Revista de la
Unión Matemática Argentina
Volumen 28 , 1977.

A NOTE ON THE SIMPLICITY OF ALTERNATING GROUPS

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In this short Note we intend to give a slightly different proof, from the known ones, of the simplicity of the alternating group A_n, $n \geq 5$. The proof results by a direct application of Sylow theorems. In the literature on the subject this is always done before proving Sylow theorems. Although, perhaps, this should be (or not) the case, we found very instructive and natural proceed "via" the Sylow theorems. The proof resulted in a course taught at the I.M.A.F. of the Universidad Nacional de Córdoba, in trying to give a simple proof of the non-existence in A_4 of subgroups or order 6. In 1. we recall this proof and using same ideas we give in 2. the proof of the simplicity of A_n, $n \geq 5$.

We wish to thank H. O'Brien his kind remarks on this Note.

1. NON EXISTENCE IN A_4 OF SUBGROUPS OF ORDER 6.

In fact, let H be a subgroup of A_4 of order 6. Then H is an invariant subgroup. Let H_3 be a 3-sylow subgroup of H. Since A_4 has order $12 = 3x4$, H_3 is also a 3-sylow subgroup of A_4. Notice that every tri-cycle (abc) in A_4 generates a 3-sylow subgroup. By the conjugacy of sylow subgroups it follows that every tri-cycle is conjugated to an element of H_3 and "a fortiori" of H. But since H is invariant in A_4 we conclude at once, that H contains *all* the tricycles in A_4. These are 8, so H has order greater than 8, a contradiction. This proves our claim.

2. SIMPLICITY OF A_n, $n \geq 5$.

We shall proceed inductively in n. We first prove it for n = 5 and next we prove for n = 6. The main ideas of the proof shall be taken from the case n = 6.

a) Simplicity of A_5.

Let H be an invariant subgroup of A_5. Let h be the order of H, $1 < h$.

We distinguish the following situations:

i) 5 divides h. Therefore H contains a 5-sylow subgroup which is also a sylow subgroup of A_5 . Thus by the argument used in 1. it follows that H contains all the elements of A_5 of order 5. These are 4! = 24.

By divisibility reasons H must have order 30 or 60. H contains therefore a 3-sylow subgroup (which is also a 3-sylow subgroup of A_5) and hence H contains the $\binom{5}{3}.2! = 20$ tricycles. This clearly implies that $H = A_5$.

ii) 3 divides h. By the same argument above, H contains the 20 tricycles of A_5, so its order might be 20, 30 or 60. But then 5 divides the order of H, and we are consequently in situation i). So again $H = A_5$.

iii) H has order a power of 2. That is to say, H has order 2 or 2^2. In the first case it follows that A_5 has center $\neq (1)$. It is easy to see that this is not so. So, H has order 4. This implies that H is a 2-sylow subgroup. Since any element of A_5 of order 2 is contained in a 2-sylow subgroup H must contain the $\frac{1}{2}.\binom{5}{3}.\binom{3}{2} = 15$ elements of order 2 in A_5, a nonsense. This concludes the proof of the simplicity of A_5.

b) Simplicity of A_6.

Assume the simplicity of A_5. Let us fix some notation. Let for any index i, $i = 1,2,3,4,5,6$, A_5^i denote the alternanting group in the letters $1,\dots,\hat{i},\dots,6$ where $\hat{i}$ means that the index i should be excluded. We identify the A_5^i 's to the corresponding subgroups of A_6. Let H be an invariant subgroup of A_6, $H \neq (1)$. Then H behaves respect to each A_5^i as follows:

$$(1) = H \cap A_5^i \quad \text{or} \quad A_5^i \subset H .$$

This clearly is consequence of the simplicity of $A_5^i \simeq A_5$. Assume that for some index i, A_5^i is contained in H. So 5 divides the order of H and by the usual argument, H contains all the 5-sylow subgroups of A_6 or the same, H contains all the elements of A_6 of order 5. These are $\binom{6}{5}.4! = 144$ elements. But as A_6 has order $\frac{1}{2}.6! = 360$, we conclude, by looking the divisors of 360, that H should have order 180 or 360. In any case this would imply that 3-sylow subgroups of A_6 would be in H. In particular all the elements of order 3 should be in H. These are as many as $\binom{6}{3}.2! + \frac{1}{2}.\binom{6}{3}.2!.2! = 80$. In conclusion H contains at least $144 + 80 = 224$. $H = A_6$ is the only possible case.

Therefore

$$A_5^i \cap H = (1) \quad \text{for all } i \ , \ i = 1,2,\dots,6$$

holds.

Call $A = \bigcup_{i=1}^{6} A_5^i$. Then $H \cap A = (1)$. It follows that the elements of H different from the identity, must be representable as product of disjoint cycles involving *all* the letters 1,2,3,4,5,6. Let $x \in H$ be an element of prime order p. Assume $p = 2$. Then x has the following representation

$$x = (ab).(cd).(ef)$$

with all distinct letters. But then x is odd, so $x \notin A_6$ a contradiction.

So $p \neq 2$. Notice that $p = 5$ is impossible, since any element of A_6 of order 5 is a cycle (abcde) omitting one letter, so can not be in H. We are therefore reduced to study the case $p = 3$. More precisely we have to consider the case where H has order a power of 3. According with the order of A_6, the order of H can be 3 or 3^2. In case 3^2, H would be a sylow subgroup, so H would contain all the elements of A_6 of order 3, so as many as 80 elements. So H might have order 3. Then would be generated by an element of the form

$$x = (abc).(efg)$$

with all distinct letters. But clearly $(abe).x.(abe)^{-1} = (bec).(afg)$ does not belong to H. We have proved the simplicity of A_6.

c) Simplicity of A_n, $n > 6$.

Let i,j be two indices in the natural interval $I_n = \{1,2,3,\dots,n\}$, $i \neq j$. We call $A^{i,j}_{n-2}$ the alternating group in the letters $1,\dots,\hat{i},\dots,\hat{j},\dots,n$ with i and j omitted, included in A_n. Let H be an invariant subgroup of A_n and $H \neq (1)$. As in b) we have

$$(1) = H \cap A^{i,j}_{n-2} \quad \text{or} \quad A^{i,j}_{n-2} \subset H .$$

Assume that some $A^{i,j}_{n-2}$ is contained in H. We claim that all the $A^{r,s}_{n-2}$ are in H. In fact, let r,s be a pair of indices in I_n, $r \neq s$.

If $r = i$, $s = j$, then $A^{r,s}_{n-2} \subset H$. If $r = i$, $s \neq j$ then

$$(jsr).A^{r,s}_{n-2}.(jsr)^{-1} = A^{i,j}_{n-2}$$

and therefore $A^{r,s}_{n-2} \subset H$. Assume $r \neq i,j$ and $s \neq i,j$. Then

$$(js).(ir).A^{r,s}_{n-2}.((js).(ir))^{-1} = A^{i,j}_{n-2} .$$

So again $A^{r,s}_{n-2} \subset H$. Hence H contains all the $A^{i,j}_{n-2}$, which implies that H contains all the tri-cycles of A_n. Since A_n is generated by tri-cycles, we conclude that $H = A_n$.

Let A^{i}_{n-1} denote the alternating group in the letters $1,\dots,\hat{i},\dots,n$. Then by the simplicity of $A_{n-1} \simeq A^{i}_{n-1}$, we have that $H \cap A^{i}_{n-1} = (1)$ or $A^{i}_{n-1} \subset H$. The latter is impossible since A^{i}_{n-1} contains all the $A^{i,j}_{n-2}$. Consequently any element of H is representable as a product of disjoint cycles involving *all* the letters $1,2,\dots,n$. Let $x \in H$ be an element of prime order p. As in b) we can exclude the case $p = 2$. We have to analyze only two possible representations of x as product of disjoint cycles. Namely

i) $x = (a_1 \dots a_p).(b_1 \dots b_p)\dots(c_1 \dots c_p)$; $p < n$,

ii) $x = (a_1 \dots a_p)$

Assume i). In case $p > 3$ we can choose an element y, in the alternating group in the letters $a_1,\dots,a_p$ such that $1 \neq [y , (a_1,\dots,a_p)]$. Therefore

$$1 \neq [y , (a_1,\dots,a_p)] = [y,x] \in H$$

a contradiction, since $[y,x]$ involves at most $p < n$ letters. Let $p=3$. Since $n > 6$, x contains at least 3 tricycles. We then repeat the previous argument by choosing an element y in the alternating group in the letters $a_1, a_2, a_3, b_1, b_2, b_3$ such that $1 \neq [y\ ,\ (a_1a_2a_3).(b_1b_2b_3)]$.

Assume ii). This means that $p = n$ and that H is a p-group. Since p is the highest power of p dividing $\frac{p!}{2} = \frac{n!}{2}$, we have that H has order p. Moreover it is a sylow subgroup of A_n, whence it contains *all* the elements of order p, which are $(p-1)!$, a number clearly greater than p.

The simplicity of A_n is completely proved. Pace e Bene.

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Recibido en octubre de 1976.