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A REMARK ON TERRIBLE POINTS

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In the study of curves in a projective plane over a field of characteristic zero, it is well known that, given any point P in the plane, there are only a finite number of tangents to the curve through P.

This result is not true in the case of fields of characteristic $p \neq 0$ (see, for instance [2], Appendix). For instance, there is an infinite number of tangents to the curve defined by $x^{p+1}-y^pz = 0$ through the point (0,1,0). These are the so called terrible points.

The object of this note is to extend this concept to higher dimensional varieties and to give a computational method to determine the terrible points.

In all this note p will be the characteristic of the field k and $Fx = x^p$ the Frobenius morphism.

1. DIEUDONNÉ DERIVATIVES.

DEFINITION 1.1. If A is a commutative K-algebra with unit, K a commutative ring with unit, and M is an A-module, then a K-linear map $\delta: A \longrightarrow M$ is called an n^{th}-order derivation if it satisfies

$$\delta(x_0,\dots,x_n) = \sum_{i=1}^{n} (-1)^{i+1}[\sum_{j_1 \dots j_i} x_{j_1} \dots x_{j_i} \delta(x_0,\dots,\hat{x}_{j_1},\dots,\hat{x}_{j_i},\dots,x_n)]$$

and $\delta(1) = 0$.

Clearly, a 1-derivation is a standard derivation.

NOTATION. Let A, K as before and let $\mu: A \otimes_k A \longrightarrow A$ denote the multiplication map (i.e., $\mu(x \otimes y) = xy$). We will set $I = \mathrm{Ker}\, \mu$, and let $T: A \longrightarrow I$ denote the K-linear map defined by $T(x) = 1 \otimes x - x \otimes 1$.

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The following result is trivial.

LEMMA 1.2. *If* $\delta: A \longrightarrow M$ *is an* n^{th}*-order derivation and if* $\gamma: M \longrightarrow N$ *is an* A*-linear map, then the composite* $\gamma\delta: A \longrightarrow N$ *is an* n^{th}*-order derivation.*

DEFINITION 1.3. The map $A \xrightarrow{\varepsilon} A \otimes A$ given by $\varepsilon(x) = x \otimes 1$ gives an A-module structure to I. Call $D_n(A/k)$ the A-module I/I^{n+1}, which will be called the module of n^{th}-order differentials. Call $d_n: A \longrightarrow D_n(A/k)$ the composite of $T: A \longrightarrow I$ with the canonical projection $I \longrightarrow I/I^{n+1} = D_n(A/k)$.

LEMMA 1.4. *If* $\delta: A \longrightarrow M$ *is an* n^{th}*-order derivation and* $a \in A$, *the map* $a\delta: A \longrightarrow M$ *defined by* $(a\delta)(x) = a\delta(x)$ *is an* n^{th}*-order derivation, hence the set* $\mathrm{Der}(A,M)$ *of* n^{th}*-order derivations from* A *to* M *has a natural* A*-module structure.*

LEMMA 1.5. *The map* $d_n: A \longrightarrow D_n(A/k)$ *is an* n^{th}*-order derivation, which induces (by composition) an* A*-isomorphism* $d_n^*: \mathrm{Hom}(D_n(A/k),M) \cong$ $\cong \mathrm{Der}_n(A,M)$. *Hence* $(d_n, D_n(A/k)$ *is universal for* n^{th}*-order derivations.*

For the proof of this lemma, see [5] or [6].

DEFINITION 1.6. It is well known ([5], [6]) that, for $A = k[x_1,\dots,x_n]$, $D_n(A/k)$ is a free A-module freely generated by the monomials in $d_n x_i$ of degree at most n. For $n = p^e$ we define a $(p^e)^{th}$ order derivation $\dfrac{\partial}{\partial F^e x_i}: A \longrightarrow A$ as the element of the dual basis which is one on $d_n(F^e x_i)$.

Since k has characteristic p, this derivation can not be obtained as a composite of lower order derivations, and it will be called a Dieudonné derivative of order p^e with respect to the variable x_i.

These derivatives have been studied by Dieudonné in [1].

LEMMA 1.7. *Let* $A = k[x_1,\dots,x_n]$ *where* k *is a field of characteristic* $p \neq 0$. *Call* A^{p^s} *to* $k[F^s x_1,\dots,F^s x_n]$. *If* $f \in A^{p^e}$ *and* $\dfrac{\partial f}{\partial(F^e x_i)} = 0$ *for* $i = 1,\dots,n$, *then* $f \in A^{p^{e+1}}$.

Proof. Because $f \in A^{p^e}$, $f = \sum_\alpha c_\alpha (F^e x_1)^{\alpha(1)} \dots (F^e x_n)^{\alpha(n)}$ with $c_\alpha \in 1$ and $\alpha = (\alpha(1),\dots,\alpha(n))$.

If $f = 0$, then $f \in A^{p^{e+1}}$. Hence we need to show that for each α for which $c_\alpha \neq 0$, $\alpha(i)$ is a multiple of p for $i = 1,\dots,n$.

Now $\dfrac{\partial f}{\partial(F^e x_i)} = \sum_\alpha c_\alpha \alpha(i)(F^e x_1)^{\alpha(1)} \dots (F^e x_i)^{\alpha(i)-1} \dots (F^e x_n)^{\alpha(n)}$ and $\dfrac{\partial f}{\partial(F^e x_i)} = 0$ imply that $c_\alpha \alpha(i) = 0$ for all i. Therefore $\alpha(i) = 0 \bmod p$

for all i when $c_\alpha \neq 0$ and hence $\alpha(i) = p\,\beta(i)$. Thus

$$f = \sum_\alpha c_\alpha (F^e x_1)^{p\beta(1)} \dots (F^e x_n)^{p\beta(n)} = \sum_\alpha c_\alpha (F^{e+1} x_1)^{\beta(1)} \dots (F^{e+1} x_n)^{\beta(n)},$$

that is, $f \in A^{p^{e+1}}$

REMARK 1.8. The same argument shows that if

$f \in k[x_1,\dots,F^e x_i,\dots,x_n]$ and $\dfrac{\partial f}{\partial (F^e x_i)} = 0$, then

$f \in k[x_1,\dots,F^{e+1} x_i,\dots,x_n]$.

LEMMA 1.9. *If* $A = k[x_1,\dots,x_n]$, *then the derivation* $\dfrac{\partial}{\partial F^s x_i}$ *is determined by the following properties:*

a) $\dfrac{\partial}{\partial F^s x_i}$ *is a* k-*linear map* $A \longrightarrow A$.

b) $\dfrac{\partial fg}{\partial F^s x_i} = f \dfrac{\partial g}{\partial F^s x_i} + g \dfrac{\partial f}{\partial F^s x_i}$ *if* $f \in k[F^s x_1,\dots,F^s x_n]$.

c) $\dfrac{\partial x_j^r}{\partial F^s x_i} = 0$ *for* $r < p^s$.

d) $\dfrac{\partial F^s x_j}{\partial F^s x_i} = \delta_{ij}$.

2. DIFFERENTIAL IDEALS.

LEMMA 2.1. *Let* I *be an ideal in* $k[x_0,\dots,x_n]$ *where* k *is a field of characteristic* p, *then* I *is closed under the derivation* $\partial/\partial x_0$ *if and only if it has a set of generators* $\{f_i\}$ *with* $f_i \in k[x_0^p,x_1,\dots,x_n]$.

Proof. Let $g_1,\dots,g_r$ be a set of generators for I and write

$g_i = \sum_{j=0}^{p-1} h_{ij} x_0^j$, $h_{ij} \in k[x_0^p,x_1,\dots,x_n]$. Now, by taking

$(\frac{\partial}{\partial x_0})^{p-1} g_i = (p-1)!\,h_{i,p-1}$ and, since I is closed under $\frac{\partial}{\partial x_0}$ we have

$h_{i,p-1} \in I$; hence $\sum_{j=0}^{p-2} h_{ij} x_0^j \in I$ and we continue, so, all $h_{ij} \in I$ and

they obviously generate I.

The converse follows from the fact that $\frac{\partial}{\partial x_0} f_i = 0$, hence if

$\sum \ell_i f_i \in I$, we have $\frac{\partial}{\partial x_0} (\sum \ell_i f_i) = \sum \frac{\partial \ell_i}{\partial x_0} f_i \in I$, so, I is closed

under $\frac{\partial}{\partial x_0}$.

LEMMA 2.2. *Let* I *be an ideal in* $k[x_0,x_1,\dots,x_n]$ *where* k *is a field of* *characteristic* p. *Then* I *is closed under the derivations* $\partial/\partial x_0$, $\partial/\partial Fx_0,\dots,\partial/\partial F^{e-1}x_0$ *if and only if it has a set of generators* $\{f_i\}$ *with* $f_i \in k[F^e x_0, x_1,\dots,x_n]$.

Proof. We will proceed by induction on e. For e = 1 we are reduced to the previous lemma. Assume it is true for e = r - 1. Then I has generators $g_1,\dots,g_r \in k[F^{r-1}x_0, x_1,\dots,x_n]$. Now, write $g_i = \sum_{j=0}^{p-1} h_{ij}(F^{r-1}x_0)^j$ and apply $\partial/\partial F^{r-1}x_0$ p-1 times, and the reasoning follows as in Lemma 2.1.

REMARK 2.3. Lemma 2.1 can be restated for characteristic zero as follows:

If I is an ideal in $k[x_0,\dots,x_n]$ where k is a field of characteristic zero, then I is closed under the derivation $\partial/\partial x_0$ if and only if it has a set of generators $\{f_i\}$ with $f_i \in k[x_1,\dots,x_n]$.

3. TANGENTS.

Let V be an algebraic variety, L a line and Q a point in $L \cap V$. If Q is a regular point in V then L is tangent to V at Q if and only if the intersection multiplicity is bigger than one.

By using an affine chunk containing Q, then the ideal of V gives an ideal J in k[t] where t is a parameter for L and the coordinate a of Q in L is a root for J. The intersection multiplicity is the multiplicity of the root a in J.

Let now V be a projective (irreducible) variety $V \subseteq P_n(k)$, defined by a (homogeneous) ideal I, P a rational point in $P_n(k)$ and Q a rational point in V, $Q \neq P$. We want to study the intersection multiplicity of $PQ \cap V$ at Q.

Choose a coordinate system such that $P = (1,0,\dots,0)$, $Q = (a_0,a_1,\dots,a_n)$. Since $Q \neq P$ at least one $a_i \neq 0$ $(i \neq 0)$. Let $a_n \neq 0$, then we consider the affine part U of $P_n(k)$ defined by $x_n \neq 0$. By restricting everything to U, the line PQ has parametric equations

$$x_0 = t \qquad x_i = a_i/a_n \qquad 1 \leqslant i \leqslant n-1.$$

If the ideal I of V is generated by $f_1,\dots,f_r \in k[x_0,\dots,x_n]$ then a_0 is a zero of $J = \{f_i(t,a_1,\dots,a_n)\} \in k[t]$, but J is a principal ideal $J = (f)$.

The following result is well known:

LEMMA 3.1. *The intersection multiplicity of* $PQ \cap V$ *at* Q *is bigger than one if and only if* a_0 *is a root of* f *and* $\frac{\partial f}{\partial f_0}$.

COROLLARY 3.2. *The conditions above imply that* $(a_0,\dots,a_n)$ *is a root for* $\frac{\partial f_i}{\partial x_0}$ $(1 \leq i \leq r)$ *and conversely.*

Proof. In fact, using the notation above, $f(x_0) = \sum g_i f_i(x_0,a_1,\dots,a_n)$, $g_i \in k[x_0]$ and $f_i(x_0,a_1,\dots,a_n) = h_i f(x_0)$, $h_i \in k[x_0]$. Since $\frac{\partial f_i}{\partial x_0}(x_0,a_1,\dots,a_n) = \frac{\partial (f_i(x_0,a_1,\dots,a_n))}{\partial x_0}$ we have that, if Q is a multiple root, then so is a_0 in f, hence a_0 is a root of $\frac{\partial f}{\partial x_0}$ and, since $\frac{\partial f_i(x_0,a_1,\dots,a_n)}{\partial x_0} = h_i \frac{\partial f}{\partial x_0} + \frac{\partial h_i}{\partial x_0} f$, then $\frac{\partial h_i}{\partial x_0}(a_0,a_1,\dots,a_n) = 0$.

The converse is obvious.

4. TERRIBLE POINTS.

DEFINITION 4.1. Let V be a projective (irreducible) variety $V \subseteq P_n(k)$ and P a rational point in $P_n(k)$, then P is called a terrible point for V if the set of rational points $Q \in V$ such that PQ is tangent to V at Q is not contained in a proper subvariety of V.

THEOREM 4.2. *Let* V *be a projective (irreducible) variety in* $P_n(k)$, P *a rational point in* $P_n(k)$, *where* k *is a field of characteristic zero. Then* P *is a terrible point for* V *if and only if* V *is a cone with vertex* P.

Proof. Take a coordinate system such that $P = (1,0,\dots,0)$ and I is the (prime) ideal of V. Let I_0 be the ideal generated by I and the derivatives $\frac{\partial f}{\partial x_0}$ for all $f \in I$ (If $f_1,\dots,f_r$ is a set of generators for I, then I_0 is generated by $f_1,\dots,f_r, \frac{\partial f_1}{\partial x_0},\dots,\frac{\partial f_r}{\partial x_0}$), so $I_0 \supseteq I$.

If $I_0 \neq I$, since I is prime, it defines a proper subvariety of V.
If $Q \in V$ and PQ is tangent to V at Q then Q is a zero for I_0 (Cor.3.2), hence if P is a terrible point the set of zeros of I_0 is not contained in a proper subvariety of V, so $I_0 = I$, hence I is closed under $\frac{\partial}{\partial x_0}$, which means (in characteristic zero) that I has a set of generators f_i with f_i independent of x_0.
But this is equivalent to the fact that V is a cone with vertex P.
The converse is trivial.

THEOREM 4.3. *Let* V *be a projective (irreducible) variety in* $P_n(k)$, P *a rational point in* $P_n(k)$, *where* k *is a field of characteristic* $p \neq 0$, *and suppose we have a coordinate system such that* $P = (1,0,\dots,0)$ *and*

I is the (prime) ideal of V. *Then* P *is a terrible point for* V *if and only if* I *has a set of generators* $\{f_i\}$ *with* $f_i \in k[x_0^p, x_1, \ldots, x_n]$.

Proof. As in the previous theorem, if $Q \in V$ and PQ is tangent to V at Q, then Q is a zero of I_0. If $I \neq I_0$ it defines a proper closed subset of V, hence P is not terrible. So $I = I_0$, i.e., I is closed under $\frac{\partial}{\partial x_0}$, and the theorem follows from Lemma 2.1.

5. POLARIZATION.

In [3] and [4], Hipps, Mount and Villamayor defined and studied the polarization map, extending to fields of positive characteristic the classical polarization of a homogeneous polynomial.

In this section we will summarize their construction and apply it to the terrible points.

Let U be a vector space over a field k and U* its dual. Call O^rU^* the r-fold symmetric product of U*. If $e_0, \ldots, e_n$ is a basis for U, call $x_0, \ldots, x_n$ its dual basis in U*, so O^rU^* has as a basis all monomials of degree r in the x_i's. Then, the rational points of $P_n(k)$ are the one dimensional subspaces of U.

DEFINITION 5.1. Let w be a homogeneous polynomial of degree r. Then the polarization map $\mathrm{Pol}_0(w)\colon U \longrightarrow O^{r-1}U^*$ is the linear map defined by $\mathrm{Pol}_0(w)(e_i) = \frac{\partial w}{\partial x_i}$.

Call $\mathrm{vertex}_0 w = \mathrm{Ker}(\mathrm{Pol}_0(w))$.

Suppose $\mathrm{Pol}_{i-1}(w)$ and $\mathrm{vertex}_{i-1}(w)$ have been defined, then we can define $\mathrm{Pol}_i(w)|_{\mathrm{vertex}_{i-1}(w)}$ by the following properties

a) $\mathrm{Pol}_i(w)(x+y) = \mathrm{Pol}_i(w)(x) + \mathrm{Pol}_i(w)(y)$ for $x,y \in U$.

b) $\mathrm{Pol}_i(w)(kx) = F^ik \ \mathrm{Pol}_i(w)(x)$ for $k \in k$, $x \in U$.

c) If $e_1, \ldots, e_s$ is a basis for $\mathrm{vertex}_{i-1}(w)$, $e_1, \ldots, e_n$ a basis for U and $x_1, \ldots, x_n$ its dual basis

$$\mathrm{Pol}_i(w)(e_j) = \frac{\partial w}{\partial F^i x_j} \quad \text{for} \quad e_j \in \mathrm{vertex}_{i-1}(w).$$

LEMMA 5.2. *The map* $\mathrm{Pol}_i(w)$ *is independent of the basis.* ([4], Lemma 2.2).

DEFINITION 5.3. If I is an ideal in $k[x_1, \ldots, x_n]$ generated by homogeneous polynomials $\omega_1, \ldots, \omega_m$ of the same degree, call $\mathrm{vertex}_s(I) = \bigcap_i \mathrm{vertex}_s(w_i)$.

THEOREM 5.4. *Let* V *a projective variety* $V \subseteq P_n(k)$ *defined by an ideal* I *generated by forms* w_i *all of the same degree. Then a rational point* $P \in P_n(k)$ *is a terrible point for* V *if and only if* $P \in \mathrm{vertex}_0(I)$. *Here we are identifying the points of* $P_n(k)$ *with the one dimensional subspaces of* U.

Proof. It follows from Th. 4.3 and [4] Th. 6.4.

6. A GENERALIZATION.

If V is a projective variety $V \subseteq P_n(k)$ over a field of characteristic $p \neq 0$, P is a terrible point for V and $Q \in V$, then the intersection multiplicity of $PQ \cap V$ at Q is always a multiple of p.

This fact suggests the idea of searching if the set of points Q such that the intersection multiplicity is bigger than p has properties similar to the set of points Q such that PQ is tangent to V at Q.

We can use the results of [4] to extend the definition of terrible points in the following way.

DEFINITION 6.1. If V is a projective (irreducible) variety $V \subseteq P_n(k)$ where k is a field of characteristic $p \neq 0$ and P is a rational point in $P_n(k)$, then P is called e-terrible for V if the set of rational points $Q \in V$ such that the intersection multiplicity $PQ \cap V$ at Q is a multiple of p^e bigger than p^e, is not contained in a proper closed subset of V.

Then, following the reasonings of §3 and §4, we can prove:

THEOREM 6.2. *Let* V *be a projective (irreducible) variety in* $P_n(k)$, $P \in P_n(k)$, *where* k *is a field of characteristic* $p \neq 0$, *and assume there is a coordinate system such that* $P = (1,0,\dots,0)$ *and* I *is the ideal of* V. *Then* P *is* e-*terrible for* V *if and only if* I *has a system of generators* $\{f_i\}$ *with* $f_i \in k[F^{e+1}x_0, x_1, \dots, x_n]$.

THEOREM 6.3. *Let* V *be a projective variety in* $P_n(k)$ *defined by an ideal* I *generated by forms* w_i *all of the same degree. Then a rational point* $P \in P_n(k)$ *is an* e-*terrible point for* V *if and only if* $P \in \mathrm{vertex}_e(I)$.

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