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CYCLIC RINGS

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ABSTRACT. Suppose that A is a ring with identity. Then A is cyclic if all modules M with a generating set $\{\ell_i | i \in w\}$ such that $\ell_i = \ell_{i+1}x_i$, $x_i \in A$, are in fact cyclic modules. Perfect rings are cyclic rings. All our examples of cyclic rings are perfect. Several properties of cyclic rings are established including other ways of characterizing cyclic rings. We believe that cyclic rings if not identical to the class of rings whose modules have minimal generating sets are very closely related to this class.

CYCLIC RINGS.

In the discussion below, all rings A have an identity and all modules are right unitary. A Steinitz ring is a local ring A whose Jacobson radical is T-nilpotent, i.e., given any sequence of elements $\{x_i | i \in w\}$, where $x_i \in R$, the Jacobson radical of A, there is some integer n such that $x_n \dots x_1 = 0$. A perfect ring is a ring A such that the Jacobson radical R of A is T-nilpotent and such that A/R is a semi-simple Artinian ring. For more information concerning these rings, see [2], [3], [4].

For a ring A, if $\{x_i | i \in w\}$ is any sequence of elements of A, then we define $F(\{x_i | i \in w\})$ to be the quotient of the free module generated by the countable set $\{u_i | i \in w\}$ modulo the free module generated by the countable set $\{v_i | i \in w\}$, where $v_i = u_i - u_{i+1}x_i$. A sequence $\{x_i | i \in w\}$ is T-nilpotent provided the module $F(\{x_i | i \in w\}) = 0$. A subset of A is seen to be T-nilpotent if and only if every sequence $\{x_i | i \in w\}$ with $x_i \in A$ for all $i \in w$, is T-nilpotent.

Thus one can certainly describe some properties of rings by giving properties of some (or all) of the modules $F(\{x_i | i \in w\})$ where $\{x_i | i \in w\}$ is a sequence in A. If $F(\{x_i | i \in w\})$ is always a cyclic module, then we shall call A a cyclic ring. It is the purpose of this paper to establish some properties of cyclic rings. We note that as we are actually talking about right Steinitz rings, right perfect rings so we are talking about right cyclic rings. With the modules taken to be right unitary we shall , suppress the terminology right cyclic ring and use the terminology cyclic ring instead.

Our main classification is contained in the following theorem

THEOREM 1. *A ring* A *is cyclic if and only if for any sequence* $\{x_i \mid i \in w\}$ *of elements of* A *there is an index* i_0, *such that for all* $i \geqslant i_0$ *there is an index* $n \geqslant i+1$ *and an element* m_i *of* A *for which* $x_n \ldots x_{i+1} = x_n \ldots x_{i+1} x_i m_i$.

From this it follows almost immediately that the epimorphic image of a cyclic ring is a cyclic ring and a direct sum of two cyclic rings is a cyclic ring. All Steinitz rings are cyclic rings. To enlarge the class of examples further one shows that

THEOREM 2. *If* A *is a ring for which there exists an integer* k *such that every properly ascending chain of principal right ideals contains at most* k *terms, then* A *is cyclic.*

From this we see that if A is any algebra over a division ring D such that A is finite dimensional as a right vector space over D, then A is a cyclic ring since right ideals are subspaces.

From this we find that any semi-simple Artinian ring is a cyclic ring. We show that over a perfect ring A, a module M has m generators if and only if M/MR has m generators, where R is the Jacobson radical of A, and so perfect rings are cyclic as well. One also shows without much difficulty that

THEOREM 3. *If* A *is a cyclic ring, then the Jacobson radical* R *of* A *is* T-*nilpotent, and* A *satisfies the ascending chain condition on right principal ideals. If* $xy = 0$ *implies* $y = 0$, *then* x *is a unit. Also, if* $xy = 1$, *then* $yx = 1$.

Another way of identifying perfect rings is by stating that A is perfect provided the descending chain condition on left principal ideals holds. Thus rings of the type described in theorem 2 are not only cyclic but also left perfect. If D is a division ring and if G is a finite group, then D G = A is an algebra over D which is finite dimensional as a right and left vector space and hence the descending chain condition on right and left principal ideals holds with an upper bound $k = |G|$. This way we can construct perfect (cyclic) rings which are neither semi-simple Artinian nor Steinitz. Indeed, let D = GF(p), the field with p elements and let G be a finite group whose order $|G|$ is divisible by p and a prime q with $(p,q) = 1$. Since G is not a p-group D G is not a Steinitz ring [1] and since G contains elements of order p, D G is not a semi-simple Artinian ring by Maschke's theorem. If A = B G is the group ring of the group G with coefficients in the ring B, then if A is a cyclic ring, B is also cyclic since it is an epimorphic image of A by the norm homomorphism (For relevant information see [5], pp. 86-87).

A local cyclic ring is a Steinitz ring. Steinitz rings are those local rings whose modules have minimal generating sets. It is a question of some interest to give conditions identifying those rings whose modules have minimal generating sets. A necessary condition is that the Jacobson radical be T-nilpotent. If we require that every generating set of a module M over A contain a minimal generating set, then the ring A is in fact cyclic, since a module $F(\{x_i | i \in w\})$ contains a minimal generating set which is a subset of $\{h_i | i \in w\}$ with $h_i = u_i + V$, V generated by $\{v_i | i \in w\}$, if and only if it is cyclic. This suggests that possibly cyclic rings are those rings for which modules have minimal generating sets. This last property would make cyclic rings a very interesting class of rings indeed.

If we let E(A) be the collection of idempotents of the ring A, and if $e \leqslant f$ provided $ef = fe = e$, then it follows easily from theorem 3 that

THEOREM 4. *If* A *is a cyclic ring, then* E(A) *equipped with the partial order* $\leqslant$ *satisfies both the ascending chain condition and the descending chain condition.*

From this we find that every cyclic ring A is in fact a unique direct sum $A = A_1 + \ldots + A_n$ of cyclic rings A_i, where A_i contains only central idempotents 0 and 1. Furthermore it follows that a commutative ring is cyclic if and only if it is a finite direct sum of Steinitz rings which is so if and only if the ring is in fact perfect. One also shows that cyclic regular rings satisfy the descending chain condition on left principal ideals and are thus perfect. Finally, we have no examples of cyclic rings which are not also perfect.

Proof of theorem 1 and consequences. Suppose that the conditions stated in theorem 1 hold, and that $\{x_i | i \in w\}$ is any sequence of A. Let $F\{x_i | i \in w\}$ be the corresponding module and suppose $i \geqslant i_0$. Let h_i be the image of u_i in $F(\{x_i | i \in w\})$, as above. Then $h_{n+1}x_n \ldots x_{i+1} = h_{n+1}x_n \ldots x_i m_i$ implies $h_{i+1} = h_i m_i$ and since $h_i = h_{i+1}x_i$, it follows that $h_{i+1}A = h_i A$, whence, since this is so for all $i \geqslant i_0$, $F(\{x_i | i \in w\})$ is generated by $\{h_1, \ldots, h_{i_0}\}$ and thus by $\{h_{i_0}\}$. Hence A is cyclic.

On the other hand, if A is cyclic, then $F(\{x_i | i \in w\}) = gA$, and since $g = h_{i_0}a$ for some i_0, we may take $g = h_{i_0}$. Then, if we use the fact that $F(\{x_i | i \in w\}) = h_{i_0}A = h_iA$ for $i \geqslant i_0$, letting $h_{i+1} = h_i m_i$, we have $h_{i+1}(1 - x_{i_0} m_i) = 0$. If $h_j a = 0$, then $u_j a$ is an element of V, whence $u_j a = \sum_{i=1}^{n} v_i a_i$, i.e., $a_1 = \dots = a_{j-1} = 0$, $a_j = a$, $x_n \dots x_j a = 0$. Applying this to $a = (1 - x_i m_i)$ with $j = i+1$, we obtain the statement $x_n \dots x_{i+1}\ (1 - x_i m_i) = 0$, which is precisely the condition given in the theorem.

Since the conditions of theorem 1 are preserved under homomorphism, it follows that the epimorphic image of a cyclic ring is also cyclic. Similarly, if A and B are cyclic rings, and if $\{(x_i, y_i) | i \in w\}$ is a sequence in A + B, then if $x_s \dots x_{i+1}(1-x_i m_i)=0$ and $y_t \dots y_{j+1}(1 - y_j n_j) = 0$ for all $i \geqslant i_0$, $j \geqslant j_0$ and for suitable $s \geqslant i+1$, $t \geqslant j+1$, selecting $k_0 = \max(i_0, j_0)$, $i,j \geqslant k_0$ and $r = \max(s,t)$, we have

$(x_r, y_r) \dots (x_{i+1}, y_{i+1})\ (1 - (x_i, y_i)(m_i, n_i)) = 0$ and A+B is also cyclic.

If A is a Steinitz ring, then A is a local ring with a T-nilpotent Jacobson radical R. Thus, if $\{x_i | i \in w\}$ is any sequence of elements of A, then either there is an index i_0 such that $i \geqslant i_0$ implies x_i is a unit, or the sequence and all segments are themselves T-nilpotent. In the first case take $m_i = x_i^{-1}$ for $i \geqslant i_0$, in the second case select n such that $x_n \dots x_{i+1} = 0$. It follows that A is a cyclic ring.

Proof of theorem 2. Suppose that A is not a cyclic ring. Then there is a sequence $\{x_i | i \in w\}$ of elements of A, such that for all i there is a $j(i) \geqslant i$ with $x_n \dots x_{j(i)+1}(1 - x_{j(i)}m) \neq 0$ for all $m \in A$ and all $n \geqslant j(i) + 1$. This means that $x_n \dots x_{j(i)} \notin x_n \dots x_{j(i)+1} x_{j(i)} A$, and thus $x_n \dots x_{j(i)+1}A \subset x_n \dots x_{j(i)}A$ (proper containment).

Suppose now we select $i_1 = 1$, $i_2 = j(i_1), \dots, i_\ell = j(i_{\ell-1})$, and $n \geqslant i_\ell + 1$. Then let $y_s = x_n \dots x_{i_{s+1}}$. It follows readily that we have a proper ascending sequence $x_n \dots x_{i_2}A \subset y_2 A \subset \dots \subset y_\ell A$ containing ℓ elements.

Hence if we let $\ell \geqslant k + 1$, we obtain a contradiction. From this the claims made above, following the statement of theorem 2, are virtually immediate.

PERFECT RINGS ARE CYCLIC.

Suppose now that M is a right A-module with R a T-nilpotent ideal. Then if M/MR has generators $g_1+MR,\dots,g_k+MR$ (as an A-module or an A/R module), it follows that if $m_0 \in M$, then some linear combination $g_1+a_{10}+\dots+g_k a_{k0}$ is congruent to m_0 module MR. Thus $m_0-(g_1 a_{10}+\dots+g_k a_{k0}) = m_1 r_1$. Repeating this process with respect to m_1, we find $m_0-(g_1(a_{10}+a_{11})+\dots+g_k(a_{k0}+a_{k1})) = m_2 r_2 r_1$. It is easy to see that we may in this way generate sequences $\{r_i\}$ $(r_i \in R)$ $\{m_i\}$ and $\{b_{1i},\dots,b_{ki}\}$ such that $m_0-(g_1 b_{1i}+\dots+g_k b_{ki}) =$ $= m_i r_i \dots r_1$. Since R is T-nilpotent, taking i such that $r_i \dots r_1 = 0$, it follows that M is generated by $\{g_1,\dots,g_k\}$. Clearly if $\{g_1,\dots,g_k\}$ generates M, then $\{g_1+MR,\dots,g_k+MR\}$ generates M/MR. Now if A is perfect and if R is its Jacobson radical, then R is T-nilpotent and A/R is a semi-simple Artininian ring, i.e., A/R is a cyclic ring. If we consider the module $F(\{x_i | i \in w\})/F(\{x_i | i \in w\})R$, then it is an A/R-module, and as an A/R-module it is isomorphic to the cyclic module $F(\{x_i+R | i \in w\})$. Since this latter module is cyclic, it follows that $F(\{x_i | i \in w\})$ is a cyclic module and thus A is also a cyclic ring.

Proof of theorem 3. An A-module M is quasi-cyclic if and only if it has a generating set $\{\ell_i | i \in w\}$ with $\ell_i = \ell_{i+1} x_i$ for some $x_i \in A$. It follows that there is a canonical epimorphism $F(\{x_i\}) \to M$ given by $h_i \to \ell_i$ for each quasi-cyclic module M. Hence A is cyclic if and only if all quasi-cyclic modules are cyclic.
Now suppose $\ell_1 A \subset \ell_2 A \subset \dots \subset \ell_i A \subset \ell_{i+1} A \dots$ is an ascending chain of principal right ideals. Then $\ell_i = \ell_{i+1} x_i$, i.e., the right ideal $\cup \ell_i A$ generated by $\{\ell_i | i \in w\}$ is a quasi-cyclic module and hence cyclic with generator $\ell_{i_0} A$. Hence A satisfied the ascending chain condition on principal right ideals.
Next, if x is not a left zero-divisor of A, then $x^{\ell}(1 - mx) = 0$ and $x^{\ell}(1 - xm) = 0$ implies $xm = mx = 1$. Also, if $x^{\ell} = x^{\ell+1}m$, then $x^{\ell+1} = x^{\ell+1}mx$, so that the first condition is a consequence of the second, for suitable ℓ. Hence if x has a left inverse it is a unit. Thus if x has a right inverse x' then x' is a unit and x is a unit. If we let $U = \{x \mid 1 - mx \text{ and } 1 - xm \text{ is a non-unit for some } m\}$, then $x \notin U$ provided for each $m \in A$, $1 - mx$ or $1 - xm$ is a unit. Thus the Jacobson radical of A is precisely the complement of U. Now, if $\{x_i | i \in w\}$ is any sequence of elements of R, then

$x_n \dots x_{i+1}(1 - x_i m) = 0$ for $i \geqslant i_0$, implies $x_n \dots x_{i+1} = 0$ and $\{x_i | i \in w\}$ is a T-nilpotent sequence. But then R is a T-nilpotent set as asserted.

Proof of theorem 4. If $e \leqslant f$, then $ef = fe = e$, and thus $eA = feA \leqslant fA$, while if $eA = fA$, then $f = ex$, whence $ef = ex = e = f$. Hence a properly ascending chain of idempotents $e_1 < e_2 < \dots < e_k$ implies a properly ascending chain of right principal ideals $e_1 A < e_2 A < \dots < e_k A$. Since A satisfies the ascending chain condition for right principal ideals, E(A) satisfies the ascending chain condition as a partially ordered set.

Also, if $e \leqslant f$, then $(1 - e)(1 - f) = 1 - e - f + ef = 1 - f = (1 - f)(1 - e)$, so that $(1 - f) \leqslant (1 - e)$. Since an ascending chain $e_1 < \dots < e_k$ gives rise to a descending chain $(1 - e_1) > (1 - e_2) > \dots > (1 - e_k)$ and conversely, it follows that E(A) also satisfies the descending chain condition.

If we have an infinite orthogonal set of central idempotents, say $\{e_i | i \in w\}$, let $f_i = e_1 + \dots + e_i$. Then $f_i f_{i+1} = f_{i+1} f_i = f_i$, and $f_1 < f_2 < \dots$ is an infinite ascending chain in E(A), an impossibility if A is cyclic. Thus there exist minimal central idempotents, and these form a finite orthogonal set, say $\{e_1, \dots, e_n\}$. It follows that $A = A_1 + \dots + A_n$, where $A_i = Ae_i$, and that A_i is a cyclic ring with no central idempotents other than 0 or 1. The uniqueness of the decomposition follows from the uniqueness of the minimal central idempotents.

If A is a commutative cyclic ring, then $A = A_1 + \dots + A_n$, where A_i is a commutative, cyclic and has no idempotents other than 0 or 1. Suppose $A = A_1$. If $x^\ell(1 - mx) = 0$, then $(x(1 - mx))^\ell = 0$, i.e., $x(1 - mx)$ is nilpotent. In particular $x(1 - mx)$ is an element of the prime radical R of A, and thus every prime ideal is maximal, while A/R is a regular ring. Hence, since $A = A_1$, it follows that A/R is a field and that R is the Jacobson radical of A. Hence $A = A_1$ is a Steinitz ring. Thus commutative cyclic rings are perfect.

If A is a regular cyclic ring, then given any element x of A, there is an element x' such that $xx'x = x$. From this, one concludes that $xx' = e$ is idempotent, and that $xA = eA$. Similarly, $Ax = Ae$ where $x'x = e$ is an idempotent. Thus, suppose $eA \leqslant fA$, where e and f are idempotents. For the idempotents $1 - e$ and $1 - f$ we have a relation $(1 - f)(1 - e) = 1 - e - f + fe = 1 - f$ since $e = fx$ implies $fe = e$, and thus $A(1 - e) \geqslant A(1 - f)$. In particular, if A does not satisfy the ascending chain condition on right principal ideals, then A is not cyclic. Hence if A is cyclic it satisfies the descending chain condition on left principal ideals. But this means that A is perfect. More directly, if

A is a regular cyclic ring, then if $x \in R$, $(x'x)^n = 0$ for some n, whence $x = x(x'x)^n = 0$ as well, i.e., A is semi-simple and perfect, i.e., A is semi-simple Artinian.

Thus it seems not at all impossible that all cyclic rings are perfect. On the other hand cyclic rings, as we mentioned above, may well be those rings whose modules have minimal generating sets. So, in conclusion, we conjecture that the class of rings whose modules have minimal generating sets is the class of cyclic rings and that this class of rings is precisely the class of perfect rings.

REFERENCES

[1] P.J. ALLEN and J. NEGGERS, *A Characterization of Steinitz Group Rings*, Proc. American Math. Soc., 49(1975), 39-42.

[2] H. BASS, *Finitistic dimension and an homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. 95(1960), 466-488.

[3] B.S. CHWE and J. NEGGERS, *On the extension of linearly independent subsets of free modules to a basis*, Proc. Amer. Math. Soc. 24(1970), 466-470.

[4] B.S. CHWE and J. NEGGERS, *Local Rings with Left Vanishing Radical*, J. London Math. Soc. (2), 4(1971), 374-378.

[5] D.S. PASSMAN, *Infinite Group Rings*, Marcel Dekker Inc., New York 1971.

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