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ON SMALL SUBMODULES IN THE TOTAL QUOTIENT RING OF A COMMUTATIVE RING

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As a generalization of Nakayama's lemma, we know the concept of small submodules and many authors have studied those submodules in projectives [1], [3], [7] and [8]. In [2] and [4], we have some applications of small submodules to hollow modules. In this short note, we shall study small submodules from a little different point of view.

Let R be a commutative ring with identity and Q the total quotient ring of R. In the first section, we shall show that R is a small submodule in Q as an R-module if and only if every maximal ideal in R contains a non zero-divisor. In the second section, we assume R is a Dedekind domain. Then we shall determine all small submodules in any direct sums of copies of Q as R-modules.

1. COMMUTATIVE RINGS.

Throughout this note we always assume that a ring R is commutative and has the identity. By Q and D(R) (or briefly D) we shall denote the total quotient ring of R and the set of zero-divisors, respectively. We call an element in R-D *regular*.

LEMMA 1. *Let* T *be an* R-*submodule in* Q *such that* Q = R + T *and* $R \cap T$ *contains a regular element* x. *Then* T = Q.

Proof. Let $x^{-1} = r + t$; $r \in R$, $t \in T$. Then $1 = rx + tx \in T$ and so $T \supseteq R$ and T = Q.

THEOREM 2. *Let* R *be a commutative ring and* Q *its total quotient ring. Then* R *is small in* Q *as an* R-*module if and only if every maximal ideal in* R *contains a regular element.*

Proof. We assume R is not small in Q as an R-module. Then Q = R + T for some R-module T ($\neq$ Q). Since $T \cap R$ is an ideal contained in D(R) by Lemma 1, TQ = Q. Hence, we may assume that T is an ideal in Q. Let T' be a maximal ideal in Q containing T and put $M = T' \cap R$.

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Since $T' = MQ$, M is a prime ideal in R which is contained in D. If there exists an ideal M' in R such that $D \supseteq M' \supseteq M$, $Q \neq M'Q \supseteq MQ = T'$. Hence $M = M'$. Next, we shall show that M is a maximal ideal in R. Let N be an ideal in R, which contains properly M. Then N contains a regular element x from the above observation. Let $x^{-1} = r + p(b/a)$; $b, r \in R$, $a \in R-D$ and $p \in M$, since $Q = R + T'$. Then $(1-rx)a = pbx \in M$ and $a \notin M$. Hence, $(1-rx) \in M \subseteq N$, which implies $N = R$. Therefore, D contains the maximal ideal M. Conversely, we assume that D contains a maximal ideal B. Let a be any regular element. Then $R = (a) + B$ and so $1 = ra + b$; $r \in R$, $b \in B$. Hence, $a^{-1} = r + ba^{-1} \in R + BQ$. Therefore, $Q = R + BQ$ and $Q \neq BQ$ since $B \subseteq D$.

COROLLARY 3. *Let* R *be a noetherian ring with* Q *the total quotient ring. Let* $\{P_i\}$ *be the set of associated prime ideals of* (0) (see [9], p. 211). *Then* R *is an* R*-small submodule in* Q *if and only if every* P_i *is not maximal.*

COROLLARY 4. *Let* R *be a semi-local ring. Then* R *is small in* Q *if and only if* J(R) *contains a regular element, where* J(R) *is the Jacobson radical.*

The following proposition is a direct consequence from Theorem 2, however we shall give a proof, which is interesting itself.

PROPOSITION 5. R *is small in* Q *if one of the following conditions is satisfied*:

1) *There exists an ideal containing properly* D.

2) $Q \neq R$ *and* $J(R) \supseteq D$.

3) $J(R) \not\subseteq D$.

Proof. We assume $Q = R + T$ for some R-submodule T. Let a be regular and $a^{-1} = r + t$; $r \in R$, $t \in T$. Then $1 = ra + ta$ and $ta = 1-ra \in R \cap T$.

1) Let a be in A-D, where A is an ideal containing D. If ta is in D, $1 = ra + ta \in A$, which is a contradiction. Therefore, ta is regular and $Q = T$ from Lemma 1.

2) Since $Q \neq R$, there exists a regular element a such that $a^{-1} \notin R$. If ta is in D, $ra = 1-ta$ has the inverse in R from the assumption. Hence, so does a, which is a contradiction. Therefore, $Q = T$ from Lemma 1.

3) Let a be in J(R)-D. Then $ta = 1-ra$ is an unit in R. Hence, $Q = T$.

COROLLARY 6. *If* $Q \neq R$ *and* R *is one of the followings*
1) R *is domain*, 2) R *is local and* 3) (0) *is primary, then* R *is small in* Q.

Proof. It is clear from the above.

We note that all conditions in Proposition 5 are independent each other and if R is artinian, then Q = R.

PROPOSITION 7. *We assume* R *is small in* Q *and* B *is an ideal containing a regular element. Then* $B^{-n} = \{x \in Q \mid B^n x \subseteq R\}$ *is small in* Q.

Proof. Let $Q = B^{-n} + T$ for some R-module T. Since QB = Q, $Q = B^n B^{-n} + B^n T \subseteq R + T \subseteq Q$. Hence, Q = T.

2. DEDEKIND DOMAINS.

In this section, we assume R is a Dedekind domain.

LEMMA 8. *Let* R *be a Dedekind domain. Then every* R-*small submodule in* Q *is contained in a small submodule of a form* $\sum P_i^{-n_i}$, *where* P_i *runs through all maximal ideals and the* n_i *is a natural number for all* i.

Proof. Let S be an R-small submodule in Q. Then so is R + S by Corollary 6. Hence, we may assume $S \supseteq R$. Now $Q/R = \sum_P \oplus \sum_n P^{-n}/R$ by [5]. Since the $\sum_n P^{-n}/R$ is an uni-serial module, every proper submodule is small in it. Hence, $S/R = \sum_{P_i} \oplus P_i^{-n_i}/R$. Since R is small in Q, $\sum P_i^{-n_i}$ is small in Q from the above.

THEOREM 9. *Let* R *be a Dedekind domain with* Q *the quotient field. We put* $Q^{(I)} = \sum_I \oplus Q_\alpha$; $Q_\alpha = Q$ *and* I *is any index set. Then every* R-*small submodule in* $Q^{(I)}$ *is contained in* $\sum \oplus S_{\alpha_i}$, *where the* S_{α_i} *is a small submodule given in Lemma 8 and the converse* (cf. [3], Proposition 1 and Remark 3).

Proof. We assume $S \not\subseteq \sum_J \oplus Q_{\alpha_i}$ for any finite subset J of I and show a contradiction. We put $Q^{(I)} = \sum_{i=1}^{\infty} \oplus Q_i \oplus \sum \oplus Q_\beta$. When we consider the projection of $Q^{(I)}$ to $\sum \oplus Q_i$, we may assume $Q^{(I)} = \sum_{i=1}^{\infty} \oplus Q_i$. Let p_i be the projection of $Q^{(I)}$ to ith component, we may assume $S \subseteq \sum \oplus R_i$ and $p_i(S) \neq 0$ for all i. First, we consider $S_1 = \{s_1 \in R \mid$ there exists $s \in S$ such that $s = s_1 + s_2 + \dots\}$. Since S_1 is an ideal in R, $S_1 = s_1^{(1)}R + s_1^{(2)}R + \dots + s_1^{(t_1)}R$, $(s_1^{(i)} \neq 0)$. Let $s^{(1,i)} = s_1^{(i)} + s_2' + \dots + s_{n(1,i)}'$ be in S. Put $m_1' = \max\{n(1,i)\}$ and take m_2 such that $m_1' < m_2$. Let $S_{m_2} = \{s_{m_2} \in R \mid$ there exists $s \in S$ such that $s = o + s_2 + \dots + s_{m_2} + \dots\}$. Since $m_1' < m_2$ and $p_{m_2}(S) \neq 0$, $S_{m_2} \neq 0$. Let $S_{m_2} = s_{m_2}^{(1)}R + s_{m_2}^{(2)}R + \dots + s_{m_2}^{(t_2)}R$ and $s^{(2,i)} =$

$= o + \ldots + s_{m_2}^{(i)} + \ldots + s_{n(2,i)}'$, $(s_{m_2}^{(i)} \neq 0)$. Put $m_2' =$ $= \max \{n(2,i)\}$ and take m_3 such that $m_3 > m_2'$. Let $S_{m_3} = \{s_{m_3} \in R \mid$ there exists $s \in S$ such that $s = o + s_2' + \ldots + s_{m_2} + \ldots + s_{m_3} + \ldots\}$.

Repeating those arguments, we obtain a sequence $m_1 = 1, m_2, m_3, \ldots$ such that $p(S) \supseteq \sum_{i=1}^{\infty} \oplus S_{m_i} \supseteq \sum_{i=1}^{\infty} \oplus s_{m_i}^{(1)}R$, where $p = \sum p_{m_i}$. Then $\sum_{i=1}^{\infty} \oplus s_{m_i}R$ must be small in $p(Q^{(I)}) = \sum_{i=1}^{\infty} \oplus Q_{m_i}$. Now, we define $f\colon \sum \oplus Q_{m_i} \to Q$ by setting $f(q_{m_i}) = x^{-i}(s_{m_i}^{(1)})^{-1}q_{m_i}$, where $x \in A-A^2$ for a fixed prime A. Then $f(\sum_{i=1}^{\infty} \oplus s_{m_i}^{(1)}R)$ is not small in Q by Lemma 8, which contradicts the assumption that S is small. Therefore, $S \subseteq \sum_{J} \oplus Q_{\alpha}$, for some finite subset J of I. The remaining parts are clear from Lemma 8.

We note that if R is a noetherian U.F.D., then we can obtain a similar results to the above.

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