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MICROLOCAL ESTIMATES FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS

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INTRODUCTION. This work is devoted to the proof of microlocal estimates for a certain class of pseudodifferential operators that had been studied in [1] . Here we give a geometric characterization of these operators in terms of the characteristic variety of their principal symbol.

We want to say two words about the relationship between the estimates proved in [1] and in this work. In the former, a canonical transformation, and a related Fourier integral operator were used, in order to bring the pseudodifferential operator in question to a particularly simple form and then energy methods were employed, yielding a priori estimates in certain cases.

Here we do not make use of Fourier integral operators, in their stead, energy methods are applied directly and a priori estimates are obtained in all cases. Estimates in [1] are essentially particular cases of estimates obtained here.

Let Ω be an open neighborhood of the origin in R^{n+1}, $T^*(\Omega)$ its tangent bundle, Π: $T^*(\Omega) \longrightarrow \Omega$ the canonical projection. We shall denote by $\dot{T}(\Omega)^*$ the complement in $T^*(\Omega)$ of the zero section.

Usually we are going to select coordinates in Ω in such a way that one of the variables will play a distinguished role. In such cases, the variable point in Ω will be denoted $(x_1,\ldots,x_n,t)$ or simply (x,t) in that system of coordinates. The variables along the fibres in $T^*(\Omega)$ will be denoted $(\xi_1,\ldots,\xi_n,\tau) = (\xi,\tau)$. The canonical symplectic form on $T^*(\Omega)$ will be called ω; in terms of the coordinates $$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j + d\tau \wedge dt$$
When no distinguished variable is needed we change the notation to $x = (x_1,\ldots,x_{n+1}) = (\xi_1,\ldots,\xi_{n+1})$.

A subset Γ of $\dot{T}(\Omega)^*$ is said to be conic if it is stable under dilations $(x,\xi) \longrightarrow (x,\rho\xi), \rho > 0$.

Consider two conic, closed, smooth submanifolds Σ_1, Σ_2 of $\dot{T}(\Omega)^*$ of codimension one, and assume that they are in general position.

That means that their normals are linearly independent at any point of the intersection $\Sigma_1 \cap \Sigma_2$. As a consequence $\Sigma_1 \cap \Sigma_2$ is a conic submanifold of $\dot{T}(\Omega)^*$ of codimension two, unless it is empty. We shall always assume that this is the case.

We wish to make two assumptions about Σ_1 and Σ_2; one involves the symplectic form and the other the projection $\Pi(\Sigma_1 \cap \Sigma_2) \subset \Omega$.

We require

a) $\Pi(\Sigma_1 \cap \Sigma_2)$ *is a smooth submanifold of codimension one of* Ω *containing the origin.*

Now take a point p in Σ_1 with $\Pi(p) = 0$. Let V_p^1 be the linear subspace of the tangent space to Σ_1 at p, defined by $X \in V_p^1 \iff d\Pi.X \in$ $\in T_o(\Pi(\Sigma_1 \cap \Sigma_2))$. That is V_p^1 is made up of the vectors wich are mapped into the tangent space to $\Pi(\Sigma_1 \cap \Sigma_2)$ at the origin by the differential of the canonical projection.

For a point p in Σ_2 with $\Pi(p) = 0$, we define V_p^2 in a similar way. Using the inclusions $\Sigma_1 \subset \dot{T}(\Omega)^*$, $\Sigma_2 \subset \dot{T}(\Omega)^*$ we identify V_p^1 and V_p^2 with linear subspaces of $T_p(\dot{T}(\Omega)^*)$.

Our second assumption is

b) *The symplectic form* ω_p *is non-degenerate when restricted to* V_p^1 (V_p^2).

That means that for every X in V_p^1 (V_p^2) there exists Y in V_p^1 (V_p^2) such that $\omega_p(X,Y) \neq 0$.

PROPOSITION 1. *Let* $\Omega \subseteq R^{n+1}$ $n \geq 2$. *Let* Σ_1, Σ_2 *be smooth submanifolds of* $\dot{T}(\Omega)^*$ *of codimension one, in general position, satisfying conditions* a) *and* b).

Then, there exist a system of coordinates (x,t) *in an open nbhd. of the origin* U *and* C^∞ *real functions* $a(x,t,\xi), b(x,t,\xi)$ *defined on* $U \times (R^n-\{0\})$ *such that*

i) a *and* b *are positive homogeneous in* ξ *of degree one*.

ii) $\Sigma_1/U = \{(x,t,\xi,\tau) \text{ s.t. } \tau = a(x,t,\xi) \quad (x,t) \textit{ in } U\}$

$\Sigma_2/U = \{(x,t,\xi,\tau) \text{ s.t. } \tau = b(x,t,\xi) \quad (x,t) \textit{ in } U\}$.

iii) $a(x,t,\xi) = b(x,t,\xi) \iff t = 0$.

REMARKS. 1) Once the coordinates (x,t) have been chosen they induce coordinates (x,t,ξ,τ) in a canonical way. By means of this trivialization of $\dot{T}(\Omega)^*/U$ we identify it with $U \times (R^{n+1}-\{0\})$.

2) Property ii) in Proposition 1 implies that for a given coordinate system (x,t,ξ,τ), the funtions a and b are uniquely determined by Σ_1 and Σ_2, so there is at most one pair of such functions.

Proof. We cnoose coordinates (x,t) in a nbhd. U' of the origin so that $U' \cap \Pi(\Sigma_1 \cap \Sigma_2) = \{(x,t) \text{ in } U' \text{ s.t. } t = 0\}$.

Let $p \in \Sigma_1$, $\Pi(p) = 0$ i.e. $p = (0,0,\xi_o,\tau_o)$. The vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p, \left.\frac{\partial}{\partial t}\right|_p, \left.\frac{\partial}{\partial \xi_1}\right|_p, \ldots, \left.\frac{\partial}{\partial \xi_n}\right|_p, \left.\frac{\partial}{\partial \tau}\right|_p$$

form a basis of the tangent space to the cotangent bundle at p and $\left.\frac{\partial}{\partial x_1}\right|_o, \ldots, \left.\frac{\partial}{\partial x_n}\right|_o$ form a basis of $T_o(\Pi(\Sigma_1 \cap \Sigma_2))$.

The linear map $d\Pi$ is given by

$$d\Pi.\left.\frac{\partial}{\partial x_i}\right|_p = \left.\frac{\partial}{\partial x_i}\right|_o \quad ; \quad d\Pi.\left.\frac{\partial}{\partial t}\right|_p = \left.\frac{\partial}{\partial t}\right|_o \quad ; \quad d\Pi.\left.\frac{\partial}{\partial \xi_i}\right|_p = d\Pi.\left.\frac{\partial}{\partial \tau}\right|_p = 0$$

Thus, if $X = \sum_{i=1}^{n} \alpha_i \left.\frac{\partial}{\partial x_i}\right|_p + \beta_i \left.\frac{\partial}{\partial \xi_i}\right|_p + \alpha \left.\frac{\partial}{\partial t}\right|_p + \beta \left.\frac{\partial}{\partial \tau}\right|_p$ we have

$X \in V_p^1$ iff $\alpha = 0$.

Hence $\omega_p(X,\left.\frac{\partial}{\partial \tau}\right|_p) = 0$ whenever $X \in V_p^1$ and we conclude that

$\left.\frac{\partial}{\partial \tau}\right|_p \notin T_p(\Sigma_1)$. In particular, Σ_1 being conic, $\xi_o \neq 0$.

Now there is a positive $\varepsilon = \varepsilon(p)$ and a unique C^∞ function $a_p(x,t,\xi)$ defined for $|x|,|t| < \varepsilon$, $|\xi-\xi_o| < \varepsilon$, such that $q \in \Sigma_1$ and q close enough to p implies $q = (x,t,\xi,a_p(x,t,\xi))$.

The uniqueness of a_p in a nbhd. of p has two consequences.

First, we see that two such functions necessarily coincide in overlapping domains so that they define a function $a(x,t,\xi)$ in an open subset of $\tau = 0$.

Secondly, using the conicalness of Σ_1 we derive that $a(x,t,\xi)$ is positive homogeneous of degree one.

In consequence we may construct $a(x,t,\xi)$ patching together a finite number of functions $a_{p_1},\ldots,a_{p_r}$ defined on cones of $U' \times (R^n-\{0\})$, after shrinking the nbhd. U'.

We can find a nbhd. U of the origin and a cone Γ such that $a(x,t,\xi)$ is defined on $U\times\Gamma$. Now we see that actually $\Gamma = R^n-\{0\}$. Fix (x,t) in U and consider the set of points such that $(x,t,\xi,a(x,t,\xi)) \in \Sigma_1$. There is an $\varepsilon > 0$ s.t. $|x-x'|$, $|t-t'|$, $|\xi-\xi'| < \varepsilon \Rightarrow (x',t',\xi',a(x',t',\xi'))$ is in Σ_1, so Γ is open. Now if ξ_j is a sequence in Γ with $|\xi_j| = 1$ and $\xi_j \longrightarrow \xi_\infty$ we find (passing through a subsequence if necessary) real numbers τ_j such that $(x,t,\xi_j,\tau_j) \in \Sigma_1$ and $(\tau_j^2 + |\xi_j|^2)^{-1/2}.\tau_j \to \tau_\infty$ as $j \longrightarrow \infty$.

Σ is conic and closed so $\left(x,t,\frac{\xi_\infty}{(|\xi_\infty|^2+\tau_\infty^2)^{1/2}}, \frac{\tau_\infty}{(|\xi_\infty|^2+\tau_\infty^2)^{1/2}}\right) \in \Sigma_1$.

Since $n \geqslant 2$, $R^n-\{0\}$ is connected and $\Gamma = R^n-\{0\}$.

The function b is determined in the same way, with Σ_2 in the place of Σ_1.

To prove iii) we observe that for $\xi \neq 0$, $a(x,t,\xi) = b(x,t,\xi) \Leftrightarrow$ $\Leftrightarrow (x,t,\xi,a(x,t,\xi)) = (x,t,\xi,b(x,t,\xi))$ belongs to $\Sigma_1 \cap \Sigma_2 \Leftrightarrow \Pi(p) = (x,t,0,0) \in \Pi(\Sigma_1 \cap \Sigma_2) \Leftrightarrow t = 0$.

Q.E.D.

REMARK. Since Σ_1 and Σ_2 are in general position we may assume, shrinking U, that $\partial_t(a-b)(x,t,\xi) \neq 0$, $(x,t) \in U$, $\xi \neq 0$.

Let us write $\varphi(x,\xi) = a(x,0,\xi) = b(x,0,\xi)$. If we consider the finite Taylor expansion of $a(x,t,\xi)$ and $b(x,t,\xi)$ with respect to the variable t, we can write

$$\begin{aligned} a(x,t,\xi) &= \varphi(x,\xi) + ta'(x,t,\xi) \\ b(x,t,\xi) &= \varphi(x,\xi) + tb'(x,t,\xi) \end{aligned} \qquad (x,t) \in U\ ,\quad \xi \in R^n-\{0\}$$

where all the functions are smooth in all arguments and positive homogeneous with respect to ξ.

In view of the remark we made after the proof of Proposition 1, we may assume that $a'(x,t,\xi) - b'(x,t,\xi) \neq 0$ for (x,t) in U and $\xi \neq 0$.

PROPOSITION 2. *Let* $n \geq 2$ *and* $\Omega \subseteq R^{n+1}$. *Let* $p:\dot{T}(\Omega)^* \longrightarrow C$ *be a* C^∞ *positive homogeneous function of degree* m. *Assume that the characteristic variety* $C_p = \{q \in \dot{T}(\Omega)^*$ s.t. $p(q) = 0\}$ *is the union of two closed conic submanifolds* Σ_1 *and* Σ_2 *satisfying the hypothesis of Proposition 1.*

If

a) p *vanishes on* $\Sigma_1 \cup \Sigma_2 - \Sigma_1 \cap \Sigma_2$ *of order one*

b) p *vanishes on* $\Sigma_1 \cap \Sigma_2$ *of order two*

then, there exist

i) *a system of coordinates* (x,t) *in a nbhd.* U *of the origin*

ii) C^∞ *real positive homogeneous functions* $\varphi(x,\xi)$, $a(x,t,\xi)$, $b(x,t,\xi)$ *of degree one, and a* C^∞ *positive homogeneous function* $e(x,t,\xi,\tau)$ *of degree* m-2, *all of them defined for* (x,t) *in* U *and* $\xi \neq 0$,

such that

$p(x,t,\xi,\tau) = (\tau-\varphi(x,\xi)-ta(x,t,\xi))(\tau-\varphi(x,\xi)-tb(x,t,\xi))e(x,t,\xi,\tau)$ *on* $\dot{T}(\Omega)^*/U$.

Furthermore

iii) $a(x,t,\xi)-b(x,t,\xi) \neq 0$ *for* (x,t) *in* U, $\xi \neq 0$.

iv) $e(x,t,\xi,\tau) \neq 0$ *for* (x,t) *in* U, $\xi \neq 0$.

v) $e(x,t,\xi,\tau)$ *is real if* $p(x,t,\xi,\tau)$ *is real*.

vi) *Once the coordinate system is fixed, the functions* a,b, $e(x,t,\xi,\tau)$ *are completely determined.*

Proof. We choose coordinates (x,t) and determine functions $\varphi(x,\xi) + ta'(x,t,\xi)$, $\varphi(x,\xi) + tb'(x,t,\xi)$ according to Proposition 1 and then drop the prime, so as to have $\varphi(x,\xi) + ta(x,t,\xi)$,

$\varphi(x,\xi) + tb(x,t,\xi)$.

In the complement of $\Sigma_1 \cup \Sigma_2$ we define $e(x,t,\xi,\tau)$ by

$$e(x,t,\xi,\tau) = \frac{p(x,t,\xi,\tau)}{(\tau-\varphi-ta)(\tau-\varphi-tb)} \qquad \xi \neq 0$$

We only need to define $e(x,t,\xi,\tau)$ in a nbhd. of $\Sigma_1 \cup \Sigma_2$.

It is enough to reason microlocally. Let $(x_o,0,\xi_o,\tau_o) \in \Sigma_1 \cap \Sigma_2$ and set $z_1 = \tau-\varphi-ta$ $z_2 = \tau-\varphi-tb$. We can find functions $z_3,\dots,z_{2n+2}$ in a nbhd. of $(x_o,0,\xi_o,\tau_o)$ W, such that $q \longmapsto f(q) = (z_1(q),\dots,z_{2n+2}(q))$ is a diffeomorphism of W onto a nbhd. W' of the origin in R^{2n+2}.

The mapping f takes Σ_1 into $z_1 = 0$ and Σ_2 into $z_2 = 0$.

The finite Taylor expansion of $p\circ f^{-1}$ in the variables z_1, z_2 around the origin gives $\tilde{f}(z_1,\dots,z_{2n+2}) = (p\circ f^{-1})(z_1,\dots,z_{2n+2}) =$

$$= z_1 \int_0^1 f_{z_1}(sz_1,sz_2,z_3,\dots,z_{2n+2})ds + z_2 \int_0^1 f_{z_2}(sz_1,sz_2,\dots,z_{2n+2})ds =$$

$= z_1F_1(z) + z_2F_2(z)$.

F_1 vanishes on $z_2 = 0$ and F_2 vanishes on $z_1 = 0$. Therefore we may express $F_1(z) = z_2G_1(z)$ and $F_2(z) = z_1G_2(z)$ with G_1, G_2 in $C^\infty(W')$, so $\tilde{f}(z) = z_1z_2H(z)$.

Since p vanishes of order one on $\Sigma_1 \cup \Sigma_2 - \Sigma_1 \cap \Sigma_2$ and of order two on $\Sigma_1 \cap \Sigma_2$ we conclude that $H(z)$ does not vanish on W'.

We define $e(x,t,\xi,\tau)$ on W as $e = H\circ f$.

In a nbhd. of point lying on $(\Sigma_1 - \Sigma_2) \cup (\Sigma_2 - \Sigma_1)$ we reason similarly.

This gives a bonafide definition of $e(x,t,\xi,\tau)$, since $e(x,t,\xi,\tau)$ is determined in the dense set $\dot{T}(\Omega)^* - \Sigma_1 \cup \Sigma_2$.

Q.E.D.

So far we have assumed that Σ_1 and Σ_2 are submanifolds of $\dot{T}(\Omega)^*$. If Σ_1, Σ_2 are only submanifolds of a conic open subset of $\dot{T}(\Omega)^*$ properties a) and b) still make sense and we have analogues of Propositions 1 and 2, valid in a conic nbhd. of the origin in $\dot{T}(\Omega)^*$. We state them without proof.

PROPOSITION 1'. *Let* $\Gamma \subseteq \dot{T}(\Omega)^*$ *be a conic open subset,* $0 \in \Pi(\Gamma)$, $\Omega \subseteq R^{n+1}$, $n \geqslant 2$. *Let* Σ_1, Σ_2 *be smooth submanifolds of* Γ *of codimension one in general position satisfying* a) *and* b).

Then, there exist a system of coordinates (x,t) *in an open nbhd.* U *of the origin and* C^∞ *real functions* $a(x,t,\xi)$, $b(x,t,\xi)$ *defined on* $U \times \Delta_n$ (Δ_n *a conic open subset of* $R^n-\{0\}$) *such that*

i) $a(x,t,\xi)$, $b(x,t,\xi)$ *are positive homogeneous in* ξ *of degree one,* $\xi \in \Delta_n$.

ii) $\Sigma_1/U = \{\tau = a(x,t,\xi)$ s.t. $(x,t) \in U,\ \xi \in \Delta_n\}$

$\Sigma_2/U = \{\tau = b(x,t,\xi)$ s.t. $(x,t) \in U,\ \xi \in \Delta_n\}$.

iii) $a(x,t,\xi) = b(x,t,\xi) \iff t = 0$.

iv) $\partial_t(a-b)(x,t,\xi) \neq 0$ *for* (x,t) *in* U *and* $\xi \in \Delta_n$.

PROPOSITION 2'. *Let* Γ *be a conic open subset of* $\dot{T}(\Omega)^*$, $\Omega \subseteq R^{n+1}$, $n \geq 2$ *and* $p:\Gamma \longrightarrow C$ *a* C^∞ *positive homogeneous function of degree* m.

Assume that $\{q \in \Gamma$ s.t. $p(q) = 0\}$ *is the union of two submanifolds* Σ_1, Σ_2 *satisfying the hypothesis of Proposition 1'.*

If

a) p *vanishes on* $\Sigma_1 \cup \Sigma_2 - \Sigma_1 \cap \Sigma_2$ *of order one*

b) p *vanishes on* $\Sigma_1 \cap \Sigma_2$ *of order two*

then, there exist

i) *a system of coordinates* (x,t) *in a nbhd.* U *of the origin*

ii) C^∞ *real positive homogeneous functions* $\varphi(x,\xi)$, $a(x,t,\xi)$, $b(x,t,\xi)$ *of degree one defined on* $U \times \Delta_n$, *for a certain open cone of* $R^n - \{0\}$, *and a* C^∞ *positive homogeneous function* $e(x,t,\xi,\tau)$ *of degree* m-2, *defined on* $U \times \Delta_{n+1}$ (Δ_{n+1} *an open cone of* $R^{n+1}-\{0\}$ *that projects into* Δ_n) *such that*
$p(x,t,\xi,\tau) = (\tau-\varphi(x,\xi)-ta(x,t,\xi))(\tau-\varphi(x,\xi)-tb(x,t,\xi))e(x,t,\xi,\tau)$ *on* $U \times \Delta_{n+1}$.

Furthermore

iii) $a(x,t,\xi)-b(x,t,\xi) \neq 0 \quad (x,t) \in U,\ \xi \in \Delta_n$.

iv) $e(x,t,\xi,\tau) \neq 0$ *on* $U \times \Delta_{n+1}$.

v) e *is real if* p *is real*.

vi) *Once the coordinate system* (x,t) *is chosen the functions* a,b,e *are completely determined.*

REMARKS. 1) When $\Gamma = \dot{T}(\Omega)^*$, Proposition 1' specializes to a weak form of Proposition 1, since the latter states that in fact Δ_{n+1} can be taken as $\{(\xi,\tau)$ s.t. $\xi \neq 0\}$. A simple example shows that in general, under the hypothesis of Proposition 1, it is not possible to extend smoothly the factorization to $\xi = 0$. For instance $p(x,t,\xi,\tau) =$
$= \tau^2 - t^2(\xi_1^2+\xi_2^2)$ (n=2) factors into $(\tau-t(\xi_1^2+\xi_2^2)^{1/2})(\tau+t(\xi_1^2+\xi_2^2)^{1/2})$
which is not smooth at $\xi = 0$.

2) We only used the hypothesis $n \geq 2$ in Proposition 1 and 2 to conclude that $R^n-\{0\}$ is connected. This fact is not used in the proof of Propositions 1' and 2', where we work microlocally, so they are still valid for n=1.

We now introduce a certain class of pseudodifferential operators defined on an open cone of $\dot{T}(\Omega)^*$, $\Omega \subseteq R^{n+1}$.

All pseudodifferential operators of order m considered here have asymptotic expansions

$$P(x,D) \sim \sum_{j=0}^{\infty} P_{m-j}(x,D)$$

where $P_{m-j}(x,\xi)$ is in $C^\infty(\Omega \times \Delta)$ for a certain conic open subset Δ of $R^{n+1}-\{0\}$ and it is positive homogeneous of degree m-j with respect to ξ. P(x,D) acts on functions of the form $v = \chi(D)u$, with u in $C_c^\infty(\Omega)$, $\chi(\xi)$ in $C^\infty(R^{n+1})$, supp$\chi \subseteq \Delta$, and χ is positive homogeneous of degree zero for $|\xi| > 1/2$.

The class of pseudodifferential operators of order m defined on $\Gamma = \Omega \times \Delta$ is denoted $\psi^m(\Gamma)$ and the class of corresponding symbols $S^m(\Gamma)$. When $\Gamma = \dot{T}(\Omega)^*$ we write $\psi^m(\Omega)$ rather than $\psi^m(\dot{T}(\Omega)^*)$.

DEFINITION 1. *Let* $\Gamma \subseteq \dot{T}(\Omega)^*$ *be an open cone,* $\Omega \subseteq R^{n+1}$, $P(x,D) \in S^m(\Gamma)$. *We say that* $P(x,D) \in \Lambda^m(\Gamma)$ *if its principal symbol* $P_m(x,\xi)$ *verifies the hypothesis of Proposition 2'.*

We observe that this definition is coordinate free.

If $P \in \Lambda^m(\Omega)$ and $Q \in \psi^{m'}(\Omega)$ is elliptic it is easy to check that the transpose ${}^tP$, the adjoint P belong to $\Lambda^m(\Omega)$ and the composites PQ,QP belong to $\Lambda^{m+m'}(\Omega)$.

Suppose that Γ_1, Γ_2 are two open cones of $\dot{T}(\Omega)^*$ such that $\Gamma_1 \cup \Gamma_2 = \dot{T}(\Omega)^*$ and $P \in \psi^m(\Omega)$ is both in $\Lambda^m(\Gamma_1)$ and $\Lambda^m(\Gamma_2)$.

In general P need not belong to $\Lambda^m(\Omega)$.

Consider the homogeneous partial differential operator of order four in two variables

$P(x,t,D_x,D_t) = (D_t-tD_x)(D_t+tD_x)(D_x-xD_t)(D_x+xD_t)$

defined on $\Omega = \{x^2+t^2 < 1/2\}$. If we set $\Sigma_1' = \Omega \times \{\tau=t|\xi|\}$

$\Sigma_1'' = \Omega \times \{\xi=x|\tau|\}$, $\Sigma_2' = \Omega \times \{\tau=-t|\xi|\}$, $\Sigma_2'' = \Omega \times \{\xi=-x|\tau|\}$

it is clear that the characteristic variety of $P(x,t,D_x,D_t)$ is the union $\Sigma_1' \cup \Sigma_2' \cup \Sigma_1'' \cup \Sigma_2''$. If we define $\Gamma_1 = \Omega \times \{|\tau| < 3/2|\xi|\}$ $\Gamma_2 = \Omega \times \{|\xi| < 3/2|\tau|\}$ then $\Gamma_1 \cup \Gamma_2 = \dot{T}(\Omega)^*$, $P/\Gamma_i \in \Lambda^4(\Gamma_i)$ for $i = 1,2$ but $P \notin \Lambda^4(\Omega)$, since it projects into the set xt = 0.

However in the simple case of a second order partial differential operator in two variables, we have

THEOREM 1. *Let* $P = A(x,t)\, D_t^2+2B(x,t)\, D_xD_t+C(x,t)\, D_x^2+ \dots$ *be in* $\Lambda^2(\Gamma)$ *for a certain cone* Γ. *Then* $P \in \Lambda^2(U)$ *for a certain nbhd. of the origin* U.

Proof. We may find a system of coordinates (x,t) in a nbhd. of the origin U = U(0); C^∞ positive homogeneous functions of degree one $\varphi(x,t)$, $a(x,t,\xi)$, $b(x,t,\xi)$ and an elliptic C^∞ positive homogeneous function $e(x,t,\xi,\tau)$ of degree zero, defined in a cone $\Gamma_o = U \times \{(\xi,\tau)$ s.t. $\xi > 0$, $|\tau - k\xi| < \varepsilon\}$, k a real number, ε positive such that

$P_2(x,t,\xi,\tau) = (\tau-\varphi(x,\xi)-ta(x,t,\xi))(\tau-\varphi-tb)\; e(x,t,\xi,\tau)$ on the cone Γ_o.

We can write $a(x,t,\xi) = a(x,t)\xi, b(x,t,\xi) = b(x,t)\xi, \varphi(x,\xi) = \varphi(x)\xi$ for $\xi > 0$. When $(x,t) \in U$, $\xi > 0$ the points $(\xi,(\varphi+ta)\xi)$ and $(\xi,(\varphi+tb)\xi)$ are roots of $P_2 = A\tau^2 + 2B\tau\xi + C\xi^2$.

Expressing the discriminant in terms of the roots we get $B^2 - AC = t^2(a-b)^2$ which is a square in $C^\infty(U)$.

When $t \neq 0$, $\xi > 0$, $(\varphi+ta)\xi$, $(\varphi+tb)\xi$ are distinct roots of P_2 as a polynomial in τ, so A cannot vanish. The sum of the roots is $(2\varphi + t(a+b))\xi = -(B/A)\xi$ for $t \neq 0$, and it follows that A divides B in $C^\infty(U)$. Therefore $(-B/A + t(a-b))\xi$, $(-B/A - t(a-b))\xi$ are C^∞ functions of x,t,ξ, homogeneous of degree one in ξ.

We can factor

$$A\tau^2 + 2B\tau\xi + C\xi^2 = A(\tau + (B/A + t(b-a)\xi)(\tau + (B/A - t(b-a)\xi)$$

and it follows right away that P is in $\Lambda^2(\Omega)$.

Q.E.D.

This section is devoted to the proof of a priori estimates for operators in $\psi(\Gamma)$. We begin with a factorization result.

PROPOSITION 3. *Let* $P \in \Lambda^m(\Gamma)$; Γ *a conic open subset of* $\dot{T}(\Omega)^*$ *such that* $0 \in \Pi(\Gamma)$, $\Omega \subseteq R^{n+1}$. *Then there exist an open subcone* $\Gamma_o \subset \Gamma$ *and a system of coordinates* (x,t) *defined on* $\Pi(\Gamma_o)$ *such that the following factorization is valid on* Γ_o

$$P(x,t,D_x,D_t) \sim E(x,t,D_x,D_t)(D_t^2 - R_1(x,t,D_x)D_t + R_2(x,t,D_x))$$

Here E *is elliptic of degree* m-2, *and* R_1, R_2 *are pseudodifferential operators in* D_x *depending smoothly on* t, *of degree one and two respectively. The principal symbols* r_1, r_2 *of* R_1, R_2 *can be written*

$$r_1(x,t,\xi) = t(a(x,t,\xi) + b(x,t,\xi)) + 2c(x,\xi)$$

$$r_2(x,t,\xi) = (ta(x,t,\xi) + c(x,\xi)).(tb(x,t,\xi) + c(x,\xi))$$

with a,b,c *positive homogeneous real functions of degree one,* $c(x,\xi)$ *independent of* t *and* $a(x,t,\xi)-b(x,t,\xi) \neq 0$ *on* Γ_o.

Proof. We use Proposition 2' to factor the principal symbol of P. This factorization of the principal symbol of P implies that $P \sim E(D_t^2 - R_1D_t + R_2)$ with E elliptic (see [2]).

Comparing principal symbols on both sides of the equivalence relation we obtain the expression for r_1 and r_2.

Q.E.D.

REMARKS. 1) If $\Gamma = \dot{T}(\Omega)^*$, Γ_o can be taken as $U \times \{\xi \neq 0\}$ with U a nbhd. of the origin in Ω.

2) The hypothesis implies that (see the proof of Proposition 1) the characteristic variety of $P_m(x,t,\xi,\tau)$ does not contain points of the form $(x,t,0,\tau)$, so Γ_o can always be taken to avoid the ray $\xi = 0$. Then the pseudodifferential operators in D_x can be regarded as pseudodifferential operators in D_x, D_t with symbol independent of τ.

In view of Proposition 3 we consider operators of the form

$L = D_t^2 - R_1(x,t,D_x)D_t + R_2(x,t,D_x)$ with $r_1 = t(a+b) + 2c$, $r_2 = (ta+c)(tb+c)$ as in Proposition 3.

For the sake of simplicity we assume that a,b,c are defined for all $\xi \neq 0$, but all reasonings would also hold if we were dealing with any open cone, say, $U \times \Delta$. In that case operators act on functions of the form $v = \chi(D)u$, with $u \in C_c^\infty(U)$ and $\chi(\xi)$ a smooth positive homogeneous function of degree zero for $|\xi| > 1/2$ and $\operatorname{supp}\chi \subseteq \Delta$. The class of such functions will be denoted $C_c^\infty(U)/\Delta$.

Define pseudodifferential operators A,B,C with principal symbols a,b,c respectively and consider the operator

$$\tilde{L} = (\partial_t - it.\frac{A+B}{2} - i\,C)^2 + \frac{(t.(A-B))^2}{4} + S(x,t,D_x)$$

with S of degree one in D_x depending smoothly on t. We see that choosing S conveniently $\tilde{L} \sim L$.

We use the following notation

$\partial = \partial_t - it.1/2.(A+B) - iC$, $2P = A-B$, $L = \partial^2 + t^2P^2 + S$

We observe that P is real elliptic in D_x. Redefining S if necessary, we may assume that $P = P^*$. All pseudodifferential operators occurring in L are assumed to be properly supported and its associated kernels to have support in a sufficiently thin nbhd. of the diagonal so that, speaking loosely, they transform functions of small support in functions of small support.

When $f,g \in C_c^\infty(R^{n+1})$ we denote

$$(f,g) = \int f\bar{g}\, dx\, dt \quad , \quad \|f\|^2 = (f,f)$$

We now list some properties of the operator which will be used in the sequel.

d_1) $\partial + \partial^* = B(x,t,D_x)$ *is of degree at most zero in* D_x*, depends smoothly on* t *and does not contain* D_t*. Briefly,* $B \in \psi^o(t,D_x)$.

d_2) *Let* $a = a(t)$ *be a real* C_c^∞ *function of one variable. Then*

$$2\mathrm{Re}(\partial u, au) = -(a'u,u) + (Bu,au) \qquad u \in C_c^\infty(U)/\Delta .$$

d_3) *If we regard* a *as a multiplication operator, we have* $[\partial, a] = a' = \partial_t a$ *where the brackets indicate the commutator of* ∂ *and* a.

d_4) $[\partial,P] \in \psi^1(t,D_x)$*, i.e.* $[\partial,P]$ *does not contain* D_t.

Our estimates for L will be expressed in terms of certain norms that we define now.

Let $u \in C_c^\infty(U)/\Delta$, we write

$$|||u|||_o^2 = \|u\|^2 + \|tPu\|^2 \qquad k=0$$

$$|||u|||_1^2 = \|\partial^2 u\|^2 + \|tPu\|^2 + \|Pu\| \qquad k=1$$

$$|||u|||_k^2 = \|\partial^{k+1}u\|^2 + \|tP\partial^k u\|^2 + \|P\partial^{k-1}u\| \qquad k=2,3,\dots$$

LEMMA 1. *Given* $\varepsilon > 0$ *and* $k \in Z^+$ *there exists a nbhd.* $U = U(\varepsilon,k)$ *such that* $|||u|||_k \leqslant \varepsilon\, |||u|||_{k+1}$ *for* u *in* $C_c^\infty(U)/\Delta$.

From now on we will write $o(1)$ to indicate a constant relating two norms defined on $C_c^\infty(U)/\Delta$, that can be taken arbitrarily small if the nbhd. U shrinks around the origin.

For instance , the conclusion of Lemma 1 is written
$|||u|||_k \leqslant o(1)\ |||u|||_{k+1}$.

Proof. k=0) It is enough to see that $\|\partial u\| \leqslant o(1)\ \|\partial^2 u\|$ since it is obvious that $\|tPu\| \leqslant o(1)\|Pu\|$ (in fact we only need to narrow U in the t-direction).

Using d_1) with $a(t) = t$ we get

$$\|u\|^2 = (Bu,tu) - 2Re(\partial u,tu) \leqslant K\|u\|\,\|tu\| + 2\|\partial u\|\,\|tu\| \leqslant$$

$$\leqslant o(1)\|u\|^2 + o(1)\|\partial u\|\,\|u\|$$

where we have used that B is bounded in $L^2(U)$ with norm K.

The last estimate implies $\|u\| \leqslant o(1)\|\partial u\|$. Applying this result to the function ∂u we get what we needed.

$k \geqslant 1$) The same argument proves that $\|\partial^{k+1}u\| \leqslant o(1)\|\partial^k u\|$
It is also clear that

$$\|tP\partial^k u\| \leqslant o(1)\|P\partial^k u\| \leqslant o(1)\ |||u|||_{k+1}$$

Finally, taking account of d_4) and the fact that P is elliptic of degree one in D_x,

$$\|P\partial^{k-1}u\| \leqslant o(1)\|\partial P\partial^{k-1}u\| \leqslant o(1)\,(\|P\partial^k u\| + \|[P,\partial]\partial^{k-1}u\|)$$

so $$\|P\partial^{k-1}u\| \leqslant o(1)\ |||u|||_{k+1}$$

Q.E.D.

Now we will look into two bilinear forms:

$$Re(\partial^k Lu,\partial^k u) \quad\text{and}\quad 2Re(\partial^k Lu,t\partial^{k+1}u) \qquad u \in C_c^\infty(U)/\Delta.$$

It is easy to verify that

$$\partial^k t^2 = t^2\partial^k + 2kt\,\partial^{k-1} + k(k-1)\,\partial^{k-2} \qquad k=1,2,\dots$$

so we obtain

B_1) $Re(\partial^k Lu,\partial^k u) = Re(\partial^{k+2}u,\partial^k u) + Re(t^2\partial^k P^2 u,\partial^k u) +$
$+ 2kRe(t\partial^{k-1}P^2 u,\partial^k u) + k(k-1)\,Re(\partial^{k-2}P^2 u,\partial^k u) + Re(\partial^k Su,\partial^k u)$

B_2) $2Re(\partial^k L,t\partial^{k+1}u) = 2Re(\partial^{k+2}u,t\partial^{k+1}u) + 2Re(t^3\partial^k P^2 u,\partial^{k+1}u) +$

$$+4k\mathrm{Re}(t^2\partial^{k-1}P^2u,\partial^{k+1}u)+2k(k-1)\mathrm{Re}(t\partial^{k-2}P^2u,\partial^{k+1}u)+2\mathrm{Re}(\partial^k Su,t\partial^{k+1}u).$$

Our objective is now to introduce suitable commutators in each one of the right-hand sides of B_1) and B_2) so as to produce those norms that appear in the definition of $|||u|||_k$.

We will make use of properties d_1), d_2), d_3) several times, to obtain the following identities

I_1) $\mathrm{Re}(\partial^{k+2}u,\partial^k u) = -\|\partial^{k+1}u\|^2 + \mathrm{Re}(B\partial^{k+1}u,\partial^k u)$

I_2) $\mathrm{Re}(t^2\partial^k P^2u,\partial^k u) = \mathrm{Re}(t^2P^2\partial^k u,\partial^k u) + \mathrm{Re}(t^2[\partial^k,P^2]u,\partial^k u) =$
$= \|tP\partial^k u\|^2 + \mathrm{Re}(t^2[\partial^k,P^2]u,\partial^k u)$

I_3) $2\mathrm{Re}(t\partial^{k-1}P^2u,\partial^k u) = 2\mathrm{Re}(tP\partial^{k-1}u,P\partial^k u)+2\mathrm{Re}(t[\partial^{k-1},P^2]u,\partial^k u) =$
$= 2\mathrm{Re}(tP\partial^{k-1}u,\partial P\partial^{k-1}u)+2\mathrm{Re}(tP\partial^{k-1}u,[P,\partial]\partial^{k-1}u)+$
$+ 2\mathrm{Re}(t[\partial^{k-1}u,P^2]u,\partial^k u) =$
$= -\|P\partial^{k-1}u\|^2+(tB\partial^{k-1}u,P\partial^{k-1}u) +$
$+ 2\mathrm{Re}(tP\partial^{k-1}u,[P,\partial]\partial^{k-1}u)+2\mathrm{Re}(t[\partial^{k-1},P^2]u,\partial^k u)$

In the same manner we obtain

I_4) $\mathrm{Re}(\partial^{k-2}P^2u,\partial^k u) = -\|P^{k-1}u\|^2-\mathrm{Re}([\partial^{k-1},P^2]u,\partial^{k-1}u) +$
$+ (B\partial^{k-2}P^2u,\partial^{k-1}u)$

I_5) $\mathrm{Re}(\partial^k Su,\partial^k u) = \mathrm{Re}(S\partial^k u,\partial^k u) + \mathrm{Re}([\partial^k,S]u,\partial^k u)$

I_6) $2\mathrm{Re}(\partial^{k+2}u,t\partial^{k+1}u) = -\|\partial^{k+1}u\|^2+(B\partial^{k+1}u,t\partial^{k+1}u)$

I_7) $2\mathrm{Re}(t^3\partial^k P^2u,\partial^{k+1}u) = -3\|tP\partial^k u\|^2+(BP\partial^k u,t^3P\partial^k u) +$
$+2\mathrm{Re}(t^3P\partial^k u,[P,\partial]\partial^k u)+2\mathrm{Re}(t^3[\partial^k,P^2]u,\partial^{k+1}u)$

I_8) $\mathrm{Re}(t^2\partial^{k-1}P^2u,\partial^{k+1}u) = \|P\partial^{k-1}u\|^2-\|tP\partial^k u\|-(tBP\partial^{k-1}u,P\partial^{k-1}u) -$
$-2\mathrm{Re}(tP\partial^{k-1}u,[P,\partial]\partial^{k-1}u)-\mathrm{Re}(t^2[\partial^k,P^2]u,\partial^k u)-$
$-2\mathrm{Re}(t[\partial^{k-1},P^2]u,\partial^k u)+(Bt^2\partial^{k-1}P^2u,\partial^k u)$

I_9) $2\mathrm{Re}(t^{k-2}P^2u,\partial^{k+1}u) = \|P\partial^{k-1}u\|^2+2\|P\partial^{k-1}u\|^2+(Bt\partial^{k-2}P^2u,\partial^k u) -$
$-(tBP\partial^{k-1}u,P\partial^{k-1}u)-2\mathrm{Re}(tP\partial^{k-1}[P,\partial]\partial^{k-1}u) -$
$-2\mathrm{Re}(t[\partial^{k-1},P^2]u,\partial^k u)-2(B\partial^{k-2}P^2u,\partial^{k-1}u) +$
$+2\mathrm{Re}([\partial^{k-1},P^2]u,\partial^{k-1}u)$

I_{10}) $\mathrm{Re}(\partial^k Su,t\partial^{k+1}u) = \mathrm{Re}(S\partial^k u,t\partial^{k+1}u)+\mathrm{Re}([\partial^k,S]u,t\partial^{k+1}u)$

We wish now to estimate all terms that appear on identities I_1),...,I_{10}).

LEMMA 2. *The following expressions can be dominated by* $o(1)\,|||u|||^2$:

E_1) $|(B\partial^{k+1}u,\partial^k u)|$

E_2) $|(t^2[\partial^k,P^2]u,\partial^k u)|$

E_3) i) $|(tBP\partial^{k-1}u,P\partial^{k-1}u)|$ ii) $|(tP\partial^{k-1}u,[P,\partial^{k-1}]u)|$

iii) $|(t[\partial^{k-1},P^2]u,\partial^k u)|$

$E_4)$ i) $|([\partial^{k-1},P^2]u,\partial^{k-1}u)|$ ii) $|(B\partial^{k-2}P^2u,\partial^{k-1}u)|$

$E_5)$ ii) $|([\partial^k,S]u,\partial^k u)|$

$E_6)$ $|(B\partial^{k+1}u,t\partial^{k+1}u)|$

$E_7)$ i) $|(BP\partial^k u,t^3P\partial^k u)|$ ii) $|(t^3P\partial^k u,[P,\partial]\partial^k u)|$

iii) $|(t^3[\partial^k,P^2]u,\partial^{k+1}u)|$

$E_8)$ i) $|(tBP\partial^{k-1}u,P\partial^{k-1}u)|$ ii) $|(tP\partial^{k-1}u,[P,\partial]\partial^{k-1}u)|$

iii) $|(t[\partial^{k-1},P^2]u,\partial^k u)|$ iv) $|(t^2[\partial^k,P^2]u,\partial^k u)|$

v) $|(Bt^2\partial^{k-1}P^2u,\partial^k u)|$

$E_9)$ i) $|(tBP\partial^{k-1}u,P\partial^{k-1}u)|$ ii) $|(tP\partial^{k-1}u,[P,\partial]\partial^{k-1}u)|$

iii) $|(t[\partial^{k-1},P^2]u,\partial^k u)|$ iv) $|([\partial^{k-1},P^2]u,\partial^{k-1}u)|$

v) $|(B\partial^{k-2}P^2u,\partial^{k-1}u)|$ vi) $|(tB\partial^{k-2}P^2u,\partial^k u)|$

$E_{10})$ ii) $|([\partial^k,S]u,t\partial^{k-1}u)|$

We observe that expressions collected under $E_i)$ $i=1,2,3,\dots,10$ are those appearing in identity $I_i)$. That makes that some expressions like $E_3)$i) and $E_9)$i) appear twice.

Proof. If $R \in \psi^2(t,D_x)$, we can express

$$[\partial^k,R] = \sum_{i=0}^{k-1} H_i\partial^i \quad \text{with} \quad H_i \in \psi^2(t,D_x) .$$

This is easily proved by induction.

Since all estimates are proved essentially in the same way, we prove some of them and leave the rest to the reader.

$E_1)$ $|(B\partial^{k+1}u,\partial^k u)| \leqslant \|B\partial^{k+1}u\|\,\|\partial^k u\| \leqslant o(1)\|\partial^{k+1}u\|^2 \leqslant o(1)\,|||u|||_k^2$

$u \in C_c^\infty(U)/\Delta$.

$E_2)$ Writing $[\partial^k,P^2] = \sum_{i=0}^{k-1} H_i\partial^i$, $H_i \in \psi^2(t,D_x)$ we get

$$|(t^2[\partial^k,P^2]u,\partial^k u)| \leqslant \sum_{i=0}^{k-1} |(t^2H_i\partial^i u,\partial^k u)|.$$

Now, for $i=1,\dots,k-1$

$$|(tH_i\partial^i u,t\partial^k u)| \leqslant \text{const.} \left(\int t^2\|\partial^i u\|_1^2(t)\, dt\right)^{1/2}\left(\int t^2\|\partial^k u\|_1^2(t)\, dt\right)^{1/2}$$

where $\|\partial^i u\|_1(t)$ denotes the Sobolev 1-norm in the x-variables.

Hence $|(tH_i\partial^i u,t\partial^k u)| \leqslant \text{const.}\|tP\partial^i u\|\,\|tP\partial^k u\| \leqslant o(1)\,|||u|||_k^2$.

$E_7)$ iii)

$$|(t^3[\partial^k,P^2]u,\partial^{k+1}u)| \leqslant \sum_{i=0}^{k-1}|(t^3H_i\partial^i u,\partial^{k+1}u)| \leqslant \sum_{i=0}^{k-1}|(3t^2H_i\partial^i,\partial^k u)| +$$

$$+ |(t^3\partial H_i\partial^i u,\partial^k u)| + |(Bt^3H_i\partial^i,\partial^k u)|$$

We already proved that $|(t^2H_i\partial^i u,\partial^k u)| \leqslant o(1)\,|||u|||_k^2$ $i=1,\dots,k-1$

Furthermore

$$|(t^3\partial H_i\partial^i u,\partial^k u)| \leqslant |(t^3 H_i\partial^{i+1}u,\partial^k u)| + (t^3[\partial,H_i]\partial^i u,\partial^k u) \leqslant$$
$$\leqslant o(1)\|tP\partial^{i+1}u\|\,\|tP\partial^k u\| + o(1)\,|||u|||_k^2 \leqslant$$
$$\leqslant o(1)\,|||u|||_k^2 \qquad i=1,\dots,k-1$$

Also

$$|(t^3BH_i\partial^i u,\partial^k u)| \leqslant o(1)\,|||u|||_k^2$$

Q.E.D.

We estimate $E_5)$i) and $E_{10})$i) in a separate lemma.

LEMMA 3. *Let* $S \in \psi^1(t,D_x)$. *Then there exist* $C > 0$, $k \in Z^+$ *and a nbhd.* U *of the origin, such that*

$E_5)$ i) $\quad |(S\partial^k u,\partial^k u)| \leqslant 2\|\partial^{k+1}u\| + (2k-1)\|tP\partial^k u\|$

$E_{10})$ i) $\quad |(S\partial^k u,t\partial^{k+1}u)| \leqslant 2\|\partial^{k+1}u\| + (2k-1)\|tP\partial^k u\| \qquad u \in C_c^\infty(U)/\Delta$

Proof. Since P is elliptic and P = P* we may find a pseudodifferential operator $Q \in \psi^1(t,D_x)$ such that $Q^* = Q$ and $Q^2 \sim P$.

We observe that

1) $\quad 2\mathrm{Re}(t\partial Q\partial^k u,Q\partial^k u) = -(Q\partial^k u,Q\partial^k u) + (BQ\partial^k u,tQ\partial^k u)$

On the other hand

2) $\quad (\partial Q\partial^k u,tQ\partial^k u) = (tP\partial^{k+1}u,\partial^k u) + (t[\partial,Q]\partial^k u,Q\partial^k u)$

Since $[\partial,Q] \in \psi^{1/2}(t,D_x)$, it is clear that

$$|(t[\partial,Q]\partial^k u,Q\partial^k u)| \leqslant C\|\partial^k u\|\,\|tP\partial^k u\| \leqslant o(1)\|\partial^{k+1}u\|\,\|tP\partial^k u\|$$

From 1) and 2) we get

3) $\quad \|Q\partial^k u\|^2 \leqslant |(P\partial^{k+1}u,t\partial^k u)| + o(1)\|\partial^{k+1}u\|\,\|tP\partial^k u\|$

$$\|Q\partial^k u\|^2 \leqslant \|\partial^{k+1}u\|\,\|tP\partial^k u\|.(1+o(1)) \leqslant 2\|\partial^{k+1}u\|\,\|tP\partial^k u\| \leqslant$$
$$\leqslant a\|\partial^{k+1}u\|^2 + a^{-1}\|tP\partial^k u\|^2$$

with $u \in C_c^\infty(U)/\Delta$ and U small enough so $o(1) \leqslant 1$.

Furthermore, using the Sobolev dual norms $\|\ \|_{1/2}$, $\|\ \|_{-1/2}$ we have

4) $\quad |(S\partial^k u,\partial^k u)| \leqslant \|S\partial^k u\|_{-1/2}\|\partial^k u\|_{1/2} \leqslant C\|Q\partial^k u\|_0^2 \qquad u \in C_c^\infty(U)/\Delta$

for a certain constant C independent of k.

From 3) and 4) it follows that

$$|(S\partial^k u,\partial^k u)| \leqslant Ca\|\partial^{k+1}u\|^2 + Ca^{-1}\|tP\partial^k u\|^2 \leqslant 2\|\partial^{k+1}u\|^2 + (2k-1)\|tP\partial^k u\|^2$$

if we choose a so that $Ca \leqslant 2$ and then $K \geqslant 1/2(Ca^{-1}+1)$.

$E_{10})$i) is proved in the same way.

Q.E.D.

THEOREM 2. *Let* $L = \partial^2+t^2P^2+S$ *be defined on a cone* $V(0) \times \Delta$, *with* ∂,P,S *defined as above. Then given* $\varepsilon > 0$, *there is a nbhd.* U *of the origin and* $k \in Z^+$ *such that*

$$|||u|||_k \leq \varepsilon \|\partial^k Lu\| \quad \textit{for every} \quad u \in C_c^\infty(U)/\Delta$$

Proof. Using $I_1),\ldots,I_{10})$ we obtain

$$4\mathrm{Re}(\partial^k Lu,\partial^k u+(1/2)\ t\partial^{k+1}u) = -5\|\partial^{k+1}u\|^2-(4k-1)\|tP\partial^k u\|^2 - $$
$$-k(k-1)\|P\partial^{k-1}u\|^2 + \gamma(u,u).$$

We indicate with $\gamma(u,u)$ all terms in $I_1),\ldots I_{10})$ which do not reduce to the square of a norm. In view of Lemmas 2 and 3 we have the estimate

$$|\gamma(u,u)| \leq 2(2\|\partial^{k+1}u\|^2 + (2k-1)\|tP\partial^k u\|^2) + o(1)\ |||u|||_k^2 \quad , \ u \in C_c^\infty(U)/\Delta$$

for a certain $k \in Z^+$.

Hence

$$|4\mathrm{Re}(\partial^k Lu,\partial^k u+(1/2)\ t\partial^{k+1}u)| \geq \|\partial^{k+1}u\|^2+\|tP\partial^k u\|^2+k(k-1)\|P\partial^{k-1}u\|^2 -$$
$$- o(1)\ |||u|||_k^2$$

Therefore

$$|||u|||_k^2 \leq \text{const.}\|\partial^k Lu\|\|\partial^k u+(1/2)t\partial^{k+1}u\| \leq o(1)\|\partial^k Lu\|\ |||u|||_k$$

so that $|||u|||_k \leq o(1)\|\partial^k Lu\|$.

Q.E.D.

REMARK. Theorem 2 implies estimates in terms of more familiar norms. For instance in the proof of Lemma 1, we saw that all norms $|||\ |||_k$ are stronger than the L^2-norm. In fact, they stronger than the norm $N(u) = (\int |\hat{u}(\xi,\tau)|\xi|^{1/2}|^2\ d\xi\ d\tau)^{1/2}$.

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