

We shall prove the second part of this theorem in several steps.

PROPOSITION 3.2. *Let* ψ: G $\longrightarrow$ End(V) *be a K-central continuous function such that* $\chi_\delta * \psi = \psi$. *Then* ψ *satisfies the functional equation*

$$\psi(x)\psi(y) = \int_K \psi(k\,x\,k^{-1}y)dk$$

if and only if the mapping ψ: $f \longmapsto \int_G f(g)\psi(g)dg$ *is a representation of* $I_{c,\delta}(G)$.

Proof. In view of 2.(2) and 2.(3) we have

$$\psi((\overline{\chi}_\delta * f^0 * \overline{\chi}_\delta) * (\overline{\chi}_\delta * h^0 * \overline{\chi}_\delta)) = \int\int_{G\times G} f^0(x)h^0(y) \left(\int_K \chi_\delta(k^{-1})\psi(xky)dk\right)dxdy =$$

$$= \int\int_{G\times G} f^0(x)h^0(y)\psi(xy)dxdy = \int\int_{G\times G} f(x)h^0(y) \left(\int_K \psi(k\ x\ k^{-1}y)dk\right)dxdy =$$

$$= \int\int_{G\times G} f(x)h(y) \left(\int_K \psi(k\ x\ k^{-1}y)dk\right)dxdy$$

and

$$\psi(\overline{\chi}_\delta * f^0 * \overline{\chi}_\delta)\psi(\overline{\chi}_\delta * h^0 * \overline{\chi}_\delta) = \int\int_{G\times G} f^0(x)h^0(y)\psi(x)\psi(y)dxdy =$$

$$= \int\int_{G\times G} f(x)h(y)\psi(x)\psi(y)dxdy$$

for all $f,h \in C_c(G)$. Now, the proposition follows immediately.

Let (V,π) be a finite dimensional K-module such that any irreducible submodule belongs to δ.

Let $\mathcal{A}$ denote the vector space of all continuous functions ϕ: $G \to \mathrm{End}(V)$ such that $\phi(k_1gk_2) = \pi(k_1)\phi(g)\pi(k_2)$ for all $k_1, k_2 \in K$, and let $\mathcal{B}$ denote the vector space of all continuous functions ψ: $G \to \mathrm{End}_K(V)$ such that ψ is K-central and $\chi_\delta * \psi = \psi$.

PROPOSITION 3.3. *Let* A *and* B *be the linear mappings defined by*

$$(A\phi)(g) = \int_K \pi(k)\phi(g)\pi(k^{-1})dk \qquad \textit{for} \qquad \phi \in \mathcal{A}\ ,$$

$$(B\psi)(g) = d(\delta)^2 \int_K \pi(k)\psi(k^{-1}g)dk \qquad \textit{for} \qquad \psi \in \mathcal{B}\ ,$$

Then A *is an isomorphism of* $\mathcal{A}$ *onto* $\mathcal{B}$ *and* B *is the inverse of* A.

Proof. It is clear that $(A\phi)(g) \in \mathrm{End}_K(V)$, $g \in G$, and that $A\phi$ is K-central. Let us check that $\chi_\delta * A\phi = A\phi$:

$$(\chi_\delta * A\phi)(g) = \int_K\int_K \chi_\delta(k)\pi(k_1)\phi(k^{-1}g)\pi(k_1^{-1})dk_1dk =$$

$$= \int_K\int_K \chi_\delta(k)\pi(k_1)\pi(k^{-1})\phi(g)\pi(k_1^{-1})dkdk_1 = (A\phi)(g)$$

since $\int_K \chi_\delta(k)\pi(k^{-1})dk = I$.

An obvious computation shows that B maps $\mathcal{B}$ into $\mathcal{A}$. For $\psi \in \mathcal{B}$ we have,

$$(A(B\psi))(g) = d(\delta)^2\int_K\int_K \pi(k)\pi(k_1)\psi(k_1^{-1}g)\pi(k^{-1})dk_1dk =$$

$$= d(\delta)^2 \int_K \int_K \pi(kk_1k^{-1})\psi(k_1^{-1}g)\,dk\,dk_1 = \psi(g)$$

since $d(\delta)^2 \int_K \pi(kk_1k^{-1})\,dk = \chi_\delta(k_1)I$.

In a similar way one proves that B is a left inverse of A, and this completes the proof of Proposition 3.3.

COROLLARY 3.4. *Let* $\psi \in \mathcal{B}$; *if* $\psi(e) = I$ *then* $\psi(k) = \chi_\delta(e)^{-1}\chi_\delta(k)I$ *for all* $k \in K$.

Proof. If $\phi \in \mathcal{A}$ then $\phi(e) \in \mathrm{End}_K(V)$, since $\pi(k)\phi(e) = \phi(k) = \phi(e)\pi(k)$ Therefore $(A\phi)(e) = \phi(e)$.

Now let $\phi = B\psi$, then $\phi(e) = \psi(e) = I$ and $\phi(k) = \pi(k)$ for all $k \in K$. From this we get

$$\psi(k) = (A\phi)(k) = \int_K \pi(k_1)\phi(k)\pi(k_1^{-1})\,dk_1 = \chi_\delta(e)^{-1}\chi_\delta(k)I\ , \qquad k \in K.$$

It may be worthwhile to point out also the following corollary:

COROLLARY 3.5. *For any* $\phi \in \mathcal{A}$ *we have* $\mathrm{tr}A\phi = \mathrm{tr}\phi$ *and for any* $\psi \in \mathcal{B}$ *we have* $\mathrm{tr}B\psi = \mathrm{tr}\psi$.

Proof. The first assertion is obvious and to prove the second let $\phi = B\psi$, then

$$\mathrm{tr}B\psi = \mathrm{tr}\phi = \mathrm{tr}A\phi = \mathrm{tr}\psi\ .$$

PROPOSITION 3.6. *Let* $\psi = A\phi$, $\phi \in \mathcal{A}$. *Then* ϕ *satisfies*

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)\,dk \tag{2}$$

if and only if ψ *satisfies*

$$\psi(x)\psi(y) = \int_K \psi(k\,x\,k^{-1}y)\,dk\ . \tag{3}$$

Proof. If we assume (3) we have

$$\phi(x)\phi(y) = (B\psi)(x)(B\psi)(y) = d(\delta)^4 \int\int_{K\times K} \pi(k_1)\psi(k_1^{-1}x)\pi(k_2)\psi(k_2^{-1}y)\,dk_1dk_2 =$$

$$= d(\delta)^4 \int\int\int_{K\times K\times K} \pi(k_1k_2)\psi(kk_1^{-1}\,x\,k^{-1}k_2^{-1}y)\,dk\,dk_1dk_2 =$$

$$= d(\delta)^4 \int\int\int_{K\times K\times K} \pi(k_1k_2)\psi(k_2^{-1}kk_1^{-1}\,x\,k^{-1}y)\,dk\,dk_1dk_2 =$$

$$= d(\delta)^2 \int\int_{K\times K} \pi(k_1)\phi(kk_1^{-1}\,x\,k^{-1}y)\,dk\,dk_1 = \int_K \chi_\delta(k)\phi(xk^{-1}y)\,dk\ .$$

Conversely, if we assume (2) we have

$$\psi(x)\psi(y) = (A\phi)(x)(A\phi)(y) =$$

$$= \int\int_{K\times K} \pi(k_1)\phi(x)\pi(k_1^{-1})\pi(k_2)\phi(y)\pi(k_2^{-1})dk_1dk_2 =$$

$$= \int\int_{K\times K} \pi(k_2k_1)\phi(x)\pi(k_1^{-1})\phi(y)\pi(k_2^{-1})dk_1dk_2 =$$

$$= \int\int\int_{K\times K\times K} \pi(k_2)\chi_\delta(k^{-1})\phi(k_1\ x\ k_1^{-1}ky)\ \pi(k_2^{-1})dkdk_1dk_2 =$$

$$= \int\int\int_{K\times K\times K} \pi(k_2)\chi_\delta(k^{-1})\phi(kk_1\ x\ k_1^{-1}y)\pi(k_2^{-1})dkdk_1dk_2 =$$

$$= \int_K \psi(k_1\ x\ k_1^{-1}y)dk_1\ .$$

PROPOSITION 3.7. *Let* ϕ: G $\longrightarrow$ End(V) *be a continuous function such that* $\phi(k_1gk_2) = \pi(k_1)\phi(g)\pi(k_2)$, *all* $k_1,k_2 \in K$. *Then* ϕ *satisfies the functional equation*

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk \tag{4}$$

if and only if the mapping ϕ: $f \longmapsto \int_G f(g)\phi(g)dg$ *is a representation of* $I_{c,\delta}(G)$.

Proof. That ϕ gives a representation of $I_{c,\delta}(G)$ whenever ϕ satisfies (4), it follows at once from Proposition 2.2.

To prove the converse let $\psi = A\phi$ and observe that

$$\phi(f) = \int\int_{G\times K} f(g)\phi(kgk^{-1})dkdg = \int_G f(g)\psi(g)dg = \psi(f) \tag{5}$$

for all $f \in I_c(G)$. Therefore by Proposition 3.2 ψ satisfies (3) which in turn implies that ϕ satisfies (4).

Proof of Theorem 3.1. The first part was already proved. Now let L: $I_{c,\delta}(G) \longrightarrow M_p(C)$ be a continuous representation such that $L(h) = I$ for some $h \in I_{c,\delta}(G)$. The composite map ψ: $C_c(G) \longrightarrow M_p(C)$ defined by $\psi(f) = L(\overline{\chi}_\delta * f^0)$ is a $M_p(C)$-valued Radon measure on G. We have

$$\psi(f) = L(\overline{\chi}_\delta * f^0) = L(\overline{\chi}_\delta * f^0 * h) = L(\overline{\chi}_\delta * (f*h)^0) = \psi(f*h) = (\psi*\check{h})(f)$$

for all $f \in C_c(G)$. Therefore $\psi = \psi*\check{h}$ is a continuous function on G which represents L. Using once more (1) we get

$$\psi(f) = \psi(f^0) = \psi(\psi*f^{0\vee})(e) = (\psi*f^{\vee 0})^0(e) = (\psi^0*f^\vee)^0(e) = \psi^0(f)$$

for any $f \in C_c(G)$, which shows that ψ is K-central. In a similar way one also establishes that $\chi_\delta * \psi = \psi$.

Let (V,π) be a K-module which is the direct sum of p irreducible modules belonging to δ. If we identify $M_p(C)$ with $\mathrm{End}_K(V)$ the function $\psi \in \mathcal{B}$. Let $\phi = B\psi$ (see Proposition 3.3). Now, $L(f) = \psi(f) = \phi(f)$ for every $f \in I_{c,\delta}(G)$ (cf. (5)). Therefore, by Proposition 3.7, ϕ satisfies the functional equation (4). To finish the proof we have to show that $\phi(e) = I$. From the fact that ψ satisfies (3) we obtain $\psi(e)\psi(g) = \psi(g) = \psi(g)\psi(e)$, $g \in G$. Since $\psi(I_{c,\delta}(G))$ coincides with the linear span of $\{\psi(g)\colon g \in G\}$ it follows that $\psi(e) = I$, which implies $\phi(e) = I$ (see Corollary 3.4).

REMARK 3.8. If $V = V_\delta \oplus \dots \oplus V_\delta$ (p-times) is a K-module as above, it is easy to verify that there is an algebra isomorphism $\iota : \mathrm{End}(V_\delta) \otimes \mathrm{End}_K(V) \to \mathrm{End}(V)$ such that $\iota(T \otimes S) = (T\oplus\dots\oplus T)S$. Let $I(\delta) = C(K)*\overline{\chi}_\delta$; of course $I(\delta)$ is a $*$-algebra isomorphic to $\mathrm{End}(V_\delta)$, more precisely, if we make use of the natural identification $\mathrm{End}(V_\delta) \simeq V_\delta \otimes V_\delta^*$ then an isomorphism ℓ can be described by $\ell(v \otimes \lambda)(k) = d(\delta)\lambda(k^{-1}.v)$ for all $v \in V_\delta$, $\lambda \in V_\delta^*$, $k \in K$. Now the relation between the linear maps $\phi\colon C_{c,\delta}(G) \to \mathrm{End}(V)$ and $\psi\colon I_{c,\delta}(G) \to \mathrm{End}_K(V)$, defined by $\phi \in A$ and $\psi = A\phi$, can be explained appealing to the following structural fact due to Dieudonné (cf. Dieudonné [1], p. 237): the bilinear map $(a,f) \to a*f$ of $I(\delta) \otimes I_{c,\delta}(G)$ into $C_c(G)$, establishes a $*$-algebra isomorphism of the tensor product $*$-algebra $I(\delta) \otimes I_{c,\delta}(G)$ with $C_{c,\delta}(G)$. Then

$$\begin{tikzcd} I(\delta) \otimes I_{c,\delta}(G) \arrow[r, "\simeq"] \arrow[d, "\ell \otimes \psi"'] & C_{c,\delta}(G) \arrow[d, "\phi"] \\ \mathrm{End}(V_\delta) \otimes \mathrm{End}_K(V) \arrow[r, "\iota"'] & \mathrm{End}(V) \end{tikzcd}$$

commutes. A simple and important consequence of this is the following: there exists a natural one-to-one correspondence between the equivalence classes of finite dimensional irreducible representations of $I_{c,\delta}(G)$ and those of $C_{c,\delta}(G)$.

REMARK 3.9. Let $\phi\colon G \to \mathrm{End}(V)$ be an irreducible spherical function of height p. Let $\varphi(g) = \mathrm{tr}\,\phi(g)$, $g \in G$, and put $\varphi_0 = d(\delta)^{-1}\varphi$. Then

$$\begin{tikzcd} I_{c,\delta}(G) \arrow[rr, "\psi"] \arrow[dr, "\varphi_0"'] & & M_p(C) \arrow[dl, "tr"] \\ & C & \end{tikzcd}$$

is commutative, where $\varphi_0(f) = \int_G f(g)\varphi_0(g)dg$ and $\psi(f) = \int_G f(g)(A\phi)(g)dg \in \mathrm{End}_K(V) \simeq M_p(C)$, $f \in I_{c,\delta}(G)$. According to Proposition 3.2 φ_0 satisfies (3) if and only if tr: $M_p(C) \to C$ is a homomorphism which corresponds to p=1. Therefore we have proved that φ satisfies

$$\varphi(x)\varphi(y) = d(\delta)\int_K \varphi(k\ x\ ky^{-1})dk$$

for arbitrary $x,y \in G$, if and only if p=1 (cf. Godement [1], p. 524).

For completeness we shall point out the following. Suppose that every topologically completely irreducible Banach representation of $C_{c,\delta}(G)$ is finite dimensional (see Warner [1], p. 228). Then the set of all irreducible spherical functions of type δ separates the points of $C_{c,\delta}(G)$. In fact, in virtue of the Gelfand-Raikov Theorem the set of all topologically irreducible unitary representations of G separates the points of $C_c(G)$. Let $f \in C_{c,\delta}(G)$, $f \neq 0$ and let U be a topologically irreducible unitary representation of G such that $U(f) \neq 0$. But $U(f) = U(\overline{X}_\delta * f * \overline{X}_\delta) = U(\overline{X}_\delta)U(f)U(\overline{X}_\delta)$, which says $\phi(f) \neq 0$, ϕ being the spherical function of type δ associated to U. As a consequence of this we have:

PROPOSITION 3.10. *The following properties are equivalent:*

(i) $I_{c,\delta}(G)$ *is commutative.*

(ii) *Every irreducible spherical function of type* δ *is of height one.*

(iii) $I_{c,\delta}(G)$ *is the center of* $C_{c,\delta}(G)$.

Proof. If (ii) holds, then $I_{c,\delta}(G)$ admits sufficiently many one dimensional representations, hence (i). Conversely, if $I_{c,\delta}(G)$ is commutative, then every finite dimensional irreducible representation of $I_{c,\delta}(G)$ is one dimensional so that every irreducible spherical function of type δ is of height one. It is clear that (iii) implies (i). To complete the proof it suffices to show that (ii) implies (iii). Let $f \in I_{c,\delta}(G)$, then for any $h \in C_{c,\delta}(G)$ and any irreducible spherical function ϕ of type δ we have $\phi(f*h) = \phi(f)\phi(h) = \phi(h)\phi(f) = \phi(h*f)$ since $\phi(f)$ is a scalar for every $f \in I_{c,\delta}(G)$. Therefore

$I_{c,\delta}(G)$ is contained in the center of $C_{c,\delta}(G)$. Furthermore, if f belongs to the center of $C_{c,\delta}(G)$ then $\phi(f)$ is a scalar in every irreducible spherical function of type δ, hence $\phi(f^0) = \phi(f)$, which proves that $f^0 = f$.

If ϕ: $G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ and height p, the function $A\phi = \psi$: $G \longrightarrow \mathrm{End}_K(V) \simeq M_p(C)$ should be considered as the other face of the same coin. Thus a spherical function ψ(on (G,K)) of type δ is also a continuous function on G with values in End(W) (W a finite dimensional vector space) such that:

$$\text{(i)}\quad \psi(e) = I.$$

$$\text{(ii)}\quad \chi_\delta * \psi = \psi.$$

$$\text{(iii)}\quad \psi(x)\psi(y) = \int_K \psi(k\,x\,k^{-1}y)dk \qquad \text{for all} \quad x,y \in G.$$

The dimension of W is the height of ψ.

PROPOSITION 3.11. *Let* ψ: $G \longrightarrow \mathrm{End}(W)$ *be a continuous* K-*central function which satisfies* (iii). *Then* ψ *can be decomposed in a unique way as the direct sum* $\psi = 0 + \sum_\delta \psi_\delta$ *of a zero function and of spherical functions* ψ_δ *of type* δ.

Proof. Note that for any $g \in G$,

$$(\chi_\delta * \psi)(g) = \int_K \chi_\delta(k)\psi(k^{-1}g)dk = \int_K\int_K \chi_\delta(k)\psi(k_1 k^{-1} k_1^{-1} g)dk_1 dk =$$

$$= \int_K \chi_\delta(k)\psi(k^{-1})\psi(g)dk = (\chi_\delta * \psi)(e)\psi(g) .$$

Because ψ is K-central, $\chi_\delta * \psi = \psi(\chi_\delta * \psi)(e)$ follows as before. Consequently, $(\chi_\delta * \psi)(e)\psi = \chi_\delta * \psi = \psi(\chi_\delta * \psi)(e)$.

Given $\delta, \delta' \in \hat{K}$ we have

$(\chi_\delta * \psi)(e)\ (\chi_{\delta'} * \psi)(e) = (\chi_{\delta'} * (\chi_\delta * \psi)(e)\psi)(e) = (\chi_{\delta'} * \chi_\delta * \psi)(e)$

showing that $(\chi_\delta * \psi)(e)$, $\delta \in \hat{K}$, are orthogonal projections, and therefore they are zero for almost all $\delta \in \hat{K}$. Hence, $\psi(k) = \sum_\delta (\chi_\delta * \psi)(k)$ all $k \in K$, and in particular $\psi(e) = \sum_\delta (\chi_\delta * \psi)(e)$.

We also have $\quad \psi(e)\psi(x) = \psi(x) = \psi(x)\psi(e) \quad$ for all $\quad x \in G$.

Therefore $\quad \psi = (I-\psi(e))\psi + \sum_\delta (\chi_\delta * \psi)(e)\psi = (I-\psi(e))\psi + \sum_\delta \chi_\delta * \psi \quad ,$

which clearly completes the proof of the proposition.

The K-central functions ψ: $G \longrightarrow \mathrm{End}(W)$ which satisfies (iii) are precisely those which give a representation of $I_c(G)$ on W.

4. DIFFERENTIAL PROPERTIES OF SPHERICAL FUNCTIONS. THE ALGEBRA $D_0(G)$ AND THEIR REPRESENTATIONS.

In this section, we assume that G is a connected Lie group.

LEMMA 4.1. *If* $\phi : G \longrightarrow \mathrm{End}(V)$ *is a spherical function, then* ϕ *is differentiable* (C^∞).

Proof. Let $\| \ \|$ be a norm on End(V) such that $\|TS\| \leq \|T\| \ \|S\|$ for all $T,S \in \mathrm{End}(V)$. Now, it is well-known that if $\|T-1\| < 1$, $T \in \mathrm{End}(V)$, then T is invertible. Since ϕ is continuous we can choose a neighborhood $\mathcal{U}$ of the identity in G such that $\|I - \phi(g)\| < 1$ for all $g \in \mathcal{U}$.

Let f be a C^∞ real valued function with compact support contained in $\mathcal{U}$ such that $f \geq 0$ and $\int_G f(g)dg = 1$. Then $\int_G f(g)\phi(g)dg$ is an automorphism of V. In fact

$$\|I - \int_G f(g)\phi(g)dg\| = \|\int_G f(g)(I - \phi(g))dg\| \leq \int_G f(g)\|I - \phi(g)\|dg < 1.$$

Finally,

$$\phi(x)\int_G f(y)\phi(y)dy = \int_G f(y) \int_K \chi_\delta(k^{-1})\phi(xky)dkdy =$$

$$= \int_K \chi_\delta(k^{-1}) \int_G f(k^{-1}x^{-1}y)\phi(y)dydk =$$

$$= \int_G \left(\int_K \chi_\delta(k^{-1})f(k^{-1}x^{-1}y)dk\right)\phi(y)dy$$

which shows that ϕ is C^∞.

Let D(G) denote the algebra of all left invariant differential operators on G and let $D_0(G)$ denote the set of operators in D(G) which are invariant under all right translations from K. Of course $D_0(G)$ is a subalgebra of D(G).

LEMMA 4.2. *Let* ϕ *be a spherical function of type* δ. *Then*

$$[D\phi](g) = \phi(g)[D\phi](e)$$

for all $D \in D_0(G)$, $g \in G$.

Proof. For each $D \in D(G)$ we get from 1.(2)

$$\phi(x)[D\phi](y) = \int_K \chi_\delta(k^{-1})[D\phi](xky)dk.$$

Putting y=e we obtain

$$\phi(x)[D\phi](e) = \int_K \chi_\delta(k^{-1})[D\phi](xk)dk.$$

If $D \in D_0(G)$, then

$$[D\phi](gk) = [D(\phi * \delta_k)](g) = [D\phi](g)\phi(k)$$

for all $g \in G$. Therefore,

$$\phi(g)[D\phi](e) = \int_k \chi_\delta(k^{-1})[D\phi](g)\phi(k)dk = [D\phi](g)\phi(e)$$

which proves the lemma.

PROPOSITION 4.3. *Any spherical function on* G *is analytic.*

Proof. Suppose $f: G \to V$ is a C^∞ function such that $[Df](g) = Tf(g)$, all $g \in G$, for some $T \in \mathrm{End}(V)$ and some $D \in D(G)$. We can find a basis $\{e_i\}$ of V so that T is given by a matrix of the form

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Let $S \in \mathrm{End}(V)$ be the linear map defined by $Se_i = \lambda_i e_i$, $i=1,2,\dots,n$. Then $(D-S)^n f = (T-S)^n f = 0$. Hence, if f_i denotes the i^{th}-component of f with respect to $\{e_i\}$ we have $(D-\lambda_i)^n f_i = 0$. If D is elliptic, by a theorem of S. Bernstein and induction on n, it follows that every solution of an equation $(D-\lambda)^n h = 0$ is analytic. Therefore, in this case our function f is analytic.

It is well-known that $D_0(G)$ contains elliptic operators (cf.Godement [1], p. 539) thus, the proposition follows now directly from Lemma 4.2.

We shall frequently use the following basic property:

LEMMA 4.4. *Let* f *be a* K-*central analytic function on* G; *then* f=0 *is equivalent to*

$$[Df](e) = 0 \quad \textit{for every} \quad D \in D_0(G).$$

Proof. Since f is analytic and since G is connected, it is clear that $f=0$ is equivalent to $[Df](e) = 0$ for all $D \in D(G)$. Let $D^{L(g)}$, $D^{R(g)}$ denote respectively the left and right translation by g of $D \in D_0(G)$. We can form the integral

$$D_0 = \int_K D^{R(k)}\, dk$$

which is an operator in $D_0(G)$. Since f is K-central we have

$$[D^{R(k)}f](e) = [D^{L(k)}f](e) = [Df](e) \quad , \quad \text{so}$$

$$[D_0 f](e) = \int_K [D^{R(k)} f](e)dk = [Df](e)$$

This proves the lemma.

PROPOSITION 4.5. *Let* ψ: G $\to$ End(V) *be a* K-*central analytic function. Then* ψ *satisfies the functional equation* 3.(3) *if and only if the mapping* ψ: D $\to$ $[D\psi](e)$ *is a representation of* $D_0(G)$.

Proof. From 3.(3) one gets, in a completely similar way as we proved Lemma 4.2, $[D\psi](g) = \psi(g)[D\psi](e)$ for every $D \in D_0(G)$. Conversely, it is also clear in virtue of Lemma 4.4, that this implies 3.(3). Invoking once more Lemma 4.4 one sees that

$$[D\psi](g) = \psi(g)[D\psi](e) \quad \text{for every} \quad D \in D_0(G)$$

is equivalent to require that ψ: $D_0(G) \to$ End(V) is a representation.

In the following proposition (V,π) will be a K-module as in Section 3.

PROPOSITION 4.6. *Let* ϕ: G $\to$ End(V) *be an analytic function such that* $\phi(kgk_1) = \pi(k_1)\phi(g)\pi(k_2)$ *(all* $k,k_1 \in K$*). Then* ϕ *satisfies the functional equation* 1.(2) *if and only if the mapping* ϕ: D $\to$ $[D\phi](e)$ *is a representation of* $D_0(G)$.

Proof. First of all let us observe that $[D\phi](e) \in \mathrm{End}_K(V)$ for all $D \in D_0(G)$. In fact, if $D \in D_0(G)$ we have

$$[D\phi](e)\pi(k) = [D\phi^{R(k^{-1})}](e) = [D^{R(k)}\phi](k) = [D^{L(k)}\phi](k) =$$

$$= [D\phi^{L(k^{-1})}](e) = \pi(k)[D\phi](e).$$

Let $\psi = A\phi$ (see Proposition 3.3), then

$$\psi(D) = \int_K \pi(k)[D\phi](e)\pi(k^{-1})dk = \phi(D)$$

for every $D \in D_0(G)$. Therefore, the proposition follows at once from Propositions3.6 and 4.5.

REMARK 4.7. Of course, combining Proposition 3.3 and Lemma 4.4 one gets the following analogue of Lemma 4.4 for analytic functions ϕ: G $\to$ End(V) which satisfies $\phi(kgk_1) = \pi(k)\phi(g)\pi(k_1)$, for all $k,k_1 \in K$, namely: $\phi = 0$ if and only if $[D\phi](e) = 0$ for all $D \in D_0(G)$.

We shall consider a topology on $D_0(G)$, introduced by Godement (cf. Godement [1], p. 538). We say that a variable $D \in D_0(G)$ converges to a given $D_0 \in D_0(G)$ if $[Df](e)$ converges to $[D_0 f](e)$ for every analytic K- central function f. This topology is precisely the weak topology defined on $D_0(G)$ by the natural pairing of $D_0(G)$ and the vector space of all K-central analytic functions on G.

We are now in a position to prove the main result of this section which is an infinitesimal counterpart to Theorem 3.1.

We recall that if (V,π) is a finite dimensional K-module which is the direct sum of p irreducible equivalent submodules, we can identify $M_p(C)$ with $\mathrm{End}_K(V)$.

THEOREM 4.8. *If* $\phi: G \to \mathrm{End}(V)$ *is a spherical function then* $\phi: D \to [D\phi](e)$ *maps* $D_0(G)$ *into* $\mathrm{End}_K(V)$, *giving a continuous representation of* $D_0(G)$. *Conversely, any continuous finite dimensional representation of* $D_0(G)$ *is the direct sum of a zero representation and ones given by spherical functions.*

Proof. That $[D\phi](e) \in \mathrm{End}_K(V)$ for every $D \in D_0(G)$ was observed during the proof of Proposition 4.6. If we put $\psi = A\phi$ (see Proposition 3.3) we have $[D\phi](e) = [D\psi](e)$, which shows that $\phi: D_0(G) \to \mathrm{End}_K(V)$ is continuous, by the very definition of the topology in $D_0(G)$. From Proposition 4.6 we get that ϕ defines a representation of $D_0(G)$.

To prove the second part, let us assume that $L: D_0(G) \to M_p(C)$ is a continuous representation. By weak duality such a linear map is defined by a K-central analytic function $\psi: G \to M_p(C)$; $L(D) = [D\psi](e)$. Now by Proposition 4.5 we know that ψ satisfies

$$\psi(x)\psi(y) = \int_K \psi(kxk^{-1}y)\,dk \qquad \text{all} \quad x,y \in G ,$$

which in turn implies our contention (cf. Proposition 3.11).

Naturally, a subspace $W \subset C^p$ is $\psi(G)$-invariant if and only if it is $\psi(D_0(G))$-invariant ($\psi(D) = [D\psi](e)$, $D \in D_0(G)$). This follows at once from Lemma 4.4. Thus, in particular, Theorem 4.8 establishes a one-to-one correspondence between the equivalence classes of continuous finite dimensional irreducible representations of $D_0(G)$ and the equivalence classes of irreducible spherical functions on G.

The relation between the spherical function $\phi: G \to \mathrm{End}(V)$ and its associated representation of $D_0(G)$, is the exact generalization of the correspondence between a finite dimensional representation of G and the derived representation of the Lie algebra of G. In fact, if we take $K = \{e\}$ then $D_0(G)$ becomes the algebra $D(G)$ of all left invariant differential operators on G, which is isomorphic to the universal enveloping algebra of the complexification of the Lie algebra of G. Since in this case the spherical functions are precisely the finite dimensional representations and moreover, there is a natural one-to-one correspondence between the set of all representations of a Lie algebra a on V and the set of all representations of the universal enveloping algebra of a on V, our assertion is clear.

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