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A NOTE ON CLIFFORD ALGEBRA PERIODICITY

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1. INTRODUCTION. In this paper we determine the Clifford algebras for a large class of quadratic spaces. The results are of a periodic nature and are used to determine a new Radon-Hurwitz function.

All quadratic spaces are nonsingular and defined over a field k of characteristic not 2. If V is a quadratic space with symmetric bilinear pairing B, then one can associate a quadratic map $q: V \longrightarrow k$ given by $q(A) = B(A,A)$, for $A \in V$. Let a be a nonzero scalar and r and s nonnegative integers. $V(r,s)$ denotes the $(r+s)$-dimensional quadratic space with orthogonal basis $A_1,\dots,A_r,A_{r+1},\dots,A_{r+s}$ satisfying $q(A_1) = \dots = q(A_r) = -a$ and $q(A_{r+1}) = \dots = q(A_{r+s}) = a$. The Clifford algebra of $V(r,s)$ will be denoted by $C(r,s)$, and its even Clifford algebra by $C_0(r,s)$. If $a,b \in k^* = k - \{0\}$, $(\frac{a,b}{k})$ denotes the 4-dimensional generalized quaternion algebra with basis $1,i,j,\ell$ satisfying $i^2 = a$, $j^2 = b$, $ij = -ji = \ell$. If D is a division ring, $M(t,D)$ is the algebra of $t \times t$ matrices with entries from D.

We will calculate $C(r,s)$ for all quadratic spaces $V(r,s)$. See Lam [2], pages 126-129, for the case $a = 1$. We will freely use the results contained in [2]. Another important source for Clifford algebras is [3], in which periodicity is discussed in a more general setting. For a complete study of Clifford modules, we refer to this interesting article.

2. PRELIMINARY RESULTS.

In the following Lemmas, $\hat{\otimes}$ is the "graded tensor product" and $\hat{M}(\ ,\ )$ denotes the "checker-board grading" ([2] pages 77 and 81).

LEMMA 1. $C(r+m,s+m) \simeq M(2^m,C(r,s))$.

Proof. Using [2] pages 105-106, one has

$$C(r+m,s+m) = C(V(r,s) \perp V(m,m)) \simeq C(r,s) \mathbin{\hat{\otimes}} C(m,m)$$

$$\simeq C(r,s) \hat{\otimes} \hat{M}(2^m,k)$$
$$\simeq M(2^m, C(r,s)).$$

LEMMA 2. $C(4,0) \simeq C(0,4)$.

Proof. Just use proposition 2.12, page 114, in [2]. For the case at hand, the signed determinant of V(4,0) is 1 and hence $C(4,0) \simeq C(0,4)$.

LEMMA 3. $C(r+8,s) \simeq C(r,s+8) \simeq M(16,C(r,s))$.

Proof. Using the previous Lemma and [2], pages 105-106, we have

$$\begin{aligned} C(r+8,s) &= C(V(r,s) \perp V(4,0) \perp V(4,0)) \\ &\simeq C(r,s) \hat{\otimes} C(4,0) \hat{\otimes} C(4,0) \\ &\simeq C(r,s) \hat{\otimes} C(4,0) \hat{\otimes} C(0,4) \\ &\simeq C(r,s) \hat{\otimes} C(4,4) \\ &\simeq C(r,s) \hat{\otimes} \hat{M}(16,k) \\ &\simeq M(16,C(r,s)). \end{aligned}$$

The proof for $C(r,s+8)$ is similar.

Therefore the periodicity is modulo 8 and we need only calculate $C(r,0)$ and $C(0,s)$ for $r,s = 0,1,\ldots,7$.

3. CALCULATIONS FOR $r,s = 0,1,\ldots,7$.

We begin by determining the Clifford algebras for some low dimensions, while at the same time introducing the letters X,Z,Y,W and G for some special algebras. In particular, $k\langle\sqrt{a}\rangle$ denotes the 2-dimensional k-algebra with basis 1,i satisfying $i^2 = a \neq 0$. If a is a square in k, then $k\langle\sqrt{a}\rangle \simeq k \times k$, the direct product of fields, while if a is a nonsquare, $k\langle\sqrt{a}\rangle \simeq k(\sqrt{a})$ a quadratic field extension of k.

$$C(0,0) = k$$
$$C(1,0) = k\langle\sqrt{-a}\rangle = X$$
$$C(0,1) = k\langle\sqrt{a}\rangle = Z$$
$$C(2,0) = \left(\frac{-a,-a}{k}\right) = Y$$
$$C(0,2) = \left(\frac{a,a}{k}\right) = W$$
$$\left(\frac{-1,-1}{k}\right) = G$$

To calculate $C(3,0)$ and $C(0,3)$, we use Corollary 2.7, page 113 [2]; here, $\otimes$ denotes the ordinary tensor product

$$\begin{aligned} C(3,0) &= C(V(2,0) \perp V(1,0)) \\ &\simeq C(2,0) \otimes C(0,1) \\ &= Y \otimes Z. \end{aligned}$$

Similarly, $C(0,3) \simeq W \otimes X$.

We now show that $C(4,0) \simeq M(2,G)$. By [1], page 203, $C_0(4,0) \simeq C_0(3,0) \times C_0(3,0)$. But, $C_0(3,0) \simeq G$ ([2], page 114, Corollary 2.10) and hence $C_0(4,0) \simeq G \times G$. Finally, the basic structure theorem for even dimensional spaces ([2], page 111, Theorem 2.5 (3)) tells us that $C(4,0) \simeq M(2,G)$.

The remaining Clifford algebras can be calculated easily, using the fact that $C(r+4,0) = C(V(r,0) \perp V(4,0)) \simeq C(V(r,0) \perp V(0,4)) = C(r,4)$. Furthermore, $C(r,4) \simeq M(2^r, C(0,4-r))$ for $0 \leq r \leq 4$. In a similar fashion one obtains $C(0,s+4) \simeq M(2^s, C(4-s,0))$. We obtain the following table:

	0	1	2	3	4	5	6	7
$C(r,0)$	k	X	Y	$Y\otimes Z$	$M(2,G)$	$M(2,X\otimes W)$	$M(4,W)$	$M(8,Z)$
$C(0,s)$	k	Z	W	$X\otimes W$	$M(2,G)$	$M(2,Y\otimes Z)$	$M(4,Y)$	$M(8,X)$

Table 1

In order to further our study and obtain Tables 2,3 and 4 below, the analysis is divided into 6 cases which depend on the nature of the scalars a and -a and the field k. It will be important to determine when the quaternion algebra $(\frac{a,a}{k})$ is a division ring and when it is isomorphic to $M(2,k)$.

LEMMA 4. *Let* $a \in k^*$ *and* $C(0,2) = (\frac{a,a}{k})$ *the Clifford algebra of the plane* $V(0,2)$. *Then the following are equivalent:*

(1) $C(0,2) \simeq M(2,k)$.

(2) *There exists a vector* $N \in V(0,2)$ *such that* $q(N) = 1$.

(3) $a = b^2 + c^2$ *for some* $b,c \in k$.

Proof. (1 ⇒ 2) There exists a nonzero element u of $C(0,2)$ such that $u^2 = 0$. Put $u = c_0 + c_1A_1 + c_2A_2 + c_3A_1\circ A_2$, $c_i \in k$ (the circle "∘" denotes the product in the Clifford algebra). The fact that $u^2 = 0$ implies

$$[c_0^2 + c_1^2 a + c_2^2 a - c_3^2 a^2] + 2c_0[c_1A_1 + c_2A_2 + c_3A_1\circ A_2] = 0.$$

We see from the second term that $c_0 = 0$ and hence

$$c_3^2a^2 = c_1^2a + c_2^2a.$$

Note that if $c_3a^2 = 0$, then $u = c_1A_1 + c_2A_2 \in V(0,2)$ and $u^2 = c_1^2a + c_2^2a = 0$. Thus $V(0,2)$ is a hyperbolic plane and 1 is represented. So suppose $c_3a \neq 0$. Let

$$N = \frac{c_2A_2 + c_1A_2}{c_3a}$$

from which $q(N) = 1$ as desired.

$(2 \Rightarrow 3)$ Let $1 = q(bA_1 + cA_2)$ which implies that

$$1 = b^2a + c^2a$$

and

$$a = b^2a^2 + c^2a^2 = (ba)^2 + (ca)^2,$$

whence a is the sum of two squares.

$(3 \Rightarrow 1)$ Suppose that $a = b^2 + c^2$. Consider the vector $N = (b/a)A_1 + (c/a)A_2 \in V(0,2)$. Clearly, $q(N) = 1$ and thus in the Clifford algebra $C(0,2)$ one has an equation of the form $(N-1)\circ(N+1) = 0$, showing that $C(0,2) \simeq M(2,k)$.

CASE 1. a and -a are both squares.
This is precisely Case 1, page 128, of [2]. The periodicity is modulo 2.

CASE 2. a is a square and -a is not a square, but -a is the sum of 2 squares.
This is Case 2, page 129 of [2]. The periodicity is modulo 4.
An example would be the quadratic space $\langle 1,\dots,1 \rangle$ over the field with 5 elements.

CASE 3. a is a square and -a is not the sum of 2 squares.
This is Case 3, page 129, of [2]. The periodicity is modulo 8.
For example, k can be the field of the real numbers.

CASE 4. a and -a are not squares, but both are sums of 2 squares.
In this case, one obtains the following simplifications of Table 1.
First of all, $X = k(\sqrt{-a})$ and $Z = k(\sqrt{a})$ are quadratic field extensions. By Lemma 4, $Y \simeq W \simeq M(2,k)$. Furthermore, a simple calculation shows that $G \simeq M(2,k)$ also. The periodicity is modulo 4.
As an example, consider the field with p elements, where p is a prime congruent to 1 modulo 4, and a is a nonsquare. For Case 4 Table 1 becomes

	0	1	2	3	4	5	6	7
C(r,0)	k	$k(\sqrt{-a})$	M(2,k)	$M(2,k(\sqrt{a}))$	M(4,k)	$M(4,k\sqrt{-a})$	M(8,k)	$M(8,k\sqrt{a})$
C(0,s)	k	$k(\sqrt{a})$	M(2,k)	$M(2,k\sqrt{-a})$	M(4,k)	$M(4,k\sqrt{-a})$	M(8,k)	$M(8,k\sqrt{a})$

Table 2

CASE 5. a and -a are not squares, while only a is the sum of 2 squares.
Here, $X = k(\sqrt{-a})$, $Z = k(\sqrt{a})$, $W = M(2,k)$, while $Y = \left(\frac{-a,-a}{k}\right)$ and G are division rings. For example, k can be the field of the rational numbers and a = 2. Table 1 becomes

	0	1	2	3	4	5	6	7
C(r,0)	k	$k(\sqrt{-a})$	Y	$\left(\frac{-a,-a}{k(\sqrt{a})}\right)$	M(2,G)	$M(4,k\sqrt{-a})$	M(8,k)	$M(8,k\sqrt{a})$
C(0,s)	k	$k(\sqrt{a})$	M(2,k)	$M(2,k\sqrt{-a})$	M(2,G)	$M(2,\left(\frac{-a,-a}{k\sqrt{a}}\right))$	M(4,Y)	$M(8,k\sqrt{-a})$

Table 3

CASE 6. Neither a nor -a is the sum of 2 squares.
Very few simplifications occur in this case. Both W and Y are division rings, while G may or may not be. For example, k can be the field of the rational numbers and a = 6. Table 1 looks like

	0	1	2	3	4	5	6	7
C(r,0)	k	$k(\sqrt{-a})$	Y	$\left(\frac{-a,-a}{k\sqrt{a}}\right)$	M(2,G)	$M(2,\left(\frac{a,a}{k\sqrt{-a}}\right))$	M(4,W)	$M(8,k(\sqrt{a}))$
C(0,s)	k	$k(\sqrt{a})$	W	$\left(\frac{a,a}{k\sqrt{-a}}\right)$	M(2,G)	$M(2,\left(\frac{-a,-a}{k\sqrt{a}}\right))$	M(4,Y)	$M(8,k\sqrt{-a})$

Table 4

4. APPLICATION.

Let us consider Case 4. We ask the question posed in [2], page 129: Given n, what is the biggest integer m such that C(m-1,0) has an n-dimensional representation over k?.

THEOREM. *In Case 4,* $m = 2b + 1$, *where* $n = 2^b \cdot n_0$ *and* n_0 *is odd.*

Proof. Write $m - 1 = 4t + i$ $(i=0,1,2,3;\ t=0,1,2,\ldots)$

$$C(m-1,0) = \begin{cases} M(2^{2t},k) & i = 0 \\ M(2^{2t},k\sqrt{-a}) & i = 1 \\ M(2^{2t+1},k) & i = 2 \\ M(2^{2t+1},k\sqrt{a}) & i = 3 \end{cases}$$

We see that C(m-1,0) maps into M(n,k) if and only if

$$i = 0:\ 2^{2t}|n \Leftrightarrow 2^{2t}|2^b.n_0 \Leftrightarrow 2t \leqslant b$$

$$i = 1:\ 2.2^t|n \Leftrightarrow (2t+1) \leqslant b$$

$$i = 2:\ 2^{2t+1}|n \Leftrightarrow (2t+1) \leqslant b$$

$$i = 3:\ 2.2^{t+1}|n \Leftrightarrow (2t+2) \leqslant b.$$

Hence, given $n = 2^b.n_0$, the biggest value for m-1 = 4t + i becomes t = b/2 and i = 0 if b is even. If b is odd, then t = [b/2] ("[ ]" is the greatest integer function), i = 2 and m-1 = 4[b/2] + 2. In either case, m = 2b + 1.

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