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INSCRIBING SIMPLICES IN CONVEX BODIES

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1. INTRODUCTION. It is a well known fact that in a plane convex body it is possible to inscribe equilateral triangles in every position provided the boundary of the convex body is smooth and rotund. The aim of this paper is to explore the possibilities of generalization of this fact, regarding the space dimension and the regularity of the inscribed simplex. This problem can also be formulated as follows: Is it possible to find equidistant (k+1)-pointed sets on the boundary of a k-dimensional convex body?. This question seems to be connected with the construction of equilateral sets in different metric spaces. See for example Blumenthal [1], Blumenthal & Kelly [2], Haantjes [3].

A convex body is a convex compact set with nonempty interior. A convex set K is smooth if for every x in bdry(K) there exists a single support hyperplane intersecting K at x. K is rotund if every support hyperplane meets K in a single point.

A k-simplex is the convex hull of k+1 affinely independent points. A regular k-simplex is one having all its edges of equal length. A k-simplex is isosceles if one its facets is a regular (k-1)-simplex and the remaining vertex is equidistant from all the other vertices. The regular facet of such simplex will be called its base and the remaining vertex its main vertex. A k-simplex T is inscribed in the convex body K if every vertex of T belongsto bdry(K).

Let H_d be the group of transformations defined by the formula:

$$h(x) = \lambda I_d x + b$$

where I_d is the identity matrix, λ is a nonnegative real number and b is a vector of E^d. Two d-simplices T and T' are similarly placed if there is a h in H_d such that $T' = h(T)$.

We define the measure of the dihedral angle limited by the hyperplanes H_1 and H_2 as the measure of the plane angle formed by the inner normals to H_1 and H_2.

2. INSCRIPTION OF REGULAR SIMPLICES: TWO APPROACHES.

In the first theorem of this section we attack the problem of inscribing a regular simplex in a fixed position into a convex body. The second theorem searches the possibility of inscription of a regular simplex having a vertex in a given boundary point of the convex set. The question about the uniqueness of the inscribed simplex is considered in both cases.

THEOREM 2.1. *Let* K *be a rotund and smooth convex body in* E^d, *and* T *a regular* d-*simplex. There exists a regular* d-*simplex* T' *inscribed in* K *and similarly placed with* T.

Proof. The proof is by induction on d. For d=1 the theorem is trivial since all the 1-simplices are similarly placed and every 1-dimensional convex body is itself a 1-simplex. Let K be a rotund and smooth k-dimensional convex body and T a regular k-simplex. Select a facet F of T, and let v be the vertex of T opposite to F. Denote by H_0 and H_1 the support hyperplanes of K parallell to F, being H_0 the first one encountered when travelling in the direction from F to v. Define $H_\alpha = \alpha H_1 + (1-\alpha)H_0$ for $0 < \alpha < 1$. Clearly H_α intersects K and the set $K_\alpha = K \cap H_\alpha$ is a rotund and smooth (k-1)-dimensional convex body. By the inductive hipothesis there exists a regular (k-1)-simplex F_α similarly placed with F and inscribed in K_α. Denote by z_α the centroid of F_α, by R_α the ray issuing from z_α, perpendicular to H_α and pointing towards H_1, and P_α the unique point of intersection of R_α with bdry(K). Finally define: $T_\alpha = \text{conv}(F_\alpha \cup \{p_\alpha\})$. T_α is an isosceles k-simplex inscribed in K and having base F_α and main vertex p_α. Denote by $a(\alpha)$ the angle formed by a pair of nonbasic edges of T_α. Clearly, the length of a basic edge of T_α is not greater than the diamater of K_α. Hence by the rotundity condition, for α close to 0 that edge has length close to 0. On the contrary, for α close to 0, a non basic edge would have length close to the width of K in the direction of R_α. As a conclusion, $\lim a(\alpha) = 0$ for α tending to 0. On the other hand, for α tending to 1: the height of T_α decreases faster than the length of a basic edge, owing to the smoothness of K. Hence T_α approaches a degenerate isosceles k-simplex, i.e. a regular (k-1)-simplex with segments joining the vertices with the centroid; and $a(\alpha)$ becomes greater than $\pi/2$. The F_α 's can be chosen in such a way that for $\alpha_n \to \alpha$ F_{α_n} converge to F_α. Hence $a(\alpha)$ results a continuous function, and by Bolzano's theorem there exists α_0 such that $a(\alpha_0) = \pi/3$. But then the basic and the nonbasic edges of T_{α_0} will have equal lenghth, and T_{α_0} will be the regular k-simplex we seek.

The following lemma will allow us to construct a counterexample to the uniqueness of the inscribed simplex.

LEMMA. *Let* P *be a convex polytope. There exists a smooth and rotund convex body* K *such that* P *is inscribed in* K.

Proof. Let ext (P) = $\{p_1;p_2;\ldots;p_k\}$ be the set of vertices of P. For each p_i let H_i be a supporting hyperplane of P meeting P only at p_i.

Let c_i be a point on the same side of H_i as P, and such that: (i) the segment $[c_i , p_i]$ is orthogonal to H_i, and (ii) the distance $d_i = d(c_i , p_i)$ is greater than the distance from c_i to the remaining vertices of P. Let B_i be the ball with center c_i and radius d_i, and consider the set $K_0 = \cap B_i$. K_0 is rotund and P is inscribed in it. Furthermore the set of points of non-smoothness of bdry(K_0) is closed and disjoint with P. Hence, there is $\xi > 0$ such that every point of non-smoothness is at distance greater than ξ from P.

For each $\delta > 0$ let $K'_\delta = \{x \in K_0 \ / \ d(x,CK_0) \geq \delta\}$ and $K_\delta = \{t/d(t,K'_\delta) \leq \delta\}$. The sets K_δ are smooth and rotund, they are included in K_0 and converge to it as δ tends to 0. Moreover, these sets modify K_0 only in the neigbourhood of points of non-smoothness. Since we have seen that such points are at a positive distance from P, there exists $\delta > 0$ such that $K_\delta \supset P$. Clearly such K_δ verifies the thesis.

Counterexample to the uniqueness.

Let T be a regular tetrahedron in E^3 and let p and q be the midpoints of two non-adjacent edges of T. Let T' be a translated of T in the direction $\overrightarrow{pq}$ and define $P = \text{conv}(T \cup T')$. P is a polytope having 8 vertices. Let K be the smooth and rotund convex body furnished by the previous lemma. Then T and T' are similarly placed and inscribed in K.

THEOREM 2.2. *Let* K *be a rotund and smooth convex body in* E^d $(d \geq 2)$, *and let* $p \in$ bdry K. *There exists a regular* d-*simplex inscribed in* K *and having* p *as a vertex. Furthermore, if* $d > 2$, *there are infinitely many of such simplices for each* p.

Proof. Our proof is by induction on the dimension. For d=2 the theorem is a well known exercise on plane convexity (See, for instance, Yaglom & Boltyansky [4]). Assume that $d \geq 3$. Denote by H the (unique) support hyperplane of K at p, and by V a (d-2)-flat included in H and containing p. For each α such that $0 < \alpha < \pi$ let H_α be a hyperplane intersecting H in V and forming with H a dihedral angle α, measured from H to H_α in a certain fixed sense. Define $S_\alpha = H_\alpha \cap K$. By the smoothness of K, for each α, S_α is a (d-1)-dimensional convex body. Hence, by the inductive hypothesis, there exists a regular (d-1)-simplex F_α inscribed in S_α and having p as a vertex. Denote by R_α the ray issuing from the centroid of F_α, normal to H_α and in the sense of growth of α, and let p_α be the only point of bdry K contained in R_α. Clearly the set $T_\alpha = \text{conv}(F_\alpha \cup \{p_\alpha\})$ is an isosceles d-simplex inscribed in K and having base F_α and main vertex p_α. Define $f_1(\alpha)$ as the length of an edge of F_α, and $f_2(\alpha)$ as the length of a non-basic edge of T_α. Finally denote $f(\alpha) = f_1(\alpha) - f_2(\alpha)$. For α close to 0 $f(\alpha)$ is negative since $f_2(\alpha)$ is close to the width of K in the direction orthogonal to H, and $f_1(\alpha)$ tends to 0 by the rotundity of K. On the other hand, for α tending to π, all the non-basic facets of T_α

approach the same hyperplane (namely H) by the smoothness assumption. Hence the distance from p_α to F_α tends to 0 faster than $f_1(\alpha)$ and $f(\alpha)$ becomes positive. But $f(\alpha)$ is a continuous function of α, hence, by Bolzano's theorem, there exists an α_0 such that $f(\alpha_0) = 0$. Clearly, $T = T_{\alpha_0}$ is a regular d-simplex verifying the thesis.
Furthermore, the selection of the flat V assures us that we can fix arbitrarily the intersection of H with the hyperplane containing one of the facets of T incident on p. Since a d-simplex has exactly d facets incident on a vertex, and if $d > 2$, there are infinitely many such flats, the second part of the thesis follows.

3. INSCRIPTION OF ARBITRARY SIMPLICES.

In this section we intend to generalize theorem 2.1. to arbitrary simplices.

THEOREM 3.1. *Let* K *be a rotund and smooth convex body in* E^d *and* T *an arbitrary* d-*simplex. There exists a* d-*simplex* T' *inscribed in* K *and similarly placed with* T.

Proof. There exists a non-singular affine transformation $A: E^d \longrightarrow E^d$ such that $T_1 = A(T)$ is a regular d-simplex. Denote by $K_1 = A(K)$. It is easy to verify that K_1 is a rotund and smooth convex body. By theorem 2.1 there is a regular d-simplex T_2 inscribed in K_1 and similarly placed with T_1. Let $T' = A^{-1}(T_2)$. Clearly, this is a d-simplex inscribed in K. Furthermore, owing to the normality of the group H_d as a subgroup of the affine group, T' and T are similarly placed.

Unfortunately, the same method will not yield a generalization of theorem 2.2. This is due to the non-normality of the subgroup of similarities in the affine group.

4. CONCLUDING REMARKS.

Some open questions related to our results are:

1. Is it possible to prove analogous to theorems 2.1 or 2.2 for sets of constant width?.

2. What is the largest family of convex bodies for which this results remain valid?.

3. Does the fact that all the inscribed regular simplices have the same edgelength characterize the n-ball?.

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