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MEROMORPHIC DIFFERENTIAL OPERATORS

E. Robert Fernholz

ABSTRACT. Meromorphic differential operators on a reduced locally irreducible complex analytic space are studied in this paper. Conditions are established for regularity of such operators.

1. INTRODUCTION.

This section contains some basic definitions and facts about meromorphic functions on complex analytic spaces.

Let X be a connected complex analytic space and $\mathcal{O}_X$ be the sheaf of germs of analytic functions on X. It will be assumed that X is reduced and locally irreducible. This means that each $x \in X$ has arbitrarily small neighborhoods U such that $(U, \mathcal{O}_X|U)$ is isomorphic as a ringed space to $(V, \mathcal{O}_W/\mathcal{J}_V)$ where W is the unit polydisc in C^N for some N, V is a closed irreducible subvariety of W, and $\mathcal{J}_V$ is the sheaf of ideals of analytic functions vanishing on V. In this case the stalks $\mathcal{O}_{X,x}$ are integral domains and $\mathcal{M}_X$, the sheaf of quotients of $\mathcal{O}_X$, is a sheaf of fields. $\mathcal{M}_X$ is the *sheaf of germs of meromorphic functions* on X. $\mathcal{O}_X$ can be considered a subsheaf of $\mathcal{M}_X$.

The results in this paper are of a local nature and the proofs are most conveniently carried out for germs. If X is an analytic space or a subvariety of an analytic space and $x \in X$, then X_x denotes the germ of X at x. For $f \in \mathcal{O}_{X,x}$, let f(x) denote the value of f at x, that is, the residue class modulo the maximal ideal. For a function $m \in \Gamma(U, \mathcal{M}_X)$ and a point $x \in U$, let m_x denote the germ of m at x.

Let U be an open subset of X and $m \in \Gamma(U, \mathcal{M}_X)$. The *singular set* of m, sing(m), is defined to be $\{x \in U: m_x \notin \mathcal{O}_{X,x}\}$. Since $m_x \in \mathcal{M}_{X,x}$ can be represented (not necessarily uniquely) by $m_x = f/g$ where $f, g \in \mathcal{O}_{X,x}$, x will be in sing(m) if and only if $g(x) = 0$ for all possible representations $m_x = f/g$. Because of this, sing(m) is an analytic subvariety of U. A germ $m_x \in \mathcal{M}_{X,x}$ can always be represented by a function m in a neighborhood of x, so $\text{sing}(m_x)$

can be defined as $(\mathrm{sing}(m))_x$.

A more thorough discussion of meromorphic functions can be found in Narasimhan [2], p. 88.

Certain facts about extensions and restrictions of holomorphic differential operators will be assumed; these can be found in [1].

2. MEROMORPHIC DIFFERENTIAL OPERATORS.

Let Diff^n_X be the sheaf of germs of holomorphic differential operators of order n on X. Diff^n_X is a coherent sheaf of $\mathcal{O}_X$-modules. A holomorphic differential operator $D \in \Gamma(X,\mathrm{Diff}^n_X)$ can be considered to be a C-homomorphism $D: \Gamma(X,\mathcal{O}_X) \longrightarrow \Gamma(X,\mathcal{O}_X)$ or a C-linear sheaf homomorphism $D: \mathcal{O}_X \longrightarrow \mathcal{O}_X$. A germ of a differential operator, $D_x \in \mathrm{Diff}^n_{X,x}$, defines a C-homomorphism $D_x: \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}$.

The *sheaf of germs of* n^{th} *order meromorphic differential operators* on X is defined to be $M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}^n_X$. An n^{th} order meromorphic differential operator on X is a section $D \in \Gamma(X, M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}^n_X)$ and can be considered to be a C-homomorphism $D: \Gamma(X,\mathcal{O}_X) \longrightarrow \Gamma(X,M_X)$ or a C-linear sheaf homomorphism $D: \mathcal{O}_X \longrightarrow M_X$. A germ $D_x \in (M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}^n_X)_x$ defines a C-homomorphism $D_x: \mathcal{O}_{X,x} \longrightarrow M_{X,x}$.

Diff^n_X can be considered a subsheaf of $M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}^n_X$.

A meromorphic differential operator D on X is *singular* at $x \in X$ if its germ at x is not in $\mathrm{Diff}^n_{X,x}$. We denote the set of such singular points by sing(D). The set sing(D) is a subvariety of X. For a germ $D_x \in (M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}^n_X)_x$, $\mathrm{sing}(D_x)$ can be defined as was done for meromorphic functions.

PROPOSITION 2.1. *Let* X *be an analytic space and* D *be a meromorphic differential operator on* X. *If* $D(\mathcal{O}_X) \subset \mathcal{O}_X$ *then* D *is holomorphic.*

Proof. Let n be the order of D. It suffices to prove this proposition locally, so it can assumed that $X \subset \mathbb{C}^N$ as a closed analytic subvariety. For $x \in X$, $\mathrm{Diff}^n_{X,x}$ is finitely generated as an $\mathcal{O}_{X,x}$-module since Diff^n_X is coherent.

Let D denote the germ of D at x. Then

$$D = m_1 D_1 + \dots + m_k D_k$$

where $D_1,\dots,D_k \in \mathrm{Diff}^n_{X,x}$ are the generators at x and $m_1,\dots,m_k \in M_{X,x}$. Each m_i can be represented by $m_i = f_i/g_i$ with $f_i, g_i \in \mathcal{O}_{X,x}$. Let $g = g_1 \cdots g_k \in \mathcal{O}_{X,x}$. Then the operator

$$D' = gD = f_1 D_1 + \dots + f_k D_k \in \mathrm{Diff}^n_{X,x}$$

and $D = 1/g\, D'$. Let $m = 1/g \in M_{X,x}$.

g can be extended to $\tilde{g} \in \mathcal{O}_{\mathbf{C}^N,x}$ and thus m can be extended to $\tilde{m} = 1/\tilde{g} \in M_{\mathbf{C}^N,x}$. Furthermore D' can be extended to $\tilde{D}' \in \mathrm{Diff}^n_{\mathbf{C}^N,x}$ and the operator $\tilde{D} = \tilde{m}\,\tilde{D}' \in (M_{\mathbf{C}^N} \otimes_{\mathcal{O}_{\mathbf{C}^N}} \mathrm{Diff}^n_{\mathbf{C}^N})_x$.

It is clear that $\mathrm{sing}(\tilde{D}) \subset \mathrm{sing}(\tilde{m})$ and since $\tilde{m}$ is an extension of $m \in M_{X,x}$, $\mathrm{sing}(\tilde{m}) \not\supset X_x$.

Let $\tilde{\varphi} \in \mathcal{O}_{\mathbf{C}^N,x}$ and let $\varphi = \tilde{\varphi}|X \in \mathcal{O}_{X,x}$.

By hypothesis $D(\varphi) = m\, D'(\varphi) \in \mathcal{O}_{X,x}$. Furthermore $\tilde{D}(\tilde{\varphi}) \in M_{\mathbf{C}^N,x}$ and $\tilde{D}(\tilde{\varphi})|X \in M_{X,x}$ because $\mathrm{sing}(\tilde{D}(\tilde{\varphi})) \subset \mathrm{sing}(\tilde{D})$ and $\mathrm{sing}(\tilde{D}) \not\supset X_x$.

Now, $\tilde{D}(\tilde{\varphi})|X = [\tilde{m}\,\tilde{D}'(\tilde{\varphi})]|X = \tilde{m}|X . \tilde{D}'(\tilde{\varphi})|X = m\, D'(\varphi) = D(\varphi)$ so if $\tilde{\varphi} \in \mathcal{O}_{\mathbf{C}^N,x}$ then $\tilde{D}(\tilde{\varphi})|X \in \mathcal{O}_{X,x}$.

$\tilde{D}'$ can be expressed in terms of the generators of $\mathrm{Diff}^n_{\mathbf{C}^N}$ as

$$D' = \sum a_{i_1 \dots i_N} \frac{\partial^{i_1 + \dots + i_N}}{\partial z_1^{i_1} \dots \partial z_N^{i_N}}$$

with $a_{i_1 \dots i_N} \in \mathcal{O}_{\mathbf{C}^N,x}$, so

$$\tilde{D} = \sum \tilde{m}\, a_{i_1} \dots a_{i_N} \frac{\partial^{i_1 + \dots + i_N}}{\partial z_1^{i_1} \dots \partial z_N^{i_N}} .$$

The functions $z_1^{j_1} \dots z_N^{j_N}$ are all in $\mathcal{O}_{\mathbf{C}^N,x}$ so $\tilde{D}(z_1^{j_1} \dots z_N^{j_N})|X \in \mathcal{O}_{X,x}$. Since all the $\tilde{m}\, a_{i_1 \dots i_N}$ are linear combinations of the $\tilde{D}(z_1^{j_1} \dots z_N^{j_N})$, $\tilde{m}\, a_{i_1 \dots i_N}|X \in \mathcal{O}_{X,x}$

Let $b_{i_1 \dots i_N} = (\tilde{m}\, a_{i_1 \dots i_N})|X \in \mathcal{O}_{X,x}$.

Let $\tilde{b}_{i_1 \dots i_N}$ be an extension of $b_{i_1 \dots i_N}$ to $\mathcal{O}_{\mathbf{C}^N,x}$.

Let
$$\bar{D} = \sum \tilde{b}_{i_1 \dots i_N} \frac{\partial^{i_1 + \dots + i_N}}{\partial z_1^{i_1} \dots \partial z_N^{i_N}} .$$

The operators $\tilde{D}$ and $\bar{D}$ agree on X_x, so for $\psi \in \mathcal{O}_{\mathbf{C}^N,x}$

$$\tilde{D}(\psi)|X = \bar{D}(\psi)|X .$$

If $\psi \in d_{X,x}$ then $\psi|X = 0 \in \mathcal{O}_{X,x}$ so $\tilde{D}(\psi)|X = 0$.

Thus $\overline{D}(\psi)|X = 0$ and since $\overline{D}$ is a holomorphic operator, $\overline{D}(d_{X,x}) \subset d_{X,x}$ and $\overline{D}$ induces a holomorphic differential operator on X_x. Since $\overline{D}$ and $\widetilde{D}$ agree on X_x, the induced operator must coincide with D and thus D is holomorphic.

3. MEROMORPHIC OPERATORS WITH STABLE IDEALS.

Let D be a differential operator on an analytic space X and let V be an analytic subvariety of X with associated sheaf of ideals d_V.

DEFINITION 3.1. d_V *is stable under* D *if for all* $x \in V$, $D(d_{V,x}) \subset d_{V,x}$.

Let V and W be analytic subvarieties of X.

DEFINITION 3.2. V *is transversal to* W *if there is an* $x \in V \cap W$ *such that* $V_x \not\subset W_x$ *and* $W_x \not\subset V_x$ *(in which case* V *is transversal to* W *at* x).

The proposition and corollary that follow deal with varieties transversal to sing(D).

PROPOSITION 3.3. *Let* D *be a meromorphic differential operator of order* n *on* X. *Let* $x \in X$. *Suppose that there is a germ* $f \in O_{X,x}$ $f \neq 0$, *such that for all* $g \in O_{X,x}$, $D(gf) \in O_{X,x}$. *Then for all* $g \in O_{X,x}$ *there is a representation* $D(g) = \frac{\varphi}{f^{n+1}}$ *where* $\varphi \in O_{X,x}$.

Proof. The proof is by induction on n, the order of D. For n=1, D = D' + h where D' is a meromorphic derivation and $h \in M_{X,x}$. For $g \in O_{X,x}$

$$D(fg) = D'(fg) + hfg \quad \in O_{X,x}$$

$$= f\, D'(g) + g\, D'(f) + hfg .$$

But $g\, D'(f) + hfg = g\, D(f) \in O_{X,x}$ so $f\, D'(g) = k \in O_{X,x}$ and $D'(g) = k/f$.

Now $\quad D(f^2) = 2f\, D'(f) + hf^2 \quad \in O_{X,x}$.

But $2f\, D'(f) \in O_{X,x}$ as above, so $hf^2 = k' \in O_{X,x}$ and $h = \frac{k'}{f^2}$.

Therefore $\quad D(g) = D'(g) + hg = \frac{k}{f} + \frac{k'g}{f^2} = \frac{\varphi}{f^2}$ where $\varphi \in O_{X,x}$.

Now suppose that order D = n and that the proposition is proved for differential operators of order $\leqslant$ n-1.

Let $g \in \mathcal{O}_{X,x}$. The operator D_f defined by

$$D_f(g) = D(fg) - f\,D(g)$$

has order $\leqslant n-1$ and satisfies the hypothesis $D_f(gf) \in \mathcal{O}_{X,x}$ for all $g \in \mathcal{O}_{X,x}$. Therefore

$$D_f(g) = \frac{\psi}{f^n} \quad \text{with} \quad \psi \in \mathcal{O}_{X,x} .$$

Now $D_f(g) = D(fg) - f\,D(g)$ with $D(fg) = k \in \mathcal{O}_{X,x}$ so $f\,D(g) = k - \frac{\psi}{f^n}$.

Thus $D(g) = \frac{\varphi}{f^{n+1}}$ where $\varphi = kf^n - \psi \in \mathcal{O}_{X,x}$.

COROLLARY 3.4. *Let* D *be a meromorphic differential operator on* X. *Let* V *be an analytic subvariety with sheaf of ideals* d_V. *If* d_V *is stable under* D *then* sing(D) *is not transversal to* V.

Proof. By hypothesis, for all $x \in V$, $D(d_{V,x}) \subset d_{V,x}$. Then for $f \in d_{V,x}$ and $g \in \mathcal{O}_{X,x}$, $D(gf) \in \mathcal{O}_{X,x}$ so by Proposition 3.3 $D(g) = \varphi f^{-N}$ with $\varphi \in \mathcal{O}_{X,x}$. Thus $d_{V,x} \subset d_{\mathrm{sing}(D),x}$ so $V_x \supset \mathrm{sing}(D_x)$. Therefore sing(D) is not transversal to V at x, and this holds for all $x \in V$.

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Universidad Nacional de Río Cuarto, Argentina.
and Hunter College, Cuny, U.S.A.

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