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LIFTING ESSENTIALLY (G_1) OPERATORS

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ABSTRACT. A large class of operators, including those corresponding to G_1 elements of the Calkin algebra can be "lifted" to a G_1 operator by means of a compact perturbation. However, the class (G_1+ compact) is nowhere dense in the algebra of all operators acting on a complex separable infinite dimensional Hilbert space.

KEY WORDS AND PHRASES. Calkin algebra, growth condition on the resolvents, (essentially) G_1 operators, normal operators, compact perturbations, (essential) numerical range, normal reducing eigenvalues, spectrum, Calkin essential spectrum, Weyl essential spectrum.

1. INTRODUCTION.

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all (bounded linear) operators acting on the complex separable infinite dimensional Hilbert space $\mathcal{H}$, let K be the ideal of compact operators and let $\pi: \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})/K = A$ be the canonical projection onto the Calkin algebra. $T \in \mathcal{L}(\mathcal{H})$ is (essentially) G_1 if $\|(\lambda - T)^{-1}\| = 1/d(\lambda)$ ($\|(\lambda - \pi(T))^{-1}\| = 1/d_E(\lambda)$, resp.) for all λ outside of the spectrum $\Lambda(T)$ (essential spectrum $E(T)$, resp.) of T, where $d(\lambda) = \mathrm{dist}[\lambda,\Lambda(T)]$ ($d_E(\lambda) = \mathrm{dist}[\lambda,E(T)]$, resp.).
Let G ($e(G)$,resp.) denote the class of all G_1 (essentially G_1, resp.) operators in $\mathcal{L}(\mathcal{H})$. In [8], Glenn R. Luecke conjectured that every $T \in e(G)$ has a compact perturbation in G. In *Section 2* a characterization of those $T \in \mathcal{L}(\mathcal{H})$ such that $T + K \in G$ for some compact K will be given. This characterization provides an affirmative answer to Luecke's conjecture but, unfortunately, this answer is not completely satisfactory (in a sense that will be made precise below). Some consequences of the main result and an example of an operator (indeed, a nilpotent of order two) that cannot be compactly perturbed to a G_1 operator are discussed in *Section 3*. Moreover, this example is used

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to show that $G + K$ is a nowhere dense subset of $\mathcal{L}(\mathcal{H})$ and $e(G)$ is a nowhere dense subset of $G + K$.

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2. A CHARACTERIZATION OF $G + K$.

Let $T \in \mathcal{L}(\mathcal{H})$; then the subsets $w(T) = \{\lambda \in \Lambda(T) : \lambda - T$ is not a Fredholm operator of index $0\}$ and $\Lambda_o(T) = \{\lambda \in \Lambda(T) : \lambda$ is isolated in $\Lambda(T)$ and the corresponding spectral subspace is finite dimensional$\}$ are called the *Weyl essential spectrum* of T and the set of *normal eigenvalues* of T. It is well known that $w(T) = \cap\{\Lambda(T+K) : K \in K\}$ [10], so that $w(T)$ can not be modified by compact perturbations.

Throughout this paper, $A \approx B$ will mean that A and B are unitarily equivalent Hilbert space operators, $\oplus$ will denote *orthogonal* direct sum and $T^{(\alpha)}$ will denote the orthogonal direct sum of α $(0 \leqslant \alpha \leqslant \infty)$ copies of the operator T. Finally, $U(T) = \{A : A \approx T\}$ is the *unitary orbit* of $T \in \mathcal{L}(\mathcal{H})$ and X^- and ∂X denote the closure and the boundary of X, respectively.

LEMMA 1. *Let* $T \in e(G)$. *Given* $\varepsilon > 0$ *there exists* $K \in K$ *such that* $T - K \approx T \oplus N$, *where* N *is a normal operator such that* $\Lambda(N) = E(N) = \partial w(T)$.

Proof: Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of bare points of $\partial w(T)$ (i.e., for each n, there exists $\lambda_n \in \underset{\sim}{C}$ and $r_n > 0$ such that $r_n = \text{dist}[\lambda_n, w(T)] = |\lambda_n - \mu_n| < |\lambda_n - \mu|$, for all $\mu \in w(T) \setminus \{\mu_n\}$) and assume that $(\{\mu_n\}_{n=1}^{\infty})^- = \partial w(T)$ and $\not{c}\{n : \mu_n = \mu_m\} = \aleph_o$, for all $m = 1, 2, \ldots$.

According to the first part of the *proof* of *Theorem 1.2* in [4], there exists a normal operator M defined by $Me_n = \mu_n e_n$ with respect to a suitable ONB $\{e_n\}_{n=1}^{\infty}$ and operators A, C, T_1 such that $E(C) \subset E(T)$, $T_o^{(\infty)} \oplus T_1 \in U(T)^-$, $T_o^{(\infty)} \oplus T_1 - T \in K$ and $\|T_o^{(\infty)} \oplus T_1 - T\| < \varepsilon$, where

$$T_o = \begin{pmatrix} M & 0 \\ A & C \end{pmatrix}$$

Clearly, $T_o^{(\infty)} \oplus T_1 \in e(G)$. Let $\lambda \notin w(T)$ and assume that $\|(\lambda - T_o)^{-1}\| > 1/d_E(\lambda)$ (In this case, $d_E(\lambda) = \text{dist}[\lambda, w(T)]$); then $\|\pi(\lambda - T)^{-1}\| = \max\{\|\pi(\lambda - T_1)^{-1}\|, \|(\lambda - T_o)^{-1}\|\} > 1/d_E(\lambda)$, a contradiction. Therefore, $\|(\lambda - T_o)^{-1}\| = 1/d_E(\lambda)$ for all $\lambda \notin w(T)$.

Let μ_n, λ_n, e_n be as above; then

$$(T_o-\lambda_n)^{-1}\begin{bmatrix} e_n \\ 0 \end{bmatrix} = \begin{bmatrix} (M-\lambda_n)^{-1} & 0 \\ -(C-\lambda_n)^{-1}A(M-\lambda_n)^{-1} & (C-\lambda_n)^{-1} \end{bmatrix}\begin{bmatrix} e_n \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} [1/(\mu_n-\lambda_n)]e_n \\ -(C-\lambda_n)^{-1}A(M-\lambda_n)^{-1}e_n \end{bmatrix}.$$

Since the norm of this vector cannot be larger than $1/|\mu_n-\lambda_n| = \|[1/(\mu_n-\lambda_n)]e_n\|$, we conclude that $0 = -(C-\lambda_n)^{-1}A(M-\lambda_n)^{-1}e_n = -[1/(\mu_n-\lambda_n)]\ (C-\lambda_n)^{-1}Ae_n$. Hence, Ae for all n = 1,2,..., and therefore A = 0.

It readily follows that $T-K = M^{(\infty)} \oplus [C^{(\infty)} \oplus T_1] = N \oplus (N \oplus C^{(\infty)} \oplus T_1] = N \oplus (T-K)$, where $N = \underset{\sim}{M}^{(\infty)}$ is normal and $\Lambda(N) = E(N) = \partial w(T)$.

LEMMA 2. *Let* $B \in \mathcal{L}(\mathcal{H})$ *and let* N *be a normal operator such that* $\Lambda(N) = E(N) = \partial w(B)$. *Then there exists* $K \in K$ *such that* $T = N \oplus B + K \in G$, $\Lambda(T) = w(T) \cup \Lambda_o(T)$ *and every* $\lambda \in \Lambda_o(T)$ *is a reducing normal eigenvalue corresponding to a one-dimensional eigenspace.*

Proof: According to [10], there exists $K_1 \in K$ such that $\Lambda(B + K_1) = w(B)$. Thus, without loss of generality we can assume that $\Lambda(B) = w(B)$ (i.e., $K_1 = 0$). Similary [2], up to a compact perturbation we can assume that $Ne_n = \nu_n e_n$ with respect to a suitable ONB $\{e_n\}_{n=1}^{\infty}$ and that $\not{c}\{n: \nu_n = \nu_m\} = \aleph_o$ for all m=1,2,... .

If $|\lambda| = \|B\|$, then $\|(\lambda-B)^{-1}\| \leqslant 1/(|\lambda|-\|B\|)$. Thus, if $\lambda_k = 2\|B\|.\exp\{(k-1)\pi i/3\}$, k = 1,2,...,6, and $L_6 \in \mathcal{L}(\underset{\sim}{C}^6)$ is the diagonal (normal) operator defined by $L_6 f_k = \lambda_k f_k$, k = 1,2,...,6, with respect to the canonical ONB of $\underset{\sim}{C}^6$, then $\|(\lambda-B)^{-1}\| \leqslant \|(\lambda-L_6)^{-1}\| = \max\{1/|\lambda-\lambda_k| : k=1,2,...,6\}$, for all $\lambda \in \underset{\sim}{C}$ with $|\lambda| \geqslant 2\|B\|, \lambda \neq \lambda_k$ (k = 1,2,...,6).

Let $\Lambda_6 = \{\lambda_k : k=1,2,...,6\}$ and $\Gamma_6 = \{\lambda \notin w(B) : \|(\lambda-B)^{-1}\| \geqslant \|(\lambda-[L_6 \oplus N])^{-1}\|\}$. Clearly, $\partial[\Gamma_6 \cup w(B)]$ is contained in the *open* disc of radius $2\|B\|$. Let $\lambda_7 \in \partial\Gamma_6$ be a point such that $\mathrm{dist}[\lambda_7, w(B)] = \max\{\mathrm{dist}[\lambda, w(B)] : \lambda \in \Gamma_6\}$ and define $L_7 \in \mathcal{L}(\underset{\sim}{C}^7)$ by $L_7 = L_6 \oplus \{\lambda_7\}$.

It readily follows that $\|(\lambda-B)^{-1}\| < \|(\lambda-L_7)^{-1}\|$ for all $\lambda \notin \Gamma_7 \cup w(B) \cup \{\lambda_7\}$, where $\Gamma_7 = \{\lambda \in \Gamma_6 \setminus \{\lambda_7\} : \|(\lambda-B)^{-1}\| \geqslant \|(\lambda-[N \oplus L_7])^{-1}\|\}$ is the complement in Γ_6 of a suitable open neighborhood of λ_7.

Define $\Lambda_7 = \Lambda_6 \cup \{\lambda_7\}$. By an obvious inductive argument, we either obtain $\Gamma_k = \emptyset$ for some $k \geq 6$, or an operator $L = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots\}$ (diagonal with respect to a suitable ONB) such that $(\{\lambda_k\}_{k=1}^{\infty}) \cap \cap\, w(B) = \emptyset$, $\mathrm{dist}[\lambda_k, w(B)]$ is non-increasing and tends to 0 $(k \to \infty)$ and $\|(\lambda-B)^{-1}\| \leq \|(\lambda-[N \oplus L])^{-1}\|$ for all $\lambda \notin w(B) \cup \Lambda(L)$, where $\Lambda(L) \subset (\{\lambda_k\}_{k=1}^{\infty}) \cup \partial w(B)$ (in the first case, we set $\lambda_j = \nu_1$ for all $j \geq k$). Set $n(1) = 1$; inductively, we define $n(k)$ as the *second* index strictly larger than $n(k-1)$ $(k=2,3,\ldots)$ such that $|\lambda_k - \nu_{n(k)}| < 2\,\mathrm{dist}[\lambda_k, w(B)]$.

Finally, define $K = \mathrm{diag}\{\lambda_1-\nu_1, 0, 0, \ldots, 0, \lambda_2-\nu_{n(2)}\langle n(2)\rangle, 0, 0, \ldots, 0, \lambda_3-\nu_{n(3)}\langle n(3)\rangle, 0, 0, \ldots, 0, \lambda_k-\nu_{n(k)}\langle n(k)\rangle, 0, \ldots\}$ with respect to the ONB $\{e_n\}_{n=1}^{\infty}$. Clearly, $K \in K$ and $N + K = \mathrm{diag}\{\lambda_1, \nu_2, \nu_3, \ldots, \nu_{n(2)-1}, \lambda_2, \nu_{n(2)+1}, \ldots, \nu_{n(k)-1}, \lambda_k, \nu_{n(k)+1}, \ldots\} \approx N \oplus L$ and therefore $\|(\lambda-B)^{-1}\| \leq \|(\lambda-[N+K])^{-1}\| = \|(\lambda-T)^{-1}\| = 1/d(\lambda)$ for all $\lambda \in \Lambda(T)$, where $T = (N+K) \oplus B$. Hence $T = (N \oplus B) + (K \oplus 0) \in G$.

As an immediate corollary of *Lemmas 1* and *2*, we have

THEOREM 1. *Let* $T \in e(G)$. *Then there exists* $K \in K$ *such that* $T+K \in G$.

THEOREM 2. *Let* $T \in \mathcal{L}(\mathcal{H})$, *then* $T \in G + K$ *if and only if there exists* $K' \in K$ *such that* $T + K' \approx N \oplus T$ *for some normal operator* N *with* $\Lambda(N) = E(N) = \partial[\mathbb{C}\setminus E(T)]_{\infty}$, *where the subindex "$\infty$" denotes the unbounded component of the set* $\mathbb{C}\setminus E(T)$.

Proof: Assume that some compact perturbation of T belongs to G. Without loss of generality, we can assume that $T \in G$. Let $\Lambda_o(T) = \{\lambda_n\}$ be the set of normal eigenvalues of T. Since λ_n is an isolated point of $\Lambda(T)$, it follows from [11] that λ_n is a reducing eigenvalue of finite multiplicity α_n, $n=1,2,\ldots$.

Let $\nu = \lim(j \to \infty)\lambda_{n(j)}$ for a suitable subsequence $\{\lambda_{n(j)}\}_{j=1}^{\infty}$ of $\{\lambda_n\}_{n=1}^{\infty}$, then $T = (\mathrm{diag}\{\lambda_{n(1)}{}^{(\alpha_{n(1)})}, \lambda_{n(2)}{}^{(\alpha_{n(2)})}, \ldots\}) \oplus T_\nu$ and it is clear (e.g., by using the arguments of [10]) that $T + K_\nu \approx T \oplus \nu I$ for a suitable $K_\nu \in K$. Combining this observation with the result of [2] and the arguments of [10], we conclude that, if

$\Gamma_o = (\partial[\underset{\sim}{C}\backslash E(T)]_\infty) \cap \Lambda_o(T)^-$, then there exist $K_o \in K$ and a normal operator N_o such that $T + K_o \approx T \oplus N_o$ and $\Lambda(N_o) = E(N_o) = \Gamma_o$.

On the other hand, if μ is a bare point of $\Gamma_1 = (\partial[\underset{\sim}{C}\backslash E(T)]_\infty)\backslash\Gamma_o$, then $\mu \in E(T)$ and we can proceed exactly as in the *proof* of *Lemma 1* in order that $T + K_\mu \approx T \oplus \mu I$ for a suitable $K_\mu \in K$. By similar arguments we conclude that $T + K_1 \approx T \oplus N_1$, for some $K_1 \in K$ and some normal operator N_1 such that $\Lambda(N_1) = E(N_1) = \Gamma_1^-$.

Therefore, $T + (K_o + K_1) \approx T \oplus (N_o \oplus N_1)$, where $K_o + K_1 \in K$ and $N = N_o \oplus N_1$ is a normal operator with the desired properties.

Conversely, if $T + K' \approx T \oplus N$ for a normal operator N such that $\Lambda(N) = E(N) = \partial[\underset{\sim}{C}\backslash E(T)]_\infty$, then there exists $K'' \in K$ such that $\Lambda(T + K'') = \underset{\sim}{C}\backslash[\underset{\sim}{C}\backslash E(T)]_\infty$ (see [1]) and we conclude that $T + K \in G$ for some $K \in K$ by applying the same arguments as in the *proof* of *Lemma 2* to $(T + K'') \oplus N$.

3. COMPLEMENTARY RESULTS.

Theorem 1 affirmatively answers *Conjecture 1* of [8], but it does not provide a satisfactory answer, in the sense that $\Lambda(T + K)$ is very different from $w(T)$, in general.

CONJECTURE 1. If $T \in e(G)$, then there exists $K \in K$ such that $T+K \in G$ and $\Lambda(T+K) = w(T)$.

A satisfactory answer to Luecke's Conjecture (i.e., an affirmative answer to *Conjecture 1* above) would involve a very deep analysis of the distance from the resolvent of a perturbated operator to K, in the lines of [1], [3], [7], [9] and [10].

Recently, P. Alson (personal communication) affirmatively answered *Conjecture 2* of [8] by showing that

(1) Given $T \in \mathcal{L}(\mathcal{H})$, there exists K_6 with rank $K_6 \leq 6$ such that $T + K_6$ is *convexoid*;

(2) Given $T \in \mathcal{L}(\mathcal{H})$, there exists $K \in K$ such that $W(T+K)^- = W_e(T)$, where $W(.)$ and $W_e(.)$ denote the *numerical range* and the *essential numerical range*, respectively (see [5], [8], [12] for definitions and properties).

Since $W(T)^-$ ($W_e(T)$, resp.) is always a compact convex set containing $\Lambda(T)$ ($E(T)$, resp.), and $W(T)^-$ ($W_e(T)$, resp.) coincides with the convex hull of $\Lambda(T)$ ($E(T)$, resp.) for all $T \in G$ ($T \in e(G)$, resp.), second Alson's result yields the following (very) partial answer to the above conjecture.

COROLLARY 1. *If* $T \in e(G)$ *and* $w(T)$ *is convex, then there exists* $K \in K$ *such that* $T+K \in G$ *and* $\Lambda(T+K) = w(T)$.

First Alson's result might suggest that *every* operator has a compact perturbation in G. This is *definitely false.*

EXAMPLE 1. Let $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2)$ and let $T = Q^{(\infty)} \in \mathcal{L}(\mathcal{H})$. Then $\mathrm{dist}[T, G+K] \geqslant 1/10$.

Proof. Assume that $\|A-T\| < 1/10$ for some $A \in G+K$. Then a straightforward computation shows that the spectrum of A is contained in the the open disc of radius 3/10 about the origin. Thus, if $\lambda \in \partial[\mathbb{C}\setminus E(A)]_\infty$ is a bare point, then $A - \lambda I - K$ has an infinite dimensional reducing subspace (use the *proof* of *Theorem 2*) for a suitable compact K and, a fortiori, $\pi([A-\lambda I] + [A-\lambda I]^*)$ cannot have an inverse in A. On the other hand, $\|I - [(A-\lambda I)+(A-\lambda I)^*]^2\| = \|(T+T^*)^2 - [(A-\lambda I) + (A-\lambda I)^*]^2\| \leqslant (\|T+T^*\| + \|(T+T^*)-[(A-\lambda I)+(A-\lambda I)^*]\|)^2 - \|T+T^*\|^2 \leqslant (1 + 2[\|T-A\| + |\lambda|])^2 - 1 < 1$, whence we obtain that $(A-\lambda I)+(A-\lambda I)^*$ is invertible in $\mathcal{L}(\mathcal{H})$, a contradiction.

COROLLARY 2. (i) $G+K$ *is nowhere dense in* $\mathcal{L}(\mathcal{H})$.
(ii) $e(G)$ *is nowhere dense in* $G+K$.

Proof. (i) According to [6, *Lemma 2*], given $A \in \mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{L}(\mathcal{H})$ such that $\|A-A_\varepsilon\| < \varepsilon$ and

$$A_\varepsilon = \begin{pmatrix} \lambda I+\varepsilon' T & B \\ 0 & C \end{pmatrix}$$

where $\varepsilon/4 < \varepsilon' < \varepsilon/2$, λ belongs to the unbounded component of $\mathbb{C}\setminus E(C)$, $\mathrm{dist}[\lambda, E(C)] = \varepsilon/2$ and T is the operator of *Example 1*. Minor modifications of the above proof show that $\mathrm{dist}[A_\varepsilon, G+K]$ is positive.

(ii) Combining the *proofs* of [6, *Lemma 2*], *Lemma 1* and *Lemma 2*, it is not difficult to show that if $A \in e(G)$, given $\varepsilon > 0$ there exists an operator $A_\varepsilon \approx A \oplus (\lambda I+\varepsilon'[0^{(\infty)} \oplus T])$, where $\varepsilon/4 < \varepsilon' < \varepsilon/2$, λ belongs to the unbounded component of $\mathbb{C}\setminus E(A)$, $\mathrm{dist}[\lambda, E(A)] = \varepsilon/2$ and T is the operator of *Example 1*, such that $\|A-A_\varepsilon\| < \varepsilon$.
By *Theorem 2*, $A_\varepsilon \in G+K$ and another modification of the *proof* given in *Example 1* shows that $\mathrm{dist}[A_\varepsilon, e(G)] = \varepsilon'$.

The proof is complete now.

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