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BESSEL EXPANSIONS ARISING FROM ANOMALOUS STURM-LIOUVILLE DIFFERENTIAL SYSTEMS

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We dedicate this paper to the memory of Prof. Dr. Carlota Szabó, neé Nagy, an exceptional teacher of our science.

ABSTRACT. Let $\{V_k\ ;\ k = 1,2,3,\ldots\}$ be the system of eigenfunctions and associated functions of Bessel's equation of order $\nu \geq 1$, $y'' + (\lambda - q)y = 0$, with boundary conditions depending polynomially on the parameter λ: $P(\lambda)\ y(1) + Q(\lambda)\ y'(1) = 0$, $y \in L^2(0,1)$. It is shown that the structure of $\{V_k\}$ is similar to that of the case where $q \in C([0,1])$ and the boundary conditions depend polynomially on λ at both ends.

This system is not a basis, although any function in L^2 can be expanded into a series of the form : $f = \sum c_k(f)\ V_k$.

0. INTRODUCTION. A solid sphere with an initial distribution of temperature symmetrical about the z-axis (for example, a linear distribution $C.r\cos\theta$), is cooled by immersion in a mass of a well-stirred liquid which has at each instant a uniform temperature throughout it. Assume that the sphere has radius one and $u(r,\theta,t)$ denotes its temperature at the instant t, while the initial distribution is of the form

I) $f(r).\cos\theta$.

The coefficients of the expansion of u obtained by separation of variables are determined by those of the following expansion of the radial part of the initial distribution of temperature

II) $g(r) = r\ f(r) = \sum_{j=1}^{\infty} B_j.y_j(r)$, $y_j(r) = r^{1/2}\ J_{3/2}(\lambda_j^{1/2} r)$, where

III) $(a - b\,\lambda - 1)\ y(1) + y'(1) = 0$, a and b constants.

The boundary condition is of the form

IV) $P(\lambda)\ y(1) + Q(\lambda)\ y'(1) = 0$, P and Q polynomials.

We shall not enter into more details in relation with this particular example (cf. [L] and [S]). In this paper our main objective is to study expansions into series of cylindrical functions as in II) satisfying a boundary condition of the type IV), but we shall restrict ourselves to Bessel functions of order $\nu \geq 1$.

The problems that we consider here and the objectives we pursue are in nature similar to those considered by R.E. Langer, [L], and C. Miranda, [M]. However our approach follows the same lines as in [B]. There, the following problem was studied:

$$(P)\quad \begin{cases} y'' - (\lambda + q(x))y = 0, & 0 < x < 1 \\ P(\lambda)\, y(0) + Q(\lambda)\, y'(0) = 0 \\ \tilde{P}(\lambda)\, y(1) + \tilde{Q}(\lambda)\, y'(1) = 0 \end{cases}$$

with $q \equiv 0$, $\tilde{P},\tilde{Q}$ polynomials with real coefficients and P,Q constants not both zero. The more general situation in (P) when $q(x)$ is real and continuous in $[0,1]$ and the four real polynomials $P,Q,\tilde{P},\tilde{Q}$ verify $\text{G.C.D.}(P,Q) \equiv 1$, $\text{G.C.D.}(\tilde{P},\tilde{Q}) \equiv 1$, was studied by E. Güichal in his doctoral thesis (cfr. [G]).

The boundary problem that we treat in this paper could become an introductory work to a more general theory of anomalous systems with $q \overline{\in} C([0,1])$.

We shall finish this introduction with an *alternative form for the boundary condition* IV).

Let us consider the differential equation:

$$y'' - f(x).y + \lambda y = 0 \quad , \quad a < x \leqslant b \quad , \quad f \in C^m((a,b]).$$

Then $\displaystyle \lambda y^{(h)} = -y^{(h+2)} + \sum_{j=0}^{h} \binom{h}{j} f^{(j)} y^{(h-j)}$ if $h = 0,1,\dots,m$.

It follows by induction that

$$\text{V)}\quad \lambda^k y(x) = (-1)^k\, y^{(2k)}(x) + \sum_{s=0}^{2k-2} f_{k,s}(x)\, y^{(s)}(x) \quad , \quad 2k \leqslant m ,$$

where $f_{k,s} \in C^1$. Differentiating V) we get

$$\text{VI)}\quad \lambda^k y'(x) = (-1)^k y^{(2k+1)}(x) + \sum_{s=0}^{2k-1} g_{k,s}(x)\, y^{(s)}(x) \quad , \quad 2k+1 \leqslant m ,$$

where $g_{k,s} \in C$.

Assume that the degrees of P and Q are p and q respectively, and that $m = 2p \vee (2q+1)$. Then there exist constants $c_0,c_1,\dots,c_m$, independent of $y(x)$, such that

$$\text{VII)}\quad P(\lambda)\, y(b) + Q(\lambda)\, y'(b) = \sum_{j=0}^{m} c_j\, y^{(j)}(b).$$

Conversely, given $c_0,\dots,c_m$, and using V) we get

$$\sum_0^m c_j\, y^{(j)}(b) = a_m\, \lambda^m\, y(b) + \sum_0^{m-1} c_j'\, y^{(j)}(b) \quad , \quad \text{if} \quad m = 2p,$$

$$\sum_0^m c_j\, y^{(j)}(b) = a_m\, \lambda^m\, y'(b) + \sum_0^{m-1} c_j'\, y^{(j)}(b) \quad , \quad \text{if} \quad m = 2q+1.$$

So, step by step we see that $\displaystyle\sum_0^m c_j\, y^{(j)}(b) = P(\lambda)\, y(b) + Q(\lambda)\, y'(b)$.

where the polynomials P and Q are of degrees p and q respectively and $m = 2p \vee (2q+1)$. These polynomials are independent of the solution $y(x)$. This means in particular that the boundary condition IV), which is the same used in [O], pp. 241-242, can be replaced by a condition of the form $\sum_0^m c_j y^{(j)}(1)=0$ for solutions of a Bessel differential equation

VIII) $\quad y''(x) + (\lambda - (\nu^2 - 1/4)/x^2)\, y(x) = 0 , \quad 0 < x \leqslant 1 .$

1. ON EXPANSIONS OF AN L^2-FUNCTION WITH RESPECT TO CERTAIN NON-ORTOGONAL SYSTEMS.

The following general results apply to several situations. They are an assembling of results that can essentially be found in [B].

THEOREM 1. *Let* $\{V_s;\ s=1,2,\dots\}$ *be a system of normalized complex functions in* L^2*, verifying* i), ii), iii), *and* iv).

i) $s \neq t$ *implies*

(1) $$(V_s,V_t) = \frac{O(1)}{\sigma(s)\ \sigma(t)} \qquad \textit{where} \qquad \sum_{s=1}^{\infty} \frac{1}{|\sigma(s)|^2} < \infty ,$$

ii) *if* $f \in L^2$ *its Fourier products* $b_s(f) = (f,V_s)$ *verify*

(2) $$\| b(f)\|_2 = \left(\sum_s |b_s(f)|^2\right)^{1/2} \leqslant K.\|f\| ,\ K \textit{ independent of } f,$$

iii) *for each* s *there exists a continuous linear functional on* L^2, $c_s(f) \neq 0$*, and a set* D *dense in* L^2 *such that* $f \in D$ *implies that* $\sum c_s(f)\, V_s$ *converges in the mean to* f*, and*

iv) *if* s *is great enough, say* $s > s_0$,

(3) $$c_s(f) = b_s(f).\eta_s \ , \quad \eta_s \textit{ a constant.}$$

Then

(4) $$f \in L^2 \textit{ implies } f = \sum c_s(f) V_s \ (L^2), \textit{ and } \eta_s \longrightarrow 1$$
when $s \longrightarrow \infty$; *also*

(5) $$\|f\|_2 \leqslant K\ \|c(f)\|_2 \leqslant M\ \|f\|_2 ,$$
$$c(f) = (c_1(f),c_2(f),\dots), \textit{ and}$$

(6) $$b_t(f) = c_t(f) + O(1)\ \|c(f)\|/\sigma(t).$$

Proof. Let $A_{st} = (V_t,V_s) = \overline{A}_{ts}$. Then if $N > M$

(7) $$\int_a^b |\sum_M^N e_s V_s|^2\, dx = \sum_M^N |e_s|^2 + \sum_{s \neq t} A_{ts}\, e_s\, \overline{e}_t$$

From (1) it follows that the last sum is not greater than
$C(\sum_M^N |e_s/\sigma(s)|)^2 \leqslant C \sum_M^N |e_s|^2 \cdot \sum_M^N |\sigma(s)|^{-2} \leqslant \varepsilon(M)\ \|e\|_2^2$
where $\varepsilon(M) = o(1)$ for $M \longrightarrow \infty$. Applying this to (7) we get

$$\text{(8)} \qquad \|\sum_M^N e_s V_s\|_2^2 = (1 + o(1)) \sum_M^N |e_s|^2 .$$

The equality (8) implies the following propositions j) and jj):

j) $\sum e_s V_s$ converges in L^2 if and only if $\{e_s\} \in 1^2$,

jj) $\|\sum e_s V_s\|_2 \leqslant K \|e\|_2$, K independent of $e = \{e_s\}$.

It also holds:

jjj) $\{e_s\} \in 1^2$ implies $(\sum e_s V_s, V_t) = e_t + O(1)\ \|e\|_2/\sigma(t)$.

In fact, making use of j) we obtain

$$(\sum e_s V_s, V_t) = \sum e_s A_{ts} = e_t + \sum_{s\neq t} O(1)\ e_s/\sigma(t)\ \sigma(s)$$

and jjj) follows from Schartz inequality. Let us see now that $\eta_s \longrightarrow 1$.
Assume that $f \in D$ and $t > s_0$. From j)-jjj) we obtain for $f = \sum c_s(f)V_s$:

$$\text{(9)} \quad b_t(f) = c_t(f) + O(1)\ \|c\|_2/\sigma(t) = \eta_t\ b_t(f) + O(1)\ \|c\|/\sigma(t).$$

But, if M is great enough and fixed

$$\text{(10)} \quad \|c\|_2^2 = \sum_{s=1}^M |c_s(f)|^2 + \sum_{M+1}^\infty |c_s(f)|^2 \leqslant \sum_1^M |c_s(f)|^2 + 2\|\sum_{M+1}^\infty c_s V_s\|^2 =$$
$$= \sum_1^M |c_s|^2 + 2\|f - \sum_1^M c_s V_s\|^2 \leqslant \sum_1^M |c_s|^2 + 2[\ \|f\| + \sum_1^M |c_s|^2]^2 \leqslant$$
$$\leqslant K_0\ \|f\|_2^2 .$$

Then from (9) we obtain for any f belonging to D: $1 - \eta_t =$
$= O(1)K_0\ \|f\|_2/\sigma(t)\ |b_t(f)|$, and therefore

$$1 - \eta_t = \frac{O(1)}{\sigma(t)} \inf_D \frac{\|f\|}{|b_t(f)|} .$$

Since $\overline{D} = L^2$, taking f's near to V_t one sees that the inf is equal to one. Since $\sigma(t) \longrightarrow \infty$ with t, it follows $1 - \eta_t = o(1)$.

Let us prove now (4). Assume that $g_n \in D$ converges to g in L^2. Then ii)implies that $b(g_m) \longrightarrow b(g)$ in 1^2. From the hypothesis on the c's and the fact that $\eta_s \longrightarrow 1$, it follows

$$\text{(11)} \qquad c(g_m) \longrightarrow c(g) \quad \text{in } 1^2.$$

In consequence, making use of jj),

$$\|g - \sum c_s(g)V_s\|_2 = \|(g - g_m) - \sum c_s(g - g_m).V_s\| \leq$$
$$\leq C \|c(g_m) - c(g)\| + \|g - g_m\|.$$

Since the right-hand side tends to zero as m tends to infinity, the left-hand side is null, and (4) follows. The first inequality in (5) is a consequence of jj) and the second one follows from (10) for $f \in D$ and in general, from an approximation argument and (11).

(6) is a consequence of jjj) and (4). Q.E.D.

REMARKS. Theorem 1 assures that the transformation T: $l^2 \ni e \longrightarrow f = \sum e_s V_s \in L^2$ is continuous and onto, since $f \longrightarrow c(f)$ is a right inverse of T.

Let us call $\Gamma = \{c(f);\ f \in L^2\}$. Γ is a subspace (closed) of l^2 (cf.(5)). $\Gamma = l^2$ *if and only if each function has a unique expansion.*

In fact, each function has a unique expansion iff T is one-to-one, and this happens iff $f \longrightarrow c(f)$ is a left inverse of T. That is, iff $\Gamma = l^2$.

THEOREM 2. *Assume that the hypothesis of theorem 1 hold.*

a) *If* A *is the gramian of the system* $\{V_s\}$: $A_{ij} = (V_j, V_i)$, *then* $A = I+T$ *where* T *is defined by a matrix of finite Hilbert-Schmidt norm.*

b) *Assume that* $A_{ki} \neq 0$ *for a pair of different subscripts* i,k, *both greater than* s_0. *Then no function of* L^2 *has a unique expansion with respect to the system* $\{V_j\}$.

c) *Let* $B = \{b(f);\ f \in L^2\}$ *be the space of all Fourier products.* B *is a subspace of* l^2 *and* $\|f\|_2$ *is equivalent to* $\|b(f)\|_2$.

d) $G = l^2 \ominus B$ *is of finite dimension* g. *Besides,* G *and* Γ *form a pair of complementary manifolds.*

Proof. a) $T_{ij} = (1 - \delta_{ij})A_{ij}$. Then,

$$\sum_{i,j} |T_{ij}|^2 \leq \sum_{i,j} O(1)/(\sigma(i)\sigma(j))^2 = O(1)(\sum_i \sigma(i)^{-2})^2 < \infty .$$

b) Take $f = V_i$. Then $c_k(f) = \eta_k b_k(f) = \eta_k A_{ki} \neq 0$, and therefore

$V_i - \sum_1^\infty c_s(f)\, V_s$ is equal to 0 but not all its coefficients vanish.

c) Observe that if $f = \sum e_s V_s$ then $b(f) = A.e = (I+T).e$ and therefore B is the range of I+T with T completely continuous. This implies $B = \bar{B}$, (cf.[A], [RS]), and since $f \longrightarrow b(f)$ is a one-to-one continuous transformation from L^2 onto B, its inverse is also bounded, i.e., $\|f\| \sim \|b(f)\|$.

d) A defines a transformation whose range is B. G is the null space of $A^* = A$, i.e., the eigenspace of T corresponding to the eigenvalue -1.

Then dim $G = g < \infty$. On the other hand, given $e \in l^2$, if $f = \sum e_s V_s$, we have $f = \sum (e_s - c_s(f))V_s + \sum c_s(f)V_s$. Then, the first summand is equal to 0 and also $A.(e-c(f)) = 0$. So, $e = c + (e-c)$ with $c \in \Gamma$ and $e-c \in G = A^{-1}(0)$. Q.E.D.

Next we apply the preceding general results to non-orthogonal expansions in series of Bessel functions.

2. EIGENVALUE EQUATION.

Consider Bessel's equation VIII), $\nu \geq 1$. $\{\sqrt{x}\, J_\nu(x\sqrt{\lambda}), \sqrt{x}\, Y_\nu\,(x\sqrt{\lambda})\}$ is a set of linearly independent solutions. The only solutions in $L^2(0,1)$ are of the form $c\,\sqrt{x}\, J_\nu\,(x\sqrt{\lambda})$, (when $\nu \geq 1$, Bessel's equation belongs to H. Weyl's "Limit point case"). Let us call

$$\Delta(x,\lambda) = J_\nu(x\sqrt{\lambda})\, Y_\nu'\,(\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu'\,(\sqrt{\lambda}) = (2i)^{-1}(H_\nu^{(1)}(x\sqrt{\lambda})H_\nu^{(2)'}(\sqrt{\lambda}) - H_\nu^{(2)}(x\sqrt{\lambda})H_\nu^{(1)'}(\sqrt{\lambda})),$$

$$D(x,\lambda) = J_\nu(x\sqrt{\lambda})Y_\nu(\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(\sqrt{\lambda}) = (2i)^{-1}(H_\nu^{(2)}(x\sqrt{\lambda})H_\nu^{(1)}(\sqrt{\lambda}) - H_\nu^{(1)}(x\sqrt{\lambda})H_\nu^{(2)}(\sqrt{\lambda})).$$

Then,

$$(12)\quad \phi(x,\lambda) = (\pi/2)\sqrt{x}\, D(x,\lambda) \quad ; \quad \theta(x,\lambda) = \phi/2 + (\pi/2)\sqrt{x\lambda}\, \Delta(x,\lambda)\ ,$$

are solutions of Bessel's equation verifying

$$\phi(1,\lambda) = 0 \quad , \quad \phi'(1,\lambda) = -1 \quad ; \qquad \theta(1,\lambda) = 1 \quad , \quad \theta'(1,\lambda) = 0.$$

Therefore: $(\theta\phi')(x,\lambda) - (\theta'\phi)(x,\lambda) \equiv \theta.\phi'(1,\lambda) - \theta'.\phi(1,\lambda) \equiv -1$.

The characteristic values λ_n are the zeroes of $J_\nu\,(\sqrt{\lambda})$ for the boundary conditions $y(1) = 0$, $y \in L^2$. An L^2-solution for $\lambda \neq \lambda_n$ $\forall$ n is of the form $\psi = \theta + m(\lambda)\phi$ where (cf. [T]),

$$(13)\qquad m(\lambda) = -\sqrt{\lambda}\ .J_\nu'\,(\sqrt{\lambda})/J_\nu(\sqrt{\lambda}) - 1/2$$

Since the wronskian $W(J_\nu(z), Y_\nu(z)) = 2/(\pi z)$, it holds

$$(14)\qquad \psi(x,\lambda) = \sqrt{x}\ .J_\nu(x\sqrt{\lambda})/J_\nu(\sqrt{\lambda})$$

Let P and Q be polynomials in λ. Consider the function

$$(15)\qquad \Phi(x,\lambda) = -(Q.\theta + P.\phi)$$

This is a solution of Bessel's equation satisfying

$$\Phi(1,\lambda) = -Q(\lambda) \quad , \quad \Phi'(1,\lambda) = P(\lambda).$$

Taking into account (12), we obtain

$$(16)\quad \Phi(x,\lambda) = -Q(\lambda)\pi\sqrt{\lambda x}\,\Delta(x,\lambda)/2 - (P+Q/2)\pi\sqrt{x}\,D(x,\lambda)/2 =$$

$$= (\pi/2)\sqrt{x}\,\{J_\nu(xs)[-Q(s^2)s\,Y'_\nu(s) - (P+Q/2)(s^2)\,Y_\nu(s)] +$$

$$+ Y_\nu(xs)\,[Q(s^2)s\,J'_\nu(s) + (P+Q/2)(s^2)\,J_\nu(s)]\} =$$

$$= (\pi/2)\sqrt{x}\,\{J_\nu(xs)\,[\ldots] + Y_\nu(xs).s^\nu\omega(s^2)\}.$$

Now we assume that P and Q are real polynomials such that GCD(P,Q)=1, and $p \vee q \geqslant 1$, where p and q are the degrees of P and Q respectively. In particular we have excluded the case where some of them is identically zero. Next we define two other polynomials, also real, $\pi(\lambda)$, $\rho(\lambda)$, by the following conditions: $\deg \pi < p$, $\deg \rho < q$ and

$$(17)\qquad \pi(\lambda)\,Q(\lambda) + \rho(\lambda)\,P(\lambda) \equiv -1.$$

Let us define now the function Θ:

$$(18)\qquad \Theta(x,\lambda) = \rho(\lambda)\,\theta(x,\lambda) - \pi(\lambda)\,\phi(x,\lambda).$$

This function verifies: $\Theta(1,\lambda) = \rho(\lambda)$, $\Theta'(1,\lambda) = \pi(\lambda)$, and is a solution of Bessel's equation. Therefore,

$$(19)\qquad \Theta.\Phi'(x,\lambda) - \Theta'.\Phi(x,\lambda) = \Theta.\Phi'(1,\lambda) - \Theta'.\Phi(1,\lambda) =$$

$$= \pi Q + \rho P = -1.$$

Let us call

$$(20)\qquad \Psi(x,\lambda) = \Theta(x,\lambda) + M(\lambda)\,\Phi(x,\lambda)\ ,$$

where M is so chosen that $\Psi \in L^2$. Then, except at the poles of $m(\lambda)$: $\Psi = C(\theta + m\phi)$. From (15), (18) and (20) it follows that $C = \rho - MQ$, $-(\pi + MP) = mC$; that is,

$$(21)\qquad M(\lambda) = \frac{\pi + m\,\rho}{-P + m\,Q} \qquad\qquad m(\lambda) = -\frac{\pi + M\,P}{\rho - M\,Q}\ .$$

M and m are *meromorphic functions* of λ. From the first formula in (21) it follows that at a pole of M, $m = P/Q$, (eventually equal to ∞ if $Q = 0$). Conversely, if $m = P/Q$ at λ then if $Q \neq 0$, $m \neq \infty$, it follows from (17): $\pi + m\rho \neq 0$. In consequence, M has a pole there. If $Q = 0$ and $m = P/Q$ then $P \neq 0$, $m = \infty$. Therefore, M ($\sim \rho/Q$) has a pole at λ. We have then

PROPOSITION 1. $M(\lambda) = \infty$ *if and only if* $m(\lambda) = P(\lambda)/Q(\lambda)$, *and the poles of* M *are the roots of the equation*

$$(22)\qquad \frac{P(\lambda)}{Q(\lambda)} = -\frac{1}{2} - \frac{\sqrt{\lambda}\,J'_\nu\,(\sqrt{\lambda})}{J_\nu\,(\sqrt{\lambda})}\ .$$

The poles of $M(\lambda)$ are exactly those λ for which $\Phi(x,\lambda) \in L^2$. That is, those λ for which the boundary problem: y solution of Bessel's equation

VIII) verifying $P(\lambda)\, y(1) + Q(\lambda)\, y'(1) = 0$, $y \in L^2$, has a non-trivial solution.

So (22) *is the equation for the eigenvalues of this problem.*

3. ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES.

Call $\lambda = s^2$. Then (22) is equal to

$$-\frac{P(s^2)}{sQ(s^2)} = \frac{(d/ds)(\sqrt{s}\, J_\nu(s))}{\sqrt{s}\, J_\nu(s)} .$$

Taking into account the asymptotic expansions of $\sqrt{z}\, J_\nu(z)$ and its derivative, which hold in $|\arg z| < \pi - \varepsilon$, we have:

$$(23)\quad -\frac{P(s^2)}{sQ(s^2)} = \frac{-\sin\{s-\nu\pi/2-\pi/4\}\ (1+O(1/s^2)) + \cos\{\dots\}\ (A/s+O(1/s^3))}{\cos\{s-\nu\pi/2-\pi/4\}\ (1+O(1/s^2)) + \sin\{\dots\}\ (A/s+O(1/s^3))}$$

where $A = (1 - 4\nu^2)/8$. Then (23) is the equation for the poles of $M(s^2)$ and the $O(1/s^j)$'s that appear there, *are real for* s *real and have asymptotic expansions with real coefficients.*

Let us write $s - \nu\pi/2 - \pi/4 = \alpha + i\beta$. From (23) it follows:

$$(24)\qquad 0 = P(s^2) + AQ(s^2) - [sQ(s^2) - P(s^2)A/s]\tan(\alpha + i\beta) +$$
$$+ [P(s^2).O(1/s^3) + sQ(s^2).O(1/s^2)]\tan(\alpha + i\beta) +$$
$$+ P(s^2).O(s^{-2}) + Q(s^2).O(s^{-2}) .$$

PROPOSITION 2. a) *If* $q \geqslant p$ *there exists a real constant* C *such that the poles of* $M(s^2)$ *with* $|\arg s| \leqslant \pi/2$ *verify*

$$(25)\qquad \tan(s - \nu\pi/2 - \pi/4) = C/s + O(s^{-3}) ;$$

b) *if* $p > q$ *they verify, also with a real constant* C:

$$(26)\qquad \cot(s - \nu\pi/2 - \pi/4) = C/s + O(s^{-3}).$$

The O*'s are real if* $s > 0$.

Proof. Assume that $Q(\lambda) = a_q\lambda^q + \dots$, $P(\lambda) = \lambda^p + \dots$. In case a) it follows from (24) that

$$s^{2p} + Aa_q\, s^{2q} + O(s^{2q-2}) - (s^{2q+1}\, a_q + O(s^{2q-1}))\tan(\alpha + i\beta) = 0.$$

Then if $q > p$:

$$\tan(\alpha + i\beta) = \frac{A\, a_q + O(s^{-2})}{s\, a_q + O(s^{-1})} = \frac{A}{s} + O(s^{-3}).$$

If $q = p$, instead of A we must put: $a_q^{-1} + A$.

In case b), (24) implies:

$$s^{2p} + O(s^{2p-2}) + (As^{2p-1} - a_q s^{2q+1} + O(s^{2p-3}))\tan(\alpha + i\beta) = 0.$$

Therefore,

$$\cot(\alpha + i\beta) = \frac{-A + a_q s^{2(q-p+1)} + O(s^{-2})}{s(1 + O(s^{-2}))} = \frac{C}{s} + O(s^{-3}). \qquad \text{Q.E.D.}$$

Proposition 2 will be used to prove that *for* $|\lambda|$ *great enough*, $M(\lambda)$ *has a pole only if* λ *is real and positive.* This is the content of next theorem. But before let us prove an auxiliary result.

LEMMA 1. *Assume that* $F(z)$ *is an analytic function defined in the square* S *with sides parallel to the axes of length* 2 *with center at the origin. If* $F(\bar{z}) = \overline{F(z)}$ *then*

$$|\operatorname{Im} F(i\beta)| \leqslant 3\beta \cdot \sup_{\partial S} |F(z)| , \quad \text{for} \quad 0 \leqslant \beta \leqslant 1/3. \tag{27}$$

Proof. $F(s) - F(\bar{s}) = -\frac{\beta}{\pi} \int_{\partial S} \frac{F(t)}{(t-s)(t-\bar{s})} dt$. But, from

$\left|\int_{\partial S} \ldots\right| \leqslant \sup_{\partial S} |F(z)| .8/(2/3)^2$ it follows $|\operatorname{Im} F(s)| \leqslant \frac{18\beta}{2\pi} \sup_{\partial S} |F(z)|$.

Q.E.D.

THEOREM 3. *Let us suppose that* $|\arg s| \leqslant \pi/2$. *Then*

i) *the poles of* $M(s^2)$ *are real if* $|s|$ *is great enough, and*

ii) *they are simple with negative residues, for* s *positive great enough.*

iii) *consecutive real poles of* $M(s^2)$, *like those of* $m(s^2)$, *tend asymptotically to be at distance* π; *besides from a moment on the two sets interlace, i.e., the poles of each occur alternately.*

Proof. i) It follows easily that

$|\operatorname{Im} \tan(\alpha + i\beta)| = |(\tanh\beta)/(\cos^2\alpha + \operatorname{sen}^2\alpha . \tanh^2\beta)| \geqslant |\tanh\beta|$,

$|\operatorname{Im} \cot(\alpha + i\beta)| \geqslant |\tanh\beta|$. Then from (25) or (26) we obtain

$$|\operatorname{Im} (\tfrac{C}{s} + O(s^{-3}))| \geqslant \tanh |\operatorname{Im} s| \quad \text{if } s \text{ is a pole of } M(s^2). \tag{28}$$

In consequence if $|s|$ is great enough: $|\operatorname{Im} s| < 1/3$. Let us call $H(z)$ the meromorphic function denoted by $O(z^{-3})$ in the rigth-hand side of (25) (or (26)), and $G(z)$, the function defined by the left-hand side minus C/z. Then s is a pole of $M(s^2)$ if and only if $G(s) = H(s)$; besides if s is a pole, $\bar{s}$ is also one (cf. (22)), and,

$$G(s) = (G(s) + \overline{G(\bar{s})})/2 = (H(s) + \overline{H(\bar{s})})/2 . \tag{29}$$

Define $F(z) = (H(z) + \overline{H(\bar{z})})/2$. (29) implies that

$\tanh |\operatorname{Im} s| < |\operatorname{Im} (C/s + F(s))| < |C \operatorname{Im} \bar{s}| . |s|^{-2} + |\operatorname{Im} F(s)|$.

A straightforward application of lemma 1 to $F(z)$ if $\beta = \operatorname{Im} s$, $0 < |\beta| < 1/3$ gives:

$$\tanh |\beta| \leqslant |Cs^{-2}.\beta| + |\beta|\, O(s^{-3}). \tag{30}$$

For these values of β, there exists a number d, positive, such that $d < \tanh \beta/\beta$. Then, (30) can be verified only for s bounded, and this proves i).

ii) $M(\lambda)$ has a simple pole at $\lambda_0 = s_0^2 \neq 0$ if and only if $M(s^2)$ has a simple pole at s_0. Assume now that $s_0 > 0$ is a pole for $M(z^2)$; then $m(s_0^2) = P(s_0^2)/Q(s_0^2)$. From (13) it follows

$$\frac{d}{ds} m(s^2) = -\frac{(s\, J_\nu'(s))'}{J_\nu(s)} + s.\frac{J_\nu'^2(s)}{J_\nu^2(s)} = (s - \frac{\nu^2}{s}) + \frac{1}{s}(m(s^2)+1/2)^2. \tag{31}$$

Therefore

$$\left.\frac{d}{ds} m(s^2)\right|_{s=s_0} = (s_0 - \frac{\nu^2}{s_0}) + \frac{1}{s_0}\left[\frac{P(s_0^2)}{Q(s_0^2)} + \frac{1}{2}\right]^2 = \tag{32}$$

$$= s_0 + \frac{P(s_0^2)}{s_0\, Q(s_0^2)}[1 + \frac{P}{Q}(s_0^2)] + (1/4 - \nu^2)\frac{1}{s_0}.$$

But $p > q$ implies $(d/ds)(P/Q)(s^2) \sim \text{constant} . s^{-1}(P/Q)(s^2)$ and $p \leqslant q$ implies $(d/ds)(P/Q)(s^2) = O(1)$. In consequence from (32) it follows that at $s = s_0$, $(d/ds)m(s^2) > (d/ds)(P/Q)(s^2)$ if s_0 is great enough. This proves that s_0 is a simple pole. Its residue is

$$\frac{\pi(s_0^2) + m(s_0^2)\,\rho(s_0^2)}{(d/ds)(mQ - P)(s_0^2)} = -\frac{1/Q(s_0^2)}{Q(s_0^2).(d/ds)(m - P/Q)(s_0^2)} < 0,$$

as it follows taking into account (17) and (21).

iii) The result for m is immediate and that for M follows from the asymptotic formulae of Proposition 2 if one remembers that the $O(s^{-3})$ that appear there are real for s real. It remains only to see that the poles of M and m interlace. Since $m(s^2)$ is real and has simple poles with negative residues on the positive real axis, the values of $m(s^2)$ run from $-\infty$ to $+\infty$ when s runs from one pole to the next one. Then for a certain s_0 in between, $m(s_0^2) = P(s_0^2)/Q(s_0^2)$. For the same reason between two consecutive poles of $M(s^2)$ there exists a point s_1 such that $M(s_1^2) = \rho(s_1^2)/Q(s_1^2)$, i.e., a pole of m (cf. (21) and (17)). Q.E.D.

4. SIMPLE POLES AND RESIDUES OF GREEN'S KERNEL.

First we shall introduce some notation and auxiliary formulae. Recall that $\psi = \theta + m\,\phi = \sqrt{x}\; J_\nu(xs)/J_\nu(s)$ if $J_\nu(s) \neq 0$, i.e. when $m(s^2) \neq \infty$, and that Ψ defined by (20) is, at least where $M(s^2) \neq \infty$, $J_\nu(s) \neq 0$, equal to

$$(33) \quad \Psi = (\rho - MQ)\sqrt{x}\, J_\nu(xs)/J_\nu(s) = (mQ-P)^{-1} \sqrt{x}\, J_\nu(xs)/J_\nu(s) =$$
$$= -\sqrt{x}\, J_\nu(xs)/s^\nu\, \omega(s^2) ,$$

where $\omega(s^2) = \{sJ_\nu'(s)Q(s^2) + [P(s^2) + Q(s^2)/2]\, J_\nu(s)\}\, /s^\nu$, is an even entire function.

On the other hand, s *is a pole of* $M(z^2)$ *iff*

$$(34) \quad \omega(s^2) = (mQ-P)(s^{-\nu}\, J_\nu(s)) = s^{-\nu}[sJ_\nu'(s)Q(s^2)+(P(s^2)+Q(s^2)/2)J_\nu(s)]=0.$$

(The factor $s^{-\nu}$ has been introduced to include the point s=0). In fact, from the definition of Ψ (cf. (20)) it follows that the poles of $M(s^2)$ are exactly the poles of $\Psi(x,s^2)$, which, because of (33), are the zeros of $\omega(s^2)$. ((34) can also be proved using proposition 1). For those points, we have (cf. (15), (16)):

$$\Phi(x,s^2) = -(\pi/2)\sqrt{x}\, J_\nu(xs)\, \{Q(s^2)s\, Y_\nu'(s) + [P(s^2)+Q(s^2)/2]\, Y_\nu(s)\}.$$

This expression must be understood as a limit when $M(0) = \infty$. Then if s_n is a pole of M, from (34) and that $W(J_\nu,Y_\nu) = 2/(\pi s)$, we get

$$(35) \quad \Phi_n = -\sqrt{x}\, J_\nu(xs_n)Q(s_n^2)/J_\nu(s_n) = \sqrt{x}\, J_\nu(xs_n)[P(s_n^2)+Q(s_n^2)/2]/J_\nu'(s_n)s_n$$

Since $M(s^2) = 0$ implies that $s^{-\nu}.J_\nu(s) = 0$ is equivalent to $Q(s^2) = 0$, if the expression in the middle of (35) is indeterminate then its right-hand side must be used. If s=0 is a pole of $M(s^2)$, formula (35) is still valid but in its limit form

$$\Phi_0(x) = x^{\nu+1/2}\,(P(0) + Q(0)/2)/\nu = -x^{\nu+1/2}\,Q(0).$$

Making use of Lommel's formula we obtain for real $s_n^2 \neq 0$:

$$(36) \quad 2\|\Phi_n\|^2 = (1 - \nu^2/s_n^2)\, Q^2(s_n^2) + [P(s_n^2) + Q(s_n^2)/2]^2\, /s_n^2 ,$$

and then, the normalized functions:

$$(37) \quad \hat{\psi}_n = \Phi_n/\|\Phi_n\| = -\sqrt{2x}\, J_\nu(xs_n)\, D_n/J_\nu(s_n) = \sqrt{2x}\, J_\nu(xs_n)\, \tilde{D}_n/J_\nu'(s_n) ,$$

$$\text{where} \quad D_n = \left[1 - \frac{\nu^2}{s_n^2} + \frac{(P(s_n^2) + Q(s_n^2)/2)^2}{s_n^2 \,.\, Q^2(s_n^2)}\right]^{-1/2} , \quad \operatorname{sgn} D_n = \operatorname{sgn} Q(s_n^2) ;$$

$$\tilde{D}_n = \left[1 + \frac{(s_n^2 - \nu^2)\, Q^2(s_n^2)}{(P + Q/2)^2\, (s_n^2)}\right]^{-1/2} , \quad \operatorname{sgn} \tilde{D}_n = \operatorname{sgn}\, (P+Q/2)/s_n .$$

Observe that $D_{n/s_n}\, Q(s_n^2) = \tilde{D}_n/(P+Q/2)(s_n^2)$ if both denominators are different from zero; on the other hand, both denominators cannot be simultaneously null, and if one of them is not null the corresponding numerator is different from zero.

The functions $\hat{\psi}_n$ are uniformly bounded. Precisely

PROPOSITION 3. $\hat{\psi}_n(x) = \sqrt{xs_n}\, J_\nu(xs_n)\, O(1)$.

Proof. If $q \geqslant p$ then $(P+Q/2)/s_n Q = o(1)$ and from (34) it follows that $J'_\nu(s_n)/J_\nu(s_n) = o(1)$. In consequence, s_n approaches zeros of J'_ν, and $|J_\nu(s_n)| > c.s_n^{-1/2}$. This together with (37) imply the thesis.
Analogously, if $q < p$ then $Q.s_n/(P+Q/2) = o(1)$ and from (34): $J_\nu(s_n)/J'_\nu(s_n) = o(1)$. Then s_n approaches zeros of J_ν, and $|J'_\nu(s_n)| > > c.s_n^{-1/2}$.
Again from (37) the thesis follows. Q.E.D.

The Green's kernel $G(x,y,\lambda)$ of the differential system we are considering is equal to $-\Psi(x,\lambda)\ \Phi(y,\lambda)$ if $x \leqslant y$ and equal to $-\Psi(y,\lambda)\ \Phi(x,\lambda)$ if $x > y$.
By definition, $G_\lambda(f)(x) = \int_0^1 G(x,y,\lambda)\ f(y)dy$. Since Θ and Φ are entire functions of λ for fixed x, it follows from the definition of Ψ that

(38) $\quad G(x,y,\lambda) = \text{entire function of } \lambda - M(\lambda)\ \Phi(x,\lambda)\ \Phi(y,\lambda).$

Therefore, the poles of $G(x,y,\lambda)$ are the same as those of $M(\lambda)$. It holds,

PROPOSITION 4. *If* $\lambda = \lambda_n$ *is a simple pole of* M, *then, for* $f \in L^2(0,1)$ *we have*

$$\operatorname*{res}_{\lambda=\lambda_n} G_\lambda(f) = r_n.\hat{\psi}_n(x) \int_0^1 \hat{\psi}_n(y)\ f(y)\ dy \quad , \quad r_n \longrightarrow 1 \quad \textit{when} \quad n \longrightarrow \infty .$$

The proof of the proposition makes use of the following lemma whose detailed proof we leave to the reader, since it is a simple application of Fubini's theorem.

LEMMA 2. *Let* $F(y,\lambda)$ *be a measurable function of* (y,λ), *holomorphic in* $|\lambda - \lambda_0| < d$, *for each* $y \in (0,1)$. *Assume also the existence of a constant* A *such that* $\int_0^1 |F(y,\lambda)|\ dy < A$ *for each* λ. *Then if for each* y, $F(y,\lambda) = \sum_{n=0}^{\infty} a_n(y)(\lambda - \lambda_0)^n$ *holds, then*

$$\int_0^1 F(y,\lambda)\ dy = \sum_{n=0}^{\infty} \left[\int_0^1 a_n(y)\ dy\right]\ (\lambda - \lambda_0)^n \quad \textit{whenever} \quad d > |\lambda - \lambda_0|.$$

Proof of proposition 4. Applying lemma 2 to the function $F(y,\lambda) = G(x,y,\lambda)\ f(y)\ (\lambda - \lambda_n)^r$ where $x \in (0,1)$ is fixed and r is the order of the pole of $M(\lambda)$ at λ_n, we obtain

(39) $$\operatorname*{res}_{\lambda=\lambda_n} G_\lambda(f) = \int_0^1 [\operatorname*{res}_{\lambda_n} G(x,y,\lambda)]\ f(y)\ dy.$$

In particular if λ_n is a simple pole then from (38) we get

$\operatorname*{res}_{\lambda_n} G(x,y,\lambda) = \Phi_n(x)\ \Phi_n(y)\ .\ \operatorname*{res}_{\lambda_n}(-M(\lambda)) = \hat{\psi}_n(x)\ \hat{\psi}_n(y)\ .\ r_n$ and

(40) $$r_n = -\operatorname*{res}_{\lambda_n} M(\lambda)\ .\ \|\Phi_n\|^2\ .$$

This implies the thesis except for $r_n \longrightarrow 1$. Assume that n is so great that λ_n is a simple pole of M and not a pole of m. Then

$$\operatorname*{res}_{\lambda_n} -M(\lambda) = -\frac{\pi + m\,\rho}{(d/d\lambda)(Qm-P)}(\lambda_n) = -\frac{\pi + \rho . P/Q}{Q.(d/d\lambda)(m-P/Q)}(\lambda_n) =$$

$$= Q^{-2}(\lambda_n).\ [m'(\lambda_n) - (P/Q)'(\lambda_n)]^{-1}.$$

From (36) and (32) it follows

$$\|\Phi_n\|^2 = \frac{Q^2(s_n)}{2\ s_n}\ .\ \frac{d}{ds}\ m(s^2)\Big|_{s=s_n} = Q^2(\lambda_n).\ \frac{d}{d\lambda}\ m(\lambda)\Big|_{\lambda=\lambda_n} \quad \text{and then}$$

(41) $$\operatorname*{res}_{\lambda_n} -M(\lambda) = \frac{1}{\|\Phi_n\|^2 - [P'Q - PQ'](\lambda_n)}\ .$$

(36) also implies that $\|\Phi_n\|^2 \sim \lambda_n^{2qv(2p-1)}$. On the other hand,

$$(P'Q - PQ')(\lambda_n) = \left\{\begin{matrix} O(\lambda_n^{p+q-1}) & \text{if} & p\neq q \\ O(\lambda_n^{p+q-2}) & \text{if} & p=q \end{matrix}\right\} = O(\lambda_n^{(2pv2q)-2}).$$

Therefore, from (40) and (41) we finally get:

$$r_n = \frac{\|\Phi_n\|^2}{\|\Phi_n\|^2 - [P'Q - PQ'](\lambda_n)} \xrightarrow[n\to\infty]{} 1\ . \qquad \text{Q.E.D.}$$

5. MULTIPLE POLES OF THE GREEN'S KERNEL.

The calculation of the residue of $G_\lambda(f)$ at a pole ξ is reduced to the computation of the residue at ξ of $-M(\lambda)\ \Phi(x,\lambda)\ \Phi(y,\lambda)$, as follows from (38) and (39). On the other hand, by means of Picard's approximation method it is not difficult to prove the following proposition:

The functions $\Phi^{(i)}(x,\lambda)$, $i=0,1,2$, *are entire in* λ *for fixed* $x \in (0,1]$, (the exponent means derivation with respect to x); *besides, they are continuous in* $(x,\lambda) \in (0,1] \times C$, *and it holds*

$$(42) \qquad \frac{\partial^j \Phi^{(i)}}{\partial \lambda^j}(x,\lambda) = \frac{\partial^i}{\partial x^i}\left[\frac{\partial^j \Phi}{\partial \lambda^j}\right](x,\lambda) , \quad i=1,2, \quad j=1,2,3,\dots ,$$

Furthermore, the functions in (42) *are also continuous in* (x,λ).
(A detailed proof is given in appendix I of [G]).
Let us define the functions $u_j(x,\lambda)$, $j = 0,1,\dots,$ by

$$(43) \qquad u_j(x,\lambda) = \frac{1}{j!}\frac{\partial^j \Phi}{\partial \lambda^j}(x,\lambda) .$$

Therefore

$$(44) \qquad \Phi(x,\lambda) = u_0(x,\xi) + (\lambda - \xi)\, u_1(x,\xi) + \dots .$$

If $M(\lambda)$ has a pole of order r at ξ, let us write

$$-(\lambda - \xi)^r M(\lambda) = c_{-r} + c_{-r+1}(\lambda - \xi) + \dots + c_{-1}(\lambda - \xi)^{r-1} + \dots .$$

In consequence,

$$-(\lambda - \xi)^r M(\lambda)\, \Phi(x,\lambda)\, \Phi(y,\lambda) =$$

$$= \dots + (\lambda - \xi)^{r-1} \sum_{j=1}^{r} c_{-j}\left[\sum_{k=0}^{j-1} u_k(x,\xi)\, u_{j-1-k}(y,\xi)\right] + \dots .$$

But $(\lambda - \xi)\, \Phi(x,\lambda) = -\Phi''(x,\lambda) + (q - \xi)\, \Phi(x,\lambda)$, $q = x^{-2}(\nu^2 - 1/4)$.

Because of (42) and (44) we get

$$\sum_{k=0}^{\infty} [- u_k''(x,\xi) + (q - \xi)\, u_k(x,\xi)]\, (\lambda - \xi)^k = \sum_0^{\infty} u_k \cdot (\lambda - \xi)^{k+1} ,$$

which implies that the functions u_k satisfy the following equations

$$(46) \qquad \begin{cases} u_0'' + (\xi - q)\, u_0 = 0 , \\ u_1'' + (\xi - q)\, u_1 = -u_0 , \\ \vdots\dots\dots\dots\dots\dots \\ u_n'' + (\xi - q)\, u_n = -u_{n-1} , \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

For each n the set of functions $u_0,\dots,u_n$, is *linearly independent on any interval* I *contained in* (0,1). In fact, $u_0(x,\xi) = \Phi(x,\xi) \not\equiv 0$ on I. Let us assume that the set under consideration be linearly dependent on I. Then there exists a first k such that

$$u_k = \sum_{m=0}^{k-1} \alpha_m u_m \quad \text{on I.}$$

Applying the operator $d^2/dx^2 + (\xi - q)$ to this identity, it follows from (46) that $u_{k-1} = \sum_{m=1}^{k-1} \alpha_m u_{m-1}$, in contradiction with the definition on k.

If r is, as before, the order of the pole ξ of $M(z)$, $\{u_0(x,\xi),\dots,u_{r-1}(x,\xi)\}$ is called the *principal chain of functions associated to* $\Phi(x,\xi)$, and from (45) we see that

$$\operatorname*{res}_{\xi} G(x,y,\lambda) = \sum_{j=1}^{r} c_{-j} \left[\sum_{k=0}^{j-1} u_k(x,\xi)\ u_{j-1-k}(x,\xi)\right] ,$$

and from (39), for $f \in L^2(0,1)$, we get

$$(47)\quad \operatorname*{res}_{\xi} G_\lambda(f) = \int_0^1 \{\sum_{j=1}^{r} c_{-j} \sum_{k=0}^{j-1} u_k(x,\xi)\, u_{j-1-k}(y,\xi)\}\, f(y)\, dy .$$

The brackets inside (47) equal to

$$\{\dots\} = \sum_{k=0}^{r-1} u_k(x,\xi)\, \hat{u}_k(y,\xi), \text{ where } \hat{u}_k(y,\xi) = \sum_{j=k+1}^{r} c_{-j}\, u_{j-1-k}(y,\xi).$$

That is

$$(48)\quad \begin{vmatrix} \hat{u}_0 \\ \cdot \\ \cdot \\ \cdot \\ \hat{u}_{r-1} \end{vmatrix} = \begin{vmatrix} c_{-r} & c_{-r+1} & \dots & c_{-1} \\ 0 & c_{-r} & \dots & c_{-2} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & c_{-r} \end{vmatrix} \begin{vmatrix} u_{r-1} \\ \cdot \\ \cdot \\ \cdot \\ u_0 \end{vmatrix} , \quad c_{-r} \neq 0.$$

Therefore, the set $\hat{u}_0,\dots,\hat{u}_{r-1}$ is also linearly independent on any $I \subset (0,1)$.

Since $\operatorname*{res}_{\xi} G_\lambda(f)$ is a finite number for any $x \in (0,1)$ and any $f \in L^2(0,1)$, we see that the bracket in (47) defines a function in L^2 $(0 < y < 1)$, $\forall\, x \in (0,1)$. The linear independence of $\{u_k\ ;\ k=0,1,\dots\}$ implies that each $\hat{u}_j \in L^2$. In fact, there exist r points in I: $x_1,\dots,x_r$, such that the matrix $(u_k(x_m))$, $k=0,\dots,r-1$, $m=1,\dots,r$, has a non-null determinant.

Since $\sum_{k=0}^{r-1} u_k(x_m,\xi)\, \hat{u}_k(y,\xi) = F_m(y) \in L^2$, then

$$\hat{u}_k = \sum A_{mk} F_m \quad \text{also belongs to } L^2.$$

Summing up, we have

PROPOSITION 5. *For* $k=0,\dots,r-1$, $u_k(x,\xi)$ *and* $\hat{u}_k(x,\xi)$ *belong to* L^2, *when* ξ *is a pole of order* r *of* $M(z)$. *If* $\gamma_k(f) = \gamma_k(f,\xi)$ *denotes the continuous linear functional on* L^2 *defined by*

$$\int_0^1 \hat{u}_k(y,\xi)\, f(y)\, dy, \quad \textit{then} \quad \operatorname*{res}_{\xi} G_\lambda(f) = \sum_{k=0}^{r-1} u_k(x,\xi)\cdot\gamma_k(f) .$$

6. EIGENFUNCTION EXPANSIONS FOR CERTAIN REGULAR FUNCTIONS.

Let $\{c_n\}$ be a family of circumferences with center 0 and radii r_n to be defined later but such that $r_n \longrightarrow \infty$ as $n \longrightarrow \infty$.
We shall prove the following result.

THEOREM 4. *Assume that* $f \in C^2(0,1)$ *is null on neighbourhoods of* 0 *and* 1. *Then, it holds uniformly on* $0 < x < 1$ *that*

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_n} G_\lambda(f)\, d\lambda = f(x) .$$

To prove this theorem we need some auxiliary results. The first of them is the well-known fact that Green's operator is the inverse of the differential operator at least on a set of functions dense in L^2.

LEMMA 3. *Let* f *be as in theorem 4, and* λ *not a pole of* G_λ. *Then, on* $0 < x < 1$,

$$(49) \qquad f(x) = \lambda G_\lambda(f) + G_\lambda(\tilde{f}) \quad \textit{if} \quad \tilde{f} = f'' + x^{-2}[1/4 - \nu^2]\, f .$$

(This follows after integrating by parts twice the integrals involved in $G_\lambda(f'')$, (cf. [T], ch. II)).

LEMMA 4. *If* $g \in L^1(0,1)$ *is null on* $(0,\varepsilon)$ *for a certain* $\varepsilon > 0$, *then it holds*

$$\lim_{n\to\infty} \int_{C_n} G_\lambda(g)\, \frac{d\lambda}{\lambda} = 0 ,$$

uniformly in $x \in (0,1)$.

Proof. The integral that must be estimated is equal to

$$(50) \qquad \int_{C_n} G_\lambda(g)\, \frac{d\lambda}{\lambda} = \int_\varepsilon^1 dy\; g(y) \int_{C_n} G(x,y,\lambda)\, \frac{d\lambda}{\lambda} .$$

Then, it will be sufficient to show that, uniformly on $y > \varepsilon$, $0 < x < 1$,

$$(51) \qquad \int_{C_n} G(x,y,\lambda)\, \frac{d\lambda}{\lambda} = 2 \int_{D_n} G(x,y,s^2)\, \frac{ds}{s} = o(1),$$

holds. Here, $D_n = \{s;\ |s| = r_n^{1/2},\ |\arg s| \leqslant \pi/2\}$.

If Δ and D are defined as in paragraph 2, from the asymptotic formulae for the Hankel functions, we get:

$$(52)\quad \begin{cases} \Delta(x,s^2) = \dfrac{2}{\pi\ s\ \sqrt{x}} \cos s(1-x) + e^{(1-x)|\mathrm{Im}\ s|}.O(1/s^2)\ , \\ D(x,s^2) = \dfrac{2}{\pi\ s\ \sqrt{x}} \sin s(1-x) + e^{(1-x)|\mathrm{Im}\ s|}.O(1/s^2). \end{cases}$$

In (52) the O's hold uniformly on $\varepsilon < x < 1$. From (16) and (52) the following estimation for Φ is obtained,

$$(53)\qquad \Phi(x,s^2) = s^{(2q+1)\ v\ 2p}\ .\ e^{(1-x)\ |\mathrm{Im}\ s|}.\ O(1/s)\ .$$

Let a and b be functions of p and q defined as follows:

$a = 0$, $b = \sqrt{2/\pi}$ if $q \geqslant p$, $a = \sqrt{2/\pi}$, $b = 0$ if $q < p$. Then, from (33) we obtain on $0 < x < 1$:

$$(54)\quad \Psi(x,s^2) = \frac{-\sqrt{x}\ J_\nu(xs)}{sQ(s^2)\ J'_\nu(s) + \{P(s^2) + Q(s^2)/2\}\ J_\nu(s)} =$$

$$= \frac{-\sqrt{xs}\ J_\nu(xs)}{s^{(2q+1)v2p}\ \{[\,a.\cos(s-\nu\pi/2 - \pi/4) - b.\sin(s-\nu\pi/2-\pi/4)] + e^{|\mathrm{Im}\ s|}O(s^{-1})\}}\ .$$

Taking as $\sqrt{r_n}$ the points in $(0,\infty)$ where the function defined by the square brackets in the second denominator in (54) takes a maximum, it is possible to complement (54) with the following estimation valid for $s \in D_n$, and O(1) independent of $x \in (0,1)$:

$$(55)\qquad \Psi(x,s^2) = \frac{\sqrt{xs}\quad J_\nu(xs)\quad O(1)}{s^{(2q+1)v2p}\ .\ e^{|\mathrm{Im}\ s|}} = \frac{e^{(x-1)\ |\mathrm{Im}\ s|}\ O(1)}{s^{(2q+1)v2p}}\ .$$

From (53) and (55) it follows that

$$(56)\quad G(x,y,s^2) = \begin{cases} -\Phi(x)\ \Psi(y) = e^{(y-x)\ |\mathrm{Im}\ s|}.\ O(1/s)\ , & \text{if}\quad x > y, \\ -\Phi(y)\ \Psi(x) = e^{(x-y)\ |\mathrm{Im}\ s|}.\ O(1/s)\ , & \text{if}\quad y > x, \end{cases}$$

where the O's hold uniformly if $\varepsilon < y < 1$, $0 < x < 1$. Then

$$\int_{D_n} G(x,y,s^2)\,\frac{ds}{s} = \int_{D_n} O(1/s^2)\ ds = O(r_n^{-1/2}) = o(1). \qquad \text{Q.E.D.}$$

Proof of theorem 4. (49) says that $G_\lambda(f)(x) = f(x)/\lambda - G_\lambda(\tilde{f})(x)/\lambda$ and therefore we get:

$$\frac{1}{2\pi i}\int_{C_n} G_\lambda(f)(x)\ d\lambda = f(x) - \frac{1}{2\pi i}\int_{C_n} G_\lambda(\tilde{f})(x)\ \frac{d\lambda}{\lambda}\ .$$

The thesis now follows from Lemma 4.

COROLLARIES TO THEOREM 4. i) *Let f be as in theorem 4. If M denotes* *the finite set of poles of* M(z) *that are multiple, null or nonreal, and* $r(\xi)$ *the multiplicity of the pole* ξ*, then it holds uniformly on* (0,1) *that*

$$(57)\qquad f(x) = \sum_{\xi\in M} \sum_{k=0}^{r(\xi)-1} U_k(x,\xi)\, \gamma_k(f,\xi) + \sum_{m=1}^{\infty} b_m(f)\cdot r_m\, \hat{\psi}_m(x) ,$$

where the last sum extends over the set of normalized eigenfunctions corresponding to non-zero, real and simple eigenvalues.

ii) *The system of functions*

$$\{V_k : k=1,2,3,\dots\} = \{U_j(x,\xi)/\|U_j\|_2 \ ,\ \hat{\psi}_m(x)\ ;\ \xi\in M ,\ j<r(\xi)\ ,\ m=1,2,3,\dots\}$$ *is complete in* $L^2(0,1)$.

These results are easy outcomes of Propositions 4 and 5, and Theorem 4. Thus, iii) of Theorem 1 is verified.

7. EXPANSION OF A SQUARE-INTEGRABLE FUNCTION.

In this section we show that the complete system of functions $\{V_k\}$ verifies the hypotheses of theorems 1 and 2. This proves in particular the possibility of expanding an L^2-function into a series of eigenfunctions and associated functions.

LEMMA 5. *The gramian of the system* $\{V_k\}$ *satisfies* i), Th.1. *More precisely, for a certain* $\alpha > 1/2$ *it holds*:

i) $$\int_0^1 \hat{\psi}_n(x)\ \hat{\psi}_m(x)\ dx = \frac{O(1)}{s_n\ s_m} ,\qquad \textit{if } n\neq m ,$$

ii) $$\int_0^1 U_k(x,\xi)\ \hat{\psi}_n(x)\ dx = \frac{O(1)}{s_n^{\alpha}} ,\qquad \textit{if } \nu \neq \textit{integer} ,$$

iii) $$\int_0^1 U_k(x,\xi)\ \hat{\psi}_n(x)\ dx = \frac{O(1)\ \log s_n}{s_n} ,\qquad \textit{if } \nu \textit{ is an integer.}$$

Proof. If $s_n^2 \neq s_e^2$, from (37) and Lommel's formula we get that

$$(\hat{\psi}_n,\hat{\psi}_e) = \int_0^1 \hat{\psi}_n(x)\ \hat{\psi}_e(x)\ dx \quad \text{is equal to:}$$

$$\frac{2}{s_n^2 - s_e^2}\left[s_e \cdot \frac{J_\nu'(s_e)}{J_\nu(s_e)} - s_n \cdot \frac{J_\nu'(s_n)}{J_\nu(s_n)}\right] D_n D_e \qquad \text{if } J_\nu(s_n) \neq 0 \neq J_\nu(s_e),$$

$$\frac{-2}{s_n^2 - s_e^2}\left[s_n \cdot \frac{J_\nu(s_e)}{J_\nu'(s_e)} - s_e \cdot \frac{J_\nu(s_n)}{J_\nu'(s_n)}\right] \tilde{D}_n \tilde{D}_e \qquad \text{if } J_\nu'(s_n) \neq 0 \neq J_\nu'(s_e),$$

$$\frac{-2\, s_e}{s_n^2 - s_e^2} \cdot D_n \tilde{D}_e \quad \text{if} \quad J_\nu(s_e) = 0 \ , \ J'_\nu(s_n) = 0 \ .$$

We define now the symmetric polynomial $V(\lambda,\mu)$ as follows,

(58) $$V(\lambda,\mu) = (P(\lambda)Q(\mu) - P(\mu)Q(\lambda))/(\lambda-\mu).$$

Then, taking into account the eigenvalue equation (34), it follows easily that $(\hat{\psi}_n,\hat{\psi}_e)$ is equal to

(59) $$\int_0^1 \hat{\psi}_n(x)\ \hat{\psi}_e(x)\ dx = 2 s_n s_e . V(s_n^2,s_e^2) . \frac{D_n}{s_n\ Q(s_n^2)} \ \frac{D_e}{s_e\ Q(s_e^2)}$$

if $J_\nu(s_n) \neq 0 \neq J_\nu(s_e)$,

$$= 2\, s_n s_e .\ V(s_n^2,s_e^2)\ .\ \frac{\tilde{D}_n}{(P+Q/2)(s_n^2)} \ \frac{\tilde{D}_e}{(P+Q/2)(s_e^2)} \quad \text{if } J'_\nu(s_n) \neq 0 \neq J'_\nu(s_e),$$

$$= 2\, s_n s_e .\ V(s_n^2,s_e^2)\ .\ \frac{D_n}{s_n\ Q(s_n^2)} \ \frac{\tilde{D}_e}{(P+Q/2)(s_e^2)} \quad \text{if } J_\nu(s_e)=0 \ , \ J'_\nu(s_n)=0.$$

Now, if $q \geqslant p$ then the degree of $V(\lambda,\mu)$ in λ or μ is not greater than $q-1$, and $|D_n| \longrightarrow 1$. In consequence $D_n/(s_n\ Q(s_n^2)) = O(s_n^{-2q-1})$. Then, from (59) we obtain

(60) $$(\hat{\psi}_n,\hat{\psi}_e) = O(s_n^{-2}\ s_e^{-2}).$$

On the other hand, if $p > q$ then $\deg_\lambda V(\lambda,\mu) = \deg_\mu V(\lambda,\mu) \leqslant p-1$ and $|D_n| \longrightarrow 1$. This implies that

(61) $$(\hat{\psi}_n,\hat{\psi}_e) = O(s_n^{-1}\ s_e^{-1}).$$

i) is thus proved. To prove ii) and iii) of lemma 5 we need a more explicit expression for $\mathcal{U}_k(x,\xi)$ which is exhibited next. Assume $\nu \neq$ integer. Then $Y_\nu(z).\sin \nu\pi = \cos \nu\pi . J_\nu(z) - J_{-\nu}(z)$, and this, together with (16) yields

(62) $$-\frac{2 \sin \nu\pi}{\pi} \cdot \frac{\Phi(x,s^2)}{\sqrt{x}} = (P+Q/2)(s^2)\ [J_\nu(xs)\ J_{-\nu}(s) - J_{-\nu}(xs)\ J_\nu(s)] + $$
$$+ Q(s^2)\ s\ [J_\nu(xs) J'_{-\nu}(s) - J_{-\nu}(xs)\ J'_\nu(s)].$$

Thus for $\nu \neq$ integer we have,

(63) $$\Phi(x,s^2) = \sqrt{x}\ \ [s^{-\nu} J_\nu(xs)\ f(s^2) + s^\nu\ J_{-\nu}(xs)\ g(s^2)] \ ,$$

where f and g are certain entire functions.

Assume now ν = integer. In this case: $Y_\nu(z) = (2/\pi)\,[\log(z/2).J_\nu(z) + H_{-\nu}(z)]$ where $H_{-\nu}(z)$ is equal to $z^{-\nu}$ times an entire function of z^2.

Therefore, again from (16) we get

$$(\pi/2)\ D(x,s^2) = -\log x . J_\nu(s)\ J_\nu(xs) + J_\nu(xs)\ H_{-\nu}(s) - J_\nu(s)\ H_{-\nu}(xs)\ ,$$

$$s.(\pi/2)\ \Delta(x,s^2) = J_\nu(xs)(J_\nu(s) - s \log x . J'_\nu(s) + s\ H'_{-\nu}(s)) - \\ - H_{-\nu}(xs). s\ J'_\nu(s).$$

Thus, for ν = integer (> 0) we get with f,g and h entire functions

$$(64)\quad \Phi(x,s^2) = \sqrt{x}\ [s^\nu \log x . J_\nu(xs)\ f(s^2) + s^{-\nu} J_\nu(xs)\ g(s^2) + \\ + s^\nu H_{-\nu}(xs)\ h(s^2)].$$

In consequence, the associated functions defined in (43), after taking into account (63) and (64), take the form

$$(65)\quad k!\ \mathcal{U}_k(x,s^2) = \left(\frac{\partial}{\partial s^2}\right)^k \Phi(x,s^2) = (\text{if } \nu \neq \text{integer}) =$$

$$= \sqrt{x} \sum_{j=0}^{k} f_j(s^2) \left(\frac{\partial}{\partial s^2}\right)^j (s^{-\nu}\ J_\nu(xs)) + g_j(s^2)\left(\frac{\partial}{\partial s^2}\right)^j (s^\nu\ J_{-\nu}(xs))$$

$$(66)\quad k!\ \mathcal{U}_k(x,s^2) = (\text{for } \nu = \text{integer}) = \sqrt{x} \sum_{j=0}^{k} \{(f_j(s^2) + g_j(s^2) \log x).$$

$$.\left(\frac{\partial}{\partial s^2}\right)^j (s^{-\nu}\ J_\nu(xs)) + h_j(s^2).\left(\frac{\partial}{\partial s^2}\right)^j (s^\nu\ H_{-\nu}(xs))\},$$

where with f_j, h_j and g_j we denote entire functions. Observe that $h_j(\xi) = 0$ if $-\nu + 2j \leqslant -1$ since $\mathcal{U}_k(x,\xi) \in L^2(0,1)$.

Next we recall some formulae from the theory of Bessel functions:

$$(67)\quad \begin{cases} \left(\frac{d}{z\,dz}\right)^m z^\nu\ J_\nu(z) = z^{\nu-m}\ J_{\nu-m}(z)\ , \\ \left(\frac{d}{z\,dz}\right)^m z^{-\nu}\ J_\nu(z) = (-1)^m\ z^{-\nu-m}\ J_{\nu+m}(z). \end{cases}$$

Let $\lambda = 0$ be a pole of $M(\lambda)$ of order r and $\nu \neq$ integer. In this situation (cf. (65)):

$$(68)\quad \mathcal{U}_k(x,0) = \sqrt{x} \sum_{j=0}^{k} \{A_j\ x^{\nu+2j} + B_j\ x^{-\nu+2j}\}\quad ,\quad k = 0,1,\dots,r-1.$$

$\mathcal{U}_k \in L^2$ implies that $B_j = 0$ for $2j - \nu \leqslant -1$. Let us see ii) when $\xi = 0$. It will be sufficient to prove (cf. Proposition 3) that

$$(69)\quad \int_0^1 x^{2j+\nu+1}\ J_\nu(xs_n)\ dx = O(s_n^{-3/2})\ ,$$

$$(70)\quad \int_0^1 x^{2j-\nu+1}\ J_\nu(xs_n)\ dx = O(s_n^{-\alpha-1/2})\ ,$$

α in (70) equals $2j_0 - \nu + 3/2$ if there is a j_0 verifying $1 > 2j_0 - \nu + 3/2 > 1/2$; $\alpha = 1$ if there is no j_0 with such a property.
In (69) and (70) the O's *are independent of* j *for fixed* ν.

In fact,

$$(71)\quad t^{2j\pm\nu+2}\int_0^1 x^{2j\pm\nu+1}\, J_\nu(xt)\, dx = \int_0^t y^{2j\pm\nu+1}\, J_\nu(y)\, dy =$$

$$= \text{(from (67), m=1)} = \pm\int_0^t y^{2j}(y^{\pm\nu+1}\, J_{\nu\pm1}(y))'\, dy =$$

$$= \pm\, y^{2j\pm\nu+1}.\, J_{\nu\pm1}(y)\Big|_0^t \mp 2j\int_0^t y^{2j\pm\nu}\, J_{\nu\pm1}(y)\, dy =$$

$$= t^{2j\pm\nu+1}.\, O(t^{-1/2}) + O(1) + 2j\int_1^t y^{2j\pm\nu-1/2}.O(1)\, dy =$$

$$= (t^{2j\pm\nu+1/2} \vee 1)\, O(1)$$

and this proves (69), and also (70), because in that case we must consider only j's such that $2j - \nu > -1$, (cf. (68)).

Let us assume next that $s^2 = \xi \neq 0$ is a pole of order r of $M(\lambda)$ and still $\nu \neq$ integer.

From (65) is obtained.

$$(72)\quad \mathcal{U}_k(x,s^2) = \sqrt{x}\sum_{j=0}^{k}\left[A_j\left(\frac{\partial}{\partial s^2}\right)^j (s^{-\nu}\, J_\nu(xs)) + B_j\left(\frac{\partial}{\partial s^2}\right)^j (s^\nu\, J_{-\nu}(xs))\right]$$

with $B_j = 0$ if $2j - \nu \leqslant -1$, and $k=0,1,\dots,r-1$. To prove ii) in this case it is sufficient to show that the following estimations hold:

$$(73)\quad \int_0^1 \left(\frac{\partial}{\partial s^2}\right)^j (s^{-\nu}\, J_\nu(xs))\, J_\nu(s_n x)\, x\, dx = O(s_n^{-3/2}) \; ,$$

$$(74)\quad \int_0^1 \left(\frac{\partial}{\partial s^2}\right)^j (s^{\nu}\, J_{-\nu}(xs))\, J_\nu(s_n x)\, x\, dx = O(s_n^{-\alpha-1/2}).$$

The last one must hold only for $2j - \nu + 1 > 0$. The left-hand sides of (73) and (74) are equal to

$$(75)\quad \left(\frac{\partial}{\partial s^2}\right)^j \int_0^1 (s^{\mp\nu}\, J_{\pm\nu}\,(xs))\, J_\nu(s_n x)\, x\, dx =$$

$$= \left(\frac{\partial}{\partial s^2}\right)^j \left\{\frac{J_{\pm\nu}(xs)\; J_\nu'(s_n x).s_n x - J_\nu(s_n x)\; J_{\pm\nu}'(xs).sx}{s^{\pm\nu}\;(s^2 - s_n^2)}\;\Bigg|_{x=0}^{x=1}\right\}$$

corresponding the upper sign to (73) and the lower one to (74). Then, left-hand side of (73) =

$$= \left(\frac{\partial}{\partial s^2}\right)^j \frac{J_\nu(s)\; J_\nu'(s_n).\; s_n - J_\nu(s_n)\; J_\nu'(s_n).\; s}{s^\nu\;(s^2 - s_n^2)} = O(1/s_n^{3/2}) \; ;$$

Left-hand side of (74) $= \left(\frac{\partial}{\partial s^2}\right)^j \Big[\dfrac{J_{-\nu}(s).s^\nu J_\nu'(s_n)s_n - J_\nu(s_n)J_{-\nu}'(s).s^{1+\nu}}{s^2 - s_n^2}$

$$- \frac{2\nu\ a_0\ b_0\ s_n^{\nu}}{s^2 - s_n^2}] = O(1/s_n^{3/2}) + O(s_n^{\nu-2j-2}) = O(1/s_n^{\alpha+1/2}) ,$$

where $a_0 = 2^{-\nu}/\Gamma(\nu+1)$, $b_0 = 2^{\nu}/\Gamma(1-\nu)$. The last equality follows from the fact that here $\nu < 2j+1$; as before, α is defined by $1 \wedge \{2j - \nu + 3/2 \ ; \ 2j - \nu + 3/2 > 1/2\}$. The proof of ii) is thus accomplished.

Let us see iii), i.e., assume ν = positive integer. If zero is a pole of order r of $M(\lambda)$, from (66) and for k=0,1,...,r-1, we obtain

$$(76) \quad u_k(x,0) = \sqrt{x}\ \{ \sum_{j=0}^{k} x^{\nu+2j}(A_j + B_j \log x) + \sum_{j=0}^{k} x^{-\nu+2j}\ C_j\} ,$$

and $C_j = 0$ if $-\nu+2j \leqslant -1$. The α defined above is now equal to one and (71) can be written as

$$\int_0^1 x^{\pm\nu+2j+1}\ J_\nu(xs_n)\ dx = O(s_n^{-3/2}) \quad \text{if } \pm\nu+2j+1 > 0.$$

Then if $\xi = 0$, iii) will be established if we show that

$$(77) \qquad \int_0^1 x^{\nu+2j+1} \log x\ .\ J_\nu(xs_n)\ dx = O(\frac{\log s_n}{s_n^{3/2}}) .$$

But, as in (71) after integrating by parts we see that

$$\int_0^1 x^{\nu+2j+1} \log x\ .\ J_\nu(xt)\ dx =$$

$$= - t^{-\nu-2j-2}\ [\int_0^t y^{\nu+1}\ J_{\nu+1}(y).y^{2j-1}(2j \log \frac{y}{t} - 1)\ dy] =$$

$$= t^{-\nu-2j-2}\ O(1).\int_0^t y^{\nu+2j-1/2}[2j(\log t - \log y)+1]\ dy = O(\frac{\log t}{t^{3/2}})$$

and this implies (77). Here again the O does not depend of j. Let us suppose, for the last step, that $\xi = s^2 \neq 0$ is a pole of order r of $M(\lambda)$. From (66) it follows

$$(78)\ u_k(x,s^2) = \sqrt{x}\ \{ \sum_{j=0}^{\infty} A_j\ x^{\nu+2j} + \sum_{j=0}^{\infty} B_j\ x^{\nu+2j}.\ \log x + \sum_{2j-\nu>-1}^{\infty} C_j x^{-\nu+2j}\},$$

where the series of coefficients A_j, B_j, and C_j are absolutely summable. Then, from the estimations (69), (70) and (77) (recall that they hold uniformly on j), (79) is obtained:

$$(79) \qquad \int_0^1 u_k(x,\xi)\ J_\nu(xs_n)\ \sqrt{x}\ dx = (\sum |A_j|)\ O(\frac{1}{s_n^{3/2}}) +$$

$$+ (\sum |B_j|)\ O(\frac{\log s_n}{s_n^{3/2}}) + (\sum |C_j|)\ O(\frac{1}{s_n^{\alpha+1/2}}) ,$$

and since in this situation $\alpha = 1$, iii) is proved . Q.E.D.

THEOREM 5. *The system $\{V_s\}$ defined at the end of section 6 verifies the hypothesis of Theorem 1.*

Proof. Lemma 5 proves i) of Th. 1; Propositions 4 and 5 and the Corollaries to theorem 4 prove iii) and iv). It remains only to check ii) and this will follow from the following proposition:

$$f \in L^2, \quad b_n(f) = (f,\hat{\psi}_n) \Rightarrow \exists\, K: \sum |b_n|^2 \leqslant K \, \|f\|_2^2 \ .$$

In fact, let σ_n be positive zeroes of $J_\nu(t)$ if $p > q$, and of $J_\nu'(t)$ if $p \leqslant q$. It is not difficult to see that if s_n is a pole of $M(s^2)$ then eventually after a renumeration of $\{\sigma_n\}$ it follows that: $|\sigma_n - s_n| \longrightarrow 0$ for $n \longrightarrow \infty$. One way of checking this is to prove, using the asymptotic formulae for J_ν and J_ν', that σ_n satisfies formula (26) if $p > q$, and (25) if $p \leqslant q$, but with different values for the constants C.

Define now $\varphi_n(x) = \delta \sqrt{x}\, J_\nu(\sigma_n x) . \|\sqrt{x}\, J_\nu(s_n x)\|_2^{-1}$ with $\delta = +1$ or -1 and such that $\hat{\psi}_n(x) = \delta \sqrt{x}\, J_\nu(s_n x) . \|\sqrt{x}\, J_\nu(s_n x)\|_2^{-1}$.

$\{\varphi_n(x)\}$ is an orthogonal family of L^2-functions, and $\|\varphi_n\|_2 \xrightarrow[n\to\infty]{} 1$, (this is a consequence of one of Lommel's formulae that asserts that

$$\int_0^1 x\, J_\nu^2(\alpha x)\, dx \sim k/\alpha \quad \text{if} \quad \alpha \longrightarrow \infty \ ).$$

Then, calling $\tilde{b}_n(f) = (f,\varphi_n)$ we have

(80) $$\sum |\tilde{b}_n(f)|^2 \leqslant K \sum |\tilde{b}_n(f)|^2/\|\varphi_n\|^4 \leqslant K.\|f\|^2.$$

ii) will follow from

(81) $$b_n(f) = \tilde{b}_n(f) + \|f\|\, O(1/n) \ , \qquad n > n_0 \ .$$

But, $|\hat{\psi}_n(x) - \varphi_n(x)| = \sqrt{x}\, |J_\nu(\sigma_n x) - J_\nu(s_n x)|/\|\sqrt{x}\, J_\nu(s_n x)\|_2 =$
$= \sqrt{x s_n}\, O(1).(\sigma_n - s_n)\, J_\nu'(\beta_n x) = \sqrt{s_n/\beta_n} \cdot (\sigma_n - s_n)\sqrt{x\beta_n}\, J_\nu'(x\beta_n) O(1)$
where β_n is a number between s_n and σ_n. Since $s_n/\sigma_n \longrightarrow 1$, it also holds that $\beta_n/s_n \longrightarrow 1$, and from the last formula we obtain

(82) $$|\hat{\psi}_n(x) - \varphi_n(x)| = |\sigma_n - s_n| . O(1) \ .$$

(82), together with next estimation, proves (81).

(83) $$|\sigma_n - s_n| = O(1/s_n) \ .$$

To establish this, first observe that from (34) it follows $J_\nu(s_n) = O(s_n^{-3/2})$ when $p > q$, and then

(84) $$O(1/s_n^{3/2}) = J_\nu(s_n) = J_\nu(s_n) - J_\nu(\sigma_n) = (s_n - \sigma_n)\, J_\nu'(t_n)$$

where t_n is a number between σ_n and s_n. On the other hand $|J_\nu(s_n)| > |J_\nu(t_n)|$ because of the monotony of J_ν between two zeroes of J'_ν. Therefore, $J_\nu(t_n) = O(s_n^{-3/2})$ and $|J'_\nu(t_n)| > C_0/\sqrt{s_n}$. After replacing in (84), (83) follows.

Assume next $p \leqslant q$. (34) implies (t_n as before) that

(85) $\quad M/s_n^{3/2} \geqslant |J'_\nu(s_n)| = |J'_\nu(s_n) - J'_\nu(\sigma_n)| = |\sigma_n - s_n| . |J''_\nu(t_n)|.$

From Bessel's equation: $J''_\nu(t_n) = -J_\nu(t_n) + O(1/t_n^{3/2})$, and therefore $|J''_\nu(t_n)| > C_1/\sqrt{s_n}$ holds. As in the preceding case (85) implies (83).

Q.E.D.

THEOREM 6. *Hypothesis b) of theorem 2 holds for the system* $\{V_s\}$ *under consideration. Therefore* $g > 0$.

Proof. From (59) we see that $\int_0^1 \hat{\psi}_n \hat{\psi}_m \, dx = 0 \Rightarrow V(s_n^2, s_m^2) = 0$.

Since P and Q have no root in common, for any value of s_n, $V(s_n^2, y)$ has only a finite number of roots. Then $\int_0^1 \hat{\psi}_n \hat{\psi}_m \, dx \neq 0$ for each s_n and infinitely many s_m.

Q.E.D.

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