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A NOTE ON HOLLOW MODULES

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Let R be a ring with identity and M a right unitary R-module. Let N be an R-submodule of M. N is called a *small submodule* in M if it satisfies the following condition: the fact that M = T + N for some R-submodule T implies T = M. If every proper submodule of M is small, we call M a *hollow module* [4]. P. Fleury studied some conditions under which the endomorphism ring of a hollow module is a local ring. We shall call M *completely indecomposable* when its endomorphism ring is local.

In § 1 we shall show that every finitely generated and uniform hollow module is completely indecomposable, when R is a left or right perfect ring. In § 2 we shall give some relations between injective hollow modules and QF-3 rings. In § 3 when R is a commutative Dedekind domain, we can completely determine all hollow modules and we know that they are completely indecomposable.

1. PERFECT RINGS.

Let R be a ring with identity and M a right unitary R-module. By J(M) we shall denote the Jacobson radical of M. Since every small submodule in M is contained in J(M), we have

LEMMA 1.1 ([4]). *M is a finitely generated hollow module if and only if* J(M) *is maximal and small in* M. *In this case* M *is cyclic.*

If M = mR, M ≈ R/A, where A is a right ideal in R. Since J(M) is always small in M whenever M is finitely generated, we have

COROLLARY 1.2 ([4]). R/A *is hollow if and only if* A *is contained in a unique maximal right ideal.*

It is clear that every hollow module is indecomposable and so a hollow module of finite length is always completely indecomposable.

THEOREM 1.3. *We assume that* J(R) *is nil and* R/J(R) *is artinian. Then every finitely generated and uniform*[2] *hollow module is completely indecomposable, where* J(R) *is the Jacobson radical of* R.

Proof. Let M be an R-module with the property above. Then M ≈ R/A for some right ideal A from Lemma 1.1. If $A \subseteq J(R)$, then R is a local ring from Corollary 1.2. It is well known that $\mathrm{End}_R(R/A) = I(A)/A$; $I(A) = \{x \in R \mid xA \subseteq A\}$. Let $\bar{x}_1, \bar{x}_2$ be non-epimorphic elements in $I(A)/A$. Then $x_iR \subseteq J(R)$. Hence, $\bar{x}_1+\bar{x}_2$ is not epimorphic. Let $\bar{y}$ be an epimorphism with $\bar{1} = \bar{x}_1+\bar{y}$. Since $x_1 \in I(A) \cap J(R)$, $(1-x_1)^{-1} = 1+x_1+\ldots+x_1^{n-1} \in$ $\in I(A)$, where $x_1^n = 0$. Hence, $\bar{y}$ is isomorphic. Let $\bar{y}_1$ and $\bar{y}_2$ be epimorphic but not isomorphic , then $y_1^{-1}(0) \cap y_2^{-1}(0) \neq (0)$ from the assumptions. Hence, $\bar{y}_1+\bar{y}_2$ is not isomorphic. Therefore, $I(A)/A$ is a local ring. Next, we assume $A \not\subseteq J(R)$. Since $R/(A+J(R))$ is semi-simple and hollow module, $A+J(R)$ is a maximal right ideal in R. Let $R/J(R) = (A+J(R))/J(R) \oplus \bar{B}$, where $\bar{B}$ is a minimal right ideal in $\bar{R} = R/J(R)$. Since $\bar{R}$ is semi-simple and artinian and $J(R)$ is nil, there exist idempotents e and f such that $e \in A$, $\bar{B} = \overline{fR}$ and $R = eR \oplus fR$. Hence, $R/A \approx fR/(fR \cap A) = fR/fC$, where $C = fR \cap A$. Then $\mathrm{End}_R(R/A) \approx \mathrm{End}_R(fR/fC) = I'(fC)/fCf$; $I'(fC) =$ $= \{x \in fRf \mid xfC \subseteq fC\}$. Now, fR/fC contains a unique maximal submodule $fJ(R)/fC$. Hence, we can prove, similarly to the first part, that $\mathrm{End}_R(fR/fC)$ is local.

COROLLARY 1.4. *If R is a left or right perfect ring*[3]*, then every finitely generated and uniform hollow module is completely indecomposable.*

2. QF-3 RINGS.

Let R be a commutative ring. If Krull dimension of R is equal to zero, R is never small in any ring extension [9]. We shall study a similar situation on R-modules. First we take any ring R, which is not necessarily commutative.

PROPOSITION 2.1. *Let M be an R-module. Then the following conditions are equivalent :*

1) *M is not a small submodule in any extension module M' of M.*

2) *M is not small in an injective hull $E(M)$ of M.*

3) *There exists an injective module E containing M such that M is not small in E.*

Proof. 1) $\rightarrow$ 2) $\leftrightarrow$ 3) are clear. 2) $\rightarrow$ 1). We assume $M' \supseteq M$. Then $E(M') = E(M) \oplus E_1$. Hence, M is not small in $E(M')$. Therefore, M is not small in M'.

If M satisfies one of three equivalent conditions in Proposition 2.1, we say M is *non-small in injectives*. It is well known that any non-zero submodule is not small in M if and only if $J(M) = (0)$.

Hence, we have

PROPOSITION 2.2. *The following conditions are equivalent*:

1) *Any non-zero module is non-small in injectives.*

2) R *is a right* V-*ring.*

Proof. See [2], p. 356.

We note that if M is non-small in injectives, then so is any module extension M' of M.

LEMMA 2.3. *Let* $M \supsetneqq M_1$ *be* R-*modules. If* M/M_1 *is non-small in injectives, then so is* M.

Proof. It is clear from the definitions and the above remark.

PROPOSITION 2.4. *A ring* R *is small in* E(R) *if and only if* E = J(E) *for any injective module* E.

Proof. If E ≠ J(E) for some injective module E, there exists a homomorphism f of R to E such that $f(R) \not\subseteq J(E)$. Hence, R is not small in E(R) by Proposition 2.1 and Lemma 2.3. Next, we assume R is not small in E(R). Then there exists a submodule T ≠ E(R) such that E(R) = R+T. Hence, E(R)/T contains a maximal submodule.

COROLLARY 2.5. *If* R *is a perfect ring,* R *is non-small in injectives as an* R-*module. If* R *is a commutative domain,* E = J(E) *for any injective module* E.

Proof. It is clear from [1], Lemma 2.6 and [8], Theorem 2.

From now on in this section, we assume R is a right perfect ring. Then there exists a complete set $\{g_i\}$ of mutually orthogonal primitive idempotents such that $1 = \sum g_i$. We shall divide $\{g_i\}$ into two parts: $\{g_i\} = \{e_i\}_{i=1}^{n} \cup \{f_j\}_{j=1}^{m}$, where the e_iR is non-small in injectives and the f_jR is small in $E(f_jR)$. We know $n \geqslant 1$ by Corollary 2.5. If we denote primitive idempotents by e and f, respectively, we mean e belongs to the first class and f does to the second.

Next, we shall consider two conditions

(*) Every non-small module in injectives contains a non-zero injective module.

and

(**) Every indecomposable injective module is hollow, namely contains a unique maximal submodule.

Let K be a field and R a K-algebra of finite dimension. Then $\mathrm{Hom}_K(\ ,K)$ is a dual functor and so the condition (**) is dual to $(**)_l$ (resp. $(**)_r$). Every indecomposable, projective left (resp. right) module contains a unique minimal submodule.

LEMMA 2.6. *Let* R *be a right perfect ring. Then* (*) *holds if and only if every indecomposable, non-small module in injectives is injective.* (**) *holds if and only if every indecomposable, injective module is of the form* e_iR/e_iA, *where* A *is a right ideal.* (*) *implies* (**).

Proof. We assume (*) and M is indecomposable, non-small in injectives. Then M is injective and hollow. Hence, (**) holds. Since $M \neq J(M)$, $M/J(M) \approx g_iR/g_iJ(R)$ by Lemma 1.1. Therefore, some e_iR is a projective cover of M by Lemma 2.3. Conversely, let M be non-small in injectives and $E = E(M)$. Since $M \not\subseteq J(E)$, we have m in $M-J(E)$. Then mR is non-small in injectives by Proposition 2.1. Since $mR/mJ(R)$ is of finite length, mR contains an indecomposable and non-small module in injectives. Hence (*) holds.

LEMMA 2.7. *Let* R *be as above. If* M *is a non-small submodule in* $\sum \oplus g_iR/g_iA_i$, *then there exists* π_i *such that* $\pi_i(M) = g_iR/g_iA_i$, *where the* A_i *is a right ideal and* π_i *is the projection on* g_iR/g_iA_i.

Proof. Since $M \not\subseteq \sum \oplus g_iJ(R)/g_iA_i$, $\pi_i(M) \not\subseteq g_iJ(R)/g_iA_i$ for some i. Hence, $\pi_i(M) = g_iR/g_iA_i$, since g_iR is hollow.

PROPOSITION 2.8. *Let* R *be a right artinian ring. Then* R *is a* QF-*ring if and only if* (*) *holds and* $e_iRf_j = (0)$ *for all* i *and* j.

Proof. Let R be a QF-ring and M non-small in injectives. Let $E = E(M)$. Then $E \approx \sum \oplus e_{i_j}R$ by [3]. Since M is not small in E, M contains a direct summand isomorphic to e_jR by Lemma 2.7. Since $f_j = 0$ for all j, $e_iRf_j = (0)$. Conversely, we assume (*). Then the e_iR is injective by Lemma 2.6. If $f_j \neq 0$, $E(f_jR) \approx \sum \oplus e_{i_k}R/e_{i_k}A_k$. Hence, $(0) \neq f_jRf_j$ implies $e_kRf_j \neq (0)$ for some k, which is a contradiction to the assumption.

LEMMA 2.9. *Let* R *be as above. If* (**) *holds, every* f_jR *is isomorphically contained in a direct sum* $\sum \oplus e_{i_k}R$ *and there exists a right ideal* A *such that* e_iR/e_iA *is non-zero injective for each* i.

Proof. Let $E = E(fR)$. Then $E = \sum_{k=1}^{t} \oplus e_{i_k}R/e_{i_k}A_k$ by Lemma 2.6.

Let φ: $fR \longrightarrow E$ be the inclusion and $\varphi(f) = \sum_1^t (e_{i_k}r_kf + e_{i_k}A_k)$.

We define ψ: $fR \longrightarrow \sum_1^t \oplus e_{i_k}R$ by setting $\psi(fx) = \sum_1^t e_{i_k}r_kfx$. It is clear that ψ is monomorphic. Let $F = E(eR)$ and $F = \sum \oplus e_{i_s}R/e_{i_s}A_s$ as above. Since eR is not small in F, eR is epimorphic to some $e_{i_s}R/e_{i_s}A_s$. Hence, $eR \approx e_{i_s}R$.

PROPOSITION 2.10. *Let* R *be right artinian. Then* R *is right* QF-3 *if either* (*) *holds or* (**) *holds and each* e_iR *contains a unique minimal submodule.*

Proof. If (*) holds, each eR is injective. Hence, R is right QF-3 by Lemma 2.9 and [10]. In the second case $E(eR) = e_iR/e_iA$ and $eR \approx e_iR$ from the proof of Lemma 2.9. Hence, $e_iA = (0)$.

COROLLARY 2.11. *Let* R *be a* K-*algebra of finite dimension over a field* K. *If* $(**)_l$ *and* $(**)_r$ *hold,* R *is* QF-3.

The examples below show that the converse is not true. Now, we shall study QF-3 rings satisfying (*) or (**).

THEOREM 2.12. *Let* R *be right artinian. When either* R *is hereditary or* $J(R)^2 = (0)$, *the following conditions are equivalent:*

1) (*) *holds.*

2) (**) *holds and each* e_iR *contains a unique minimal submodule.*

3) R *is a right* QF-3 *ring.*

Proof. 1) $\to$ 2) $\to$ 3) are clear. 3) $\to$ 1). First, we assume that R is hereditary. We may assume R is basic and two-sided indecomposable. Then R is a ring of upper tri-angular matrices over a division ring by [6], Theorem 2. Hence, only one e_1R is injective and $f_jR/f_jJ(R)$ is isomorphic to submodule of e_1R/e_1A. Therefore, every injective module is isomorphic to a direct sum of some e_1R/e_1A_i, where the A_i is a right ideal. Let M be non-small in injectives. Then we have an epimorphism $f: M \longrightarrow e_1R/e_1A_i$ from Lemma 2.7. Hence, we have $h: e_1R \longrightarrow M$ such that $fh \neq 0$. Since R is hereditary, M contains an injective module. Next, we assume $J(R)^2 = (0)$ Since R is right QF-3, some e_iR is injective. Let $\{e_iR\}_1^t$ be the set of such an injective right ideal. We assume $t < n$. Then e_nR is non-small in an injective module $\sum \oplus e_pR$; $e_pR \in \{e_iR\}_1^t$ by [10]. Hence, e_nR is isomorphic to some e_iR from Lemma 2.7, which is a contradiction. Since $f_iR \subseteq \sum \oplus e_{i_q} J(R)$, f_iR is simple. Hence, f_iR is monomorphic to some e_kR. We assume $e_1R/e_1J(R)$ is not injective. Then $E = E(e_1R/e_1J(R))$ is indecomposable. Take $a \in E - J(E)$. Since $a = \sum ag_i$, $ag_i \notin J(E)$ for some i. Hence, we may assume $a \in Ee_k - J(E)$ by Lemma 2.3. Then we have either $aR \approx e_kR$ or $e_kR/e_kJ(R)$. Since $a \notin J(E) \supseteq$ $\supseteq e_1R/e_1J(R)$, $aR \approx e_kR$ is injective. Hence $E \approx e_kR$. Thus we have proved that any indecomposable injective module is isomorphic either to some e_iR or $e_jR/e_jJ(R)$. Let M be indecomposable, non-small in injectives and E its injective hull. Let S(M) be the socle of M. Then

$E = E(S(M))$ and $S(E) = S(M)$. Let $E = \sum_{\ell} \oplus e_{i_\ell} R \oplus \sum_{k} \oplus e_{i_k} R/e_{i_k} J(R)$.
Since $\sum \oplus e_{i_k} R/e_{i_k} J(R) \subseteq S(E) \subseteq M$, $M = \sum \oplus e_{i_k} R/e_{i_k} J(R) \oplus M \cap (\sum \oplus e_{i_\ell} R)$.
If $K \neq \emptyset$, $M \approx e_k R/e_k J(R)$. If $K = \emptyset$, M is not small in $\sum \oplus e_{i_\ell} R$.
Hence, $M \approx e_i R$ by Lemma2.7. Therefore, (*) holds.

EXAMPLES. 1) Let K be a field, M a K-vector space of finite dimension and $M^* = \mathrm{Hom}_K(M,K)$. We put

$$R = \begin{pmatrix} K & M^* & K \\ & K & M \\ 0 & & K \end{pmatrix}$$

Then R is a QF-3 ring by the natural multiplication $M^* \otimes_K M \longrightarrow K$ (see [7]). If $[M:K] \geqslant 2$, (**) does not hold, since Re_{22} contains two minimal submodules.

2) Put

$$R = \begin{pmatrix} K & K & K & K \\ & K & 0 & 0 \\ & & K & 0 \\ 0 & & & K \end{pmatrix}$$

Then (**) holds but R is not QF-3.

3) Let S be the ring of upper triangular matrices over K with degree n and R a K-subalgebra of S containing $\{e_{ii}\}_{i=1}^n$. We assume R is a two-sided indecomposable ring.

Then R is QF-3 if and only if (**) holds and $e_{11} R$ contains a unique minimal submodule. R is QF-3 and hereditary if and only if (*) holds.

Proof. First, we assume $e_{11}R$ is injective. Then we shall show that $e_{ii}R$ is not injective for all $i \geqslant 2$. Let $\{e_{i_t i_t}R; e_{i_1 i_1}R = e_{11}R\}_{t=1}^s$ be the set of such an injective right ideal. We note if $e_{kk}Re_{ii} \neq (0)$, $e_{ii}R$ is monomorphic to $e_{kk}R$. Hence, since $e_{11}R$ is indecomposable, $e_{11}Re_{i_t i_t} = (0)$ for $t \geqslant 2$. Let $e_{11}Re_{pp} \neq (0)$ and $e_{i_t i_t}Re_{qq} \neq (0)$ for $t \geqslant 2$. Then $e_{pp}Re_{qq} = e_{qq}Re_{pp} = (0)$, because if $e_{pp}Re_{qq} \neq (0)$, $e_{qq}R$ is monomorphic to $e_{11} R$ and so $e_{11}Re_{i_t i_t} \neq (0)$, since $e_{11}R$ is injective. Therefore, R is a direct sum of two ideals A_i such that $A_1 = \sum \oplus e_{pp}R$; $e_{11}Re_{pp} \neq (0)$ and $A_2 = \sum \oplus e_{qq}R$; $e_{i_t i_t}Re_{qq} \neq (0)$ for some $t \geqslant 2$. Since R is indecomposable, $s = 1$. We assume R is QF-3. Then $e_{11}R$ is only one injective ideal among $e_{ii}R$. Hence, $e_{11}Re_{ii} = e_{ii}Re_{nn} = K$ for all i.
We shall show $E(e_{ii}R/e_{ii}J(R))$ is isomorphic to $e_{11}R/e_{11}A$ for some right

ideal A. We assume $e_{j_1j_1}Re_{ii} = e_{j_2j_2}Re_{ii} = \ldots = e_{j_tj_t}Re_{ii} = (0)$ and $e_{j_{t+1}j_{t+1}}Re_{ii} = \ldots = e_{j_ij_i}Re_{ii} = K$, where $\{j_1 < j_2 < \ldots < j_t,\ 1 = j_{t+1} < \ldots < j_i = i\} \equiv \{1,2,\ldots,i\}$.
Put $e_{11}A = e_{1j_1}K + e_{1j_2}K + \ldots + e_{1j_t}K + e_{1i+1}R$. Then $e_{11}A$ is a right ideal and $e_{11}R/e_{11}A \approx e_{1j_{t+1}}K + e_{ij_{t+2}}K + \ldots + e_{1i}K + e_{1i+1}R/e_{1i+1}R$. Hence, $e_{11}R/e_{11}A \approx \mathrm{Hom}_K(Re_{ii},K)$ is injective and $e_{11}R/e_{11}A = E(e_{ii}R/e_{ii}J(R))$.

Therefore, (**) holds. The converse is clear from the first part and Proposition 2.10. If R is QF-3 and hereditary, (*) holds by Theorem 2.12. We assume (*) holds. Then R is QF-3 by Proposition 2.10. Let $E = E(e_{11}R/e_{1i}R)$ and $E \approx \sum_k \oplus\, e_{11}R/e_{11}A_k$. If $e_{11}R/e_{1i}R$ is small in E, $e_{11}R/e_{1i}R \subseteq \sum_k \oplus\, e_{12}R/e_{11}A_k$. However, $e_{11}Re_{11} \not\subseteq e_{1i}R$ and so $e_{11}R/e_{1i}R$ is non-small in injectives. Hence, $e_{11}R/e_{1i}R$ is injective by (*). Since $\mathrm{Hom}_K(e_{11}R/e_{1i}R,K)$ is projective and isomorphic to Re_{i-1i-1}, $e_{ii}Re_{jj} = K$ for all $i \leqslant j$. Therefore, R is hereditary by [6], Theorem 2.

Concerning with Example 3, we have

PROPOSITION 2.13. *Let* R *be right artinian and right* QF-3. *Then* R *is hereditary if and only if* e_iR/e_iA *is injective for all* i *and any right ideal* A.

Proof. Since R is QF-3, $\{e_iR\}_1^n$ is a complete set of indecomposable, injective right ideals (see the first part in the proof of Theorem 2.12). Hence, "only if" part is clear. Conversely, we assume e_iR/e_iA is injective for each i and A. Let E be an injective module and $a \in E\text{-}J(E)$. Then ae_kR is injective for some k from the assumption and Lemma 2.3. Hence, R satisfies (**). We shall show E/M is injective for any submodule M. Let S(M) be the socle of M. We define Loewy series $S^i(M)$ as follows: $S^i(M)/S^{i-1}(M) = S(M/S^{i-1}(M))$. We show the above fact by induction on $S^i(M)$. Let $E = E(M) \oplus E_1$ and $E_2 = E(M) = \sum \oplus\, e_{i_j}R/e_{i_j}A_j$. Since $S(M) = S(E_2)$, $E_2/S(E_2) \supseteq M/S(M)$ and $E_2/S(E_2)$ is injective from the assumption. Hence, if $M = S(M)$, E/M is injective. We assume E'/N' is injective for $E' \supseteq N'$ whenever E' is injective and $S^i(N') = N'$. Let $M = S^{i+1}(M)$. Then $E/S(M)$ is injective and $S^i(M/S(M)) = M/S(M)$. Hence, $E/M \approx (E/S(M))/(M/S(M))$ is injective by the induction.

3. MINIMAL NON-SMALL MODULES.

Since any extension of a non-small module in injectives is always non-small in injectives, we are interested in a minimal one among non-small modules in injectives.

PROPOSITION 3.1. *If* M *is minimal one among non-small modules in injectives, then* M *is a maximal one among hollow modules.*

Proof. Let $E = E(M)$. Then a proper submodule M_1 of M is small in E from Proposition 2.1. If $M = M_1 + M_2$ and $M_2 \neq M$, M is small in E from the above, which is a contradiction. It is clear that M is maximal one among hollow modules.

We do not know whether the converse of Proposition 3.1 is true. We shall show an affirmative answer when R is a commutative Dedekind domain.

From now on, we assume that R is a commutative domain.

PROPOSITION 3.2. *Let* M *be a torsion-free and maximal hollow module. Then* M *is isomorphic to the quotient field* Q *of* R.

Proof. We may assume $M \subseteq M \otimes_R Q$. Let $t \neq 0$ be in R. Then $t^{-1}M$ ($\subseteq M \otimes Q$) is also a hollow module containing M. Hence, $M = t^{-1}M$ and M is injective and indecomposable. Therefore, $M \approx Q$.

THEOREM 3.3. *Let* R *be a Dedekind domain. Then a hollow module is isomorphic to one of the following:*

1) R/p^n, 2) $E(R/p)$, *where* p *is a prime ideal and* 3) R *or* Q *when* R *is local. In this case every hollow module is completely indecomposable.*

Proof. Let M be a hollow module. If M is not torsion-free, M contains a direct summand isomorphic to either $E(E/p)$ of R/p^n by [11], Theorem 9. Hence, M is isomorphic to one of them. We assume M is torsion-free. Then $E(M) = \sum_I \oplus u_iQ$ by [11], Theorem 7. We put $M \cap u_iQ = u_iM_i \neq (0)$. If $M = \sum_I \oplus u_iM_i$, I consists of one element and we may assume $R \subseteq M \subseteq Q$. Let p and q be prime ideals in R. Since M/pq is a torsion hollow module, p = q by the above argument. Hence, R is local and $M \approx R$ or $M \approx Q$. If $M \neq \sum_I \oplus u_iM_i$, $\sum_I \oplus u_iM_i$ is a small submodule in M. Since $M/\sum_I \oplus u_iM_i$ is torsion and hollow, $M/\sum \oplus u_iM_i$ is isomorphic to $E(R/p)$ or R/p^n. When $M/\sum \oplus u_iM_i \approx E(R/p)$, $M = aM + \sum \oplus u_iM_i$ for any $a \neq 0$ in R. Hence, M is injective and so $M \approx Q$. When $M/\sum \oplus u_iM_i \approx R/p^n$, $b^nM \subseteq \sum_I \oplus u_iM_i$ for $b \neq 0 \in p$. If $u_iM_i = u_iQ \subseteq M$ for some i, $M \approx Q$. Hence, we may assume $u_iM_i \neq u_iQ$ for all i. Now, let π_1 be the projection of M to u_1Q, then $\pi_1(M)$ is a non-zero hollow module in u_1Q. Hence, R is local from the above. Accordingly, every u_iM_i is projective and so M is projective. Therefore, $M \approx R$.

From Theorem 3.2 and Proposition 3.1 we have

THEOREM 3.3. *Let* R *be a Dedekind domain. Then the following conditions are equivalent for an* R*-module* M.

1) M *is a minimal one among non-small modules in injectives.*

2) M *is a maximal one among hollow modules.*

3) M *is isomorphic to* E(R/p) *or to* Q *if* R *is local, where* p *is a prime ideal and* Q *is the quotient field of* R.

REMARK. Let R be a Dedekind domain which is not local. Then Q is not small in injectives, however Q does not contain a minimal non-small module in injectives.

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