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REMARKS ON A PROBLEM IN THE FLOW OF HEAT FOR A SOLID IN CONTACT WITH A FLUID

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ABSTRACT. The main objective of this paper is to exhibit the role played by the null series of an ordinary differential irregular boundary problem obtained by separation of variables in the solution of the original partial differential equation. In this case we treat a problem already studied by R.E. Langer where the heat equation is involved.

1. We consider the following problem studied by R.E. Langer, [L]. A right cylindrical solid has its lateral surface insulated against the passage of heat and has an initial distribution of temperature depending only on the longitudinal coordinate x. Let x=0 and x=1 be the coordinates of the plane terminal faces. The face at x=0 may be insulated or may permit the passage of heat. The face at x=1 at the time t=0 is placed in contact with a well-stirred liquid that at each instant will have a uniform temperature throughout it. There may or may not be passage of heat from the liquid to the surrounding medium. The problem is to determine the temperature of the liquid and the distribution of temperature in the solid at each instant $t > 0$.

The differential equation

$$\alpha^2 \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

where u(x,t) is the temperature at x at the time t and α is a physical constant, controls the one dimensional flow of heat in the cylinder.

Let v(t) be the temperature of the liquid, T_1 the constant temperature of the medium that surrounds the liquid and T_0 the constant temperature of the medium that is placed in contact with the terminal face at x=0.

Let $u_0(x)$ be the initial temperature in the cylinder and v_0 that of the liquid.

Then, following Langer [L], we must have:

$$(2) \qquad (1-\mu)\left.\frac{\partial u}{\partial x}\right|_{x=0+} = \mu\{u(0+,t) - T_o\} \;;\; 0 \leqslant \mu \leqslant 1,\ \mu \text{ constant}$$

$$(3) \qquad \left.\frac{\sigma}{\alpha^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \nu\,(u - T_1)\right|_{x=1-} = 0 ,$$

where $\sigma > 0$, $\nu \geqslant 0$ are constants.

The initial conditions of this problem are:

$$(4) \qquad \lim_{t\to 0+} u(x,t) = u_o(x) \ ,\ 0 < x < 1 ,$$

$$(5) \qquad \lim_{t\to 0+} v(t) = v_o \quad , \quad v(t) = u(1-,t) .$$

If $u(x,t) = z(x) + w(x,t)$, with

$$(6) \qquad z''(x) = 0 \quad , \quad (1-\mu)\ z'(0) - \mu z(0) = -\mu T_o \ , \quad z'(1)+\nu z(1) = \nu T_1$$

then

$$(7) \quad \alpha^2\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} \ ,\ \left[(1-\mu)\frac{\partial w}{\partial x} - \mu w\right]_{x=0+} = 0 \ ,\ \left[\frac{\sigma}{\alpha^2}\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} + \nu w\right]_{x=1-} = 0 .$$

Separation of variables in (7) yields, if $w(x,t) = \psi(t)\ \varphi(x)$,

$$(8) \qquad \psi'(t) - s^2\alpha^2\psi(t) = 0 ,$$

$$(9) \qquad \begin{cases} \varphi''(x) - s^2\varphi(x) = 0 \\ (1-\mu)\ \varphi'(0) - \mu\ \varphi(0) = 0 , \\ \varphi'(1) + [\nu + \sigma s^2]\ \varphi(1) = 0 . \end{cases}$$

For the discussion that follows we shall assume $\alpha^2 = 1$, $T_o = 0$. This does not introduce a qualitative change in the problem.

To simplify still more the situation we shall assume $\nu = 0$, that is, the liquid insulated against the passage of heat to or from its surrounding medium.

In this case the problem is already homogeneous and we put $z \equiv 0$ and $u = w$. The constant σ depends on the cylinder and the liquid and we shall suppose that $\sigma = 1$. Thus, the system to discuss is given by:

$$
(10)\quad \begin{cases}
\text{i)} & \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \;,\quad t > 0 \;,\quad 0 < x < 1 \;, \\[2mm]
\text{ii)} & (1-\mu)\,\dfrac{\partial u}{\partial x}\,(0+,t) - \mu u(0+,t) = 0 \;,\quad 0 \leqslant \mu \leqslant 1 \;,\quad t > 0 \;, \\[2mm]
\text{iii)} & \dfrac{\partial \dot{u}}{\partial t} + \left.\dfrac{\partial u}{\partial x}\right|_{x=1-} = 0 \;,\quad t > 0 \;, \\[2mm]
\text{iv)} & u(x,t) \xrightarrow[t\to 0+]{} u_o(x) \;,\quad 0 < x < 1 \;,\; t > 0 \;,\; u_o(x) \text{ real} \;, \\[2mm]
\text{v)} & u(1-,t) = v(t) \xrightarrow[t\to 0+]{} v_o \;,\quad t > 0 \;,\quad v_o \text{ real} \;.
\end{cases}
$$

Separation of variables gives:

$$
(11)\qquad \psi'(t) - \lambda\,\psi(t) = 0 \;,\quad s^2 = \lambda \;,
$$

$$
(12)\quad \begin{cases}
\text{a)} & \varphi'' - \lambda\,\varphi = 0 \\
\text{b)} & (1-\mu)\,\varphi'(0) - \mu\,\varphi(0) = 0 \;, \\
\text{c)} & \varphi'(1) + s^2\,\varphi(1) = 0 \;,\quad s^2 = \lambda \;.
\end{cases}
$$

The boundary problem (12) was studied in detail by Langer in his paper [L]. In connection with it, cf. also [M], [Ch], [F] and [WJ]. In the references of the last paper other interesting works about this subject are mentioned.

2. In [L], § 4, it is proved that the eigenfrequencies s_n, $s_n \neq 0$, are pure imaginary and simple. The eigenvalue 0 appears exactly when $\mu = 0$, that is, when the terminal face at x=0 is insulated. From [L], (22), it follows that

$$
s_{\pm n} = \pm\,(n - \frac{1 - [\mu]}{2})\,\pi\, i + O(\frac{1}{n}) \;.
$$

Besides $\varphi_n \equiv \varphi_{-n}$. Since we are interested in expansions in eigenfunctions we shall ignore in the future the eigenfrequencies s_{-n} , $n = 1,2,3,\ldots$.

Also

$$
\varphi_n(x) = (1-\mu)\cos\, is_n x \;+\; \frac{\mu}{is_n}\sin\, is_n x \qquad \text{for}\quad 0 < \mu \leqslant 1
$$

and then we must expect in this situation that the solution of (10) can be written in the form:

$$
(13)\qquad u(x,t) = \sum_{n=1}^{\infty} c_n\,\varphi_n(x)\; e^{-|s_n|^2\cdot t} \;.
$$

We shall exclude the case $\mu = 0$ and therefore the eigenvalue 0. (If $\mu = 0$ the series in (13) would be equal to $u(x,t) - z_o$, $z_o = \dfrac{\int_0^1 u_o\, dx + \sigma v_o}{1 + \sigma}$, (cf. [L], (28)), and the necessary changes in what follows are obvious).

Because of the initial conditions we must have:

$$(14) \qquad \begin{cases} \sum_1^\infty c_n \varphi_n(x) = u_o(x) \quad , \quad 0 \leqslant x < 1 \quad , \\ \sum_1^\infty c_n \varphi_n(1) = v_o \; . \end{cases}$$

Langer proves for $u_o \in L^1$ that $\sum_1^\infty c_n \varphi_n(x)$ is *uniformly equiconvergent with the Fourier series* of $u_o(x)$ on $[\varepsilon, 1-\varepsilon]$, $\forall\, \varepsilon > 0$, if

$$(15) \qquad c_n = \frac{\int_0^1 u_o \varphi_n\, dx + v_o \varphi_n(1)}{\int_0^1 \varphi_n^2\, dx + \varphi_n^2(1)} \quad ,$$

and also that with this choice of coefficients *the second relation in* (14) *holds*.

Let us call $V_n = \varphi_n/\|\varphi_n\|_2$, $C_n = c_n\|\varphi_n\|_2$. From $\varphi_n(x) = k_n \sin is_n(x-h_n)$ we see that

$$(16) \qquad V_n(x) = \varphi_n(x)/\|\varphi_n\|_2 = O(1) \sin is_n(x-h_n)$$

and also that $\varphi_n'(1)/\|\varphi_n\| = O(s_n)$. Therefore

$$V_n(1) = \varphi_n(1)/\|\varphi_n\| = -\varphi_n'(1)/\|\varphi_n\|\cdot s_n^2 = O(1/s_n) \; .$$

Besides $C_n = ((u_o,V_n) + v_o V_n(1))/(1 + \varphi_n^2(1)/\|\varphi_n\|^2)$ as it follows from (15). If $u_o \in L^2$ then $\{(u_o,V_n)\} \in l^2$, ([BP], §1). Also $\{V_n(1)\} \in l^2$ and in consequence $\{C_n\} \in l^2$. Therefore, from [BP], Th.1, *the first series in* (14), *which is equal to* $\sum C_n V_n(x)$, *converges in* L^2 *to* $u_o(x)$. $\sum b_n V_n$ is called a *null series* if it converges to 0 in $L^2(0,1)$. The dimension of the subspace of l^2 of sequences of coefficients of null series will be called *the degrees of freedom of the system* $\{V_n\}$.

Denoting with g the degrees of freedom of the system, we know from [G], Ch.V, that $g=1$ and from Ch.III, Th. 4, that a certain expansion of $u_o(x)$, its Orr expansion, *converges to* 0 *at* $x=1$.

Then, 1) each null series is uniquely determined by its sum at $x=1$, 2) the expansion (14) is obtained summing to the Orr series of $u_o(x)$ the null series that is equal to v_o at $x=1$. In fact, if $u_o \equiv 0$ then the series $\sum C_n V_n = \sum c_n \varphi_n$ is a null series that converges to v_o at $x=1$. Because of $g=1$, 1) is proved, and 2) follows immediately. From what we said above it follows that null series converge uniformly in compact sets of $(0,1)$. (In relation with Orr and null series cf. Appendix of this note).

3. If $u_o \in L^2$, $u_o = \sum c_n \varphi_n(L^2)$, $v_o = \sum c_n \varphi_n(1)$, then (13) satisfies iv) (10). In fact, from [BP], Th. 1, we have

$$\int_0^1 \left|\sum_M^{M'} C_n e^{-|\lambda_n|t} \cdot V_n\right|^2 dx \leqslant (1 + O(1/M)) \sum_M^{M'} \left|C_n\right|^2$$

and from this it follows that $\sum C_n V_n \exp(-|\lambda_n|t)$ converges in $L^2((0,1)\times(0,N))$ to a function $u(x,t)$. Each function $\varphi_n(x) \exp(-|\lambda_n|t)$ is a solution of $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$ on $(-\infty,\infty)\times(0,\infty)$, and therefore $u(x,t)$ is a solution of this hypoelliptic equation in the sense of distributions and must be a $C^\infty(x,t)$ - function. (This is proved later by a straightforward computation). It holds that ([BP], Th. 1):

$$\int_0^1 \left|u_o(x) - u(x,t)\right|^2 dx \leqslant K \sum_1^\infty \left|C_n(1 - e^{-|\lambda_n|t})\right|^2 \xrightarrow[t\to 0]{} 0 .$$

It follows from § 2 that if $u_o(x)$ is not only square summable but is also a continuous function of bounded variation (that satisfies a Hölder condition) on $(0,1)$ then $\sum c_n \varphi_n(x)$ converges (uniformly in compact sets of $(0,1)$) to $u_o(x)$, $0 < x < 1$. From a well-known proper ty of Dirichlet series ([Wi], 2.5) we see that
$u(x,t) = \sum c_n \varphi_n(x) \exp(-|\lambda_n|t)$ tends (uniformly in compact sets of $(0,1)$) to $u_o(x)$ for $t \longrightarrow 0$, $0 < x < 1$.

That is, iv) (10) is verified and in different topologies depending of the regularity properties of $u_o(x)$.

(16) implies that $|V_n^{(j)}(x)| = |s_n|^j \cdot O(1)$ and also that the series in

$$(17) \quad \frac{\partial^{j+k}}{\partial x^j\ \partial t^k} u(x,t) = \sum_1^\infty C_n\{V^{(j)}(x).(-|s_n|^2)^k\} e^{-|s_n|^2 t}$$

converges uniformly in compact sets of $(-\infty < x < \infty) \times (0 < t < \infty)$, (in fact, $C_n = O(1)$, $\{.\} = O(1).|s_n|^{2k+j}$).

In particular we have for $t > 0$ that $u(x,t) = \sum C_n V_n(x).e^{-|\lambda_n|.t}$ pointwise and then that $v(t) = u(1-,t) = u(1,t) = \sum_1^\infty c_n \varphi_n(1) e^{-|\lambda_n|.t}$ $\longrightarrow \sum_1^\infty c_n \varphi_n(1) = v_o$ for $t \longrightarrow 0+$. Thus v) (10) holds.

ii) (10) follows from b) (12) since because of (17) and what we said above it is sufficient to check it term by term.

iii) (10) is proved in an analogous way using i) (10) and a) and c) (12). Thus we have proved i) of the following theorem.

THEOREM 1. i) *There exists a* C^∞ - *solution for problem* (10) *such that* $\|u(.,t) - u_o\|_2 \xrightarrow[t\to 0+]{} 0$.

ii) *There exists at most one solution of* (10) *such that* $u(x,t) \in C^2([0,1] \times (0,\infty))$, $\|u(.,t) - u_o\|_2 \xrightarrow[t\to 0]{} 0$.

Proof of ii). It is enough to prove uniqueness for real solutions. Let us define:

$$G(t) = \int_0^1 u^2(x,t)\, dx .$$

Then: $\frac{1}{2} G'(t) = \int_0^1 \frac{\partial u}{\partial t} . u\, dx = \int_0^1 \frac{\partial^2 u}{\partial x^2} . u\, dx = \frac{\partial u}{\partial x} . u\Big|_0^1 - \int_0^1 (\frac{\partial u}{\partial x})^2\, dx$.

From (10), ii) and iii), we obtain:

$$(18) \quad \frac{1}{2} G'(t) = -\frac{1}{2}\frac{\partial}{\partial t} u^2(x,t)\Big|_{x=1-} - \frac{1-\mu}{\mu}\left(\frac{\partial u}{\partial x}\right)^2\Big|_{x=0+} - \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx .$$

Then $\frac{d}{dt}[G(t) + u^2(1-,t)] \leqslant 0$. Calling $F(t)$ the function inside the brackets, we have, by hypothesis: $F(t) \longrightarrow \|u_o\|_2^2 + v_o^2 \geqslant 0$. Therefore $F(t)$ is a non-negative non-increasing function such that $F(0+) = \|u_o\|_2^2 + v_o$. If $u_o = v_o = 0$ then $F \equiv 0$ and in consequence $u \equiv 0$. Taking into account that problem (10) is linear, ii) follows. QED.

4. From what we said in the proof of ii) Theorem 1 it follows that

$$\int_0^T \int_0^1 u^2(x,t)\, dx\, dt \leq (\|u_o\|^2 + v_o^2).T \; .$$

This implies *continuity on the initial data* in a certain sense. In fact, let us call $W = [0,1] \times [0,T]$. If $u_o^{(n)}(x)$ tends to $u_o(x)$ in $L^2(0,1)$ and $v_o^{(n)}$ tends to v_o then $\iint_W |u^{(n)}(x,t) - u(x,t)|^2\, dx\, dt$ tends to 0 for $n \longrightarrow \infty$.

We call $C_o = \{0\} \times (0,T]$, $C = [0,1] \times \{0\}$, $C_1 = \{1\} \times (0,T]$.

The following result implies *continuity on the initial data* in the ordinary sense.

THEOREM 2. *Assume that* $u_o(x)$ *is continuous on* $[0,1]$ *and in case* $\mu=1$, $u_o(0) = 0$. *Then there exists a constant* K *such that the solution whose existence is asserted in Theorem 1 verifies:*

$$\sup_{\substack{0\leq x\leq 1\\ 0\leq t<\infty}} |u(x,t)| \leq K(\|u_o\|_\infty + |v_o|) \; .$$

In the proof of this theorem we use the following lemmas.

LEMMA 1. *Let* $0 < \mu < 1$, u_o *absolutely continuous with* $u_o' \in L^2(0,1)$. *Then there exists a constant* K *independent of* W *such that*

$$\sup_W |u(x,t)| \leq K(\|u_o\|_\infty + |v_o|) \; . \tag{19}$$

Proof. First we assume $v_o = u_o(1)$. In this case the first series in (14) with coefficients (15), $\sum C_n V_n(x)$, converges uniformly to $u_o(x)$ in $[0,1]$ as is proved by Churchill ([Ch], Th. 2). Therefore u(x,t) is continuous on W ([Wi], 2.5) and since it is a solution of the heat equation the maximum of u on W must occur on $C_o \cup C_1 \cup C$, ([W], p.60). Then ,

$$|u(x,t)| \leq \sup_{C_o\cup C_1\cup C} |u| \quad , \quad (x,t) \in W \; . \tag{20}$$

If $|u(0,t)|$, $0 < t \leq T$, has a positive maximum in a point $\tau \in (0,T]$ then from the boundary condition ii) (10), it follows that $u(0,\tau). \frac{\partial u}{\partial x}(0,\tau) > 0$. In consequence, there exists $\xi \in (0,1)$ such that $|u(\xi,\tau)| > |u(0,\tau)|$, and from (20) we get:

$$\sup_{C_o} |u| < \sup_{C_o\cup C_1\cup C} |u| = \sup_{C_1\cup C} |u| \; . \tag{21}$$

Besides, with the notation of Theorem 1 we have:

$$\|u_o\|_\infty^2 + v_o^2 = F(0+) \geqslant F(t) = G(t) + u^2(1,t) \geqslant u^2(1,t) \ , \tag{22}$$

$$\sup_{C_1} |u| \leqslant \sqrt{\|u_o\|_\infty^2 + |v_o|^2} \leqslant \|u_o\|_\infty + |v_o| . \tag{23}$$

From (20), (21) and (23) we get (19) with K=1. Secondly we assume $u_o(x) = 0$ on $[0,1]$. Then $u(x,t)$ is the solution of (10) obtained from a null series $\sum \alpha_n V_n(x)$ such that $\sum \alpha_n V_n(1) = v_o$. To prove that

$$\sup_W |u(x,t)| \leqslant K' . |v_o| \tag{24}$$

it is enough to show that $u(x,t)$ is bounded when $v_o = 1$ since null series form a one-dimensional linear space.

An examination of the proof of [F], Th. 2, shows that in any case the partial sums of a null series are uniformly bounded. From a well - known property of Dirichlet series it follows that $u(x,t)$ is bounded on W. Since the solution in the general case is the sum of two solutions of the types already considered, (19) follows easily. QED.

LEMMA 2. *Let* $\mu=1$ *and* u_o *absolutely continuous,* $u_o' \in L^2(0,1)$ *and* $u_o(0) = 0$. *Then,* (19) *holds with a constant* K *independent of* W.

Proof. The proof of the preceding lemma can be repeated step by step but taking care of using instead of Churchill's theorem there mentioned the following modification of it.

PROPOSITION 1. *Let* $y_i(x)$, $i = 1,2,\dots$, *be the characteristic functions of the problem*

$$\begin{cases} y'' + (\lambda+q)\, y = 0 \ , \ q \text{ real and continuous on } 0 \leqslant x \leqslant 1 \ , \\ y(0) = 0 \ , \quad a_1\, y(1) + y'(1) + b_1\, y''(1) = 0 \ , \quad b_1 > 0 \ . \end{cases} \tag{25}$$

If $f(x)$ *is absolutely continuous on* $[0,1]$, $f(0) = 0$, *and* $f'(x) \in L^2$, *then the series* $\sum [B(f,y_i)/B(y_i,y_i)]\ y_i(x)$ *converges uniformly in* $[0,1]$ *to* $f(x)$, *where* $B(f,g) = \int_0^1 f(x)\, g(x)\, dx + b_1\, f(1)\, g(1)$.

Proof. The proposition is a consequence of the principle of reflection and the following differential boundary problem for which Churchill's result holds:

$$
(26) \quad \begin{cases} y'' + (\lambda+\tilde{q})y = 0 \quad , \quad -1 \leqslant x \leqslant 1 \quad , \quad \tilde{q}(x) = q(|x|) \; , \\ a_1\, y(-1) - y'(-1) + b_1\, y''(-1) = 0 \quad , \\ a_1\, y(1) + y'(1) + b_1\, y''(1) = 0 \; . \end{cases}
$$

The eigenfunctions of (26) are either even or odd functions. The restrictions to $[0,1]$ of the odd ones are all the characteristic functions of problem (25). If $\tilde{f}(x)$ is odd and equal to $f(x)$ for $0 \leqslant x \leqslant 1$, then Churchill's expansion theorem applies to $\tilde{f}$ and only odd characteristic functions of (26) appear. This yields the proposition. QED.

If $q=0$, proposition 1 holds even when $f(x)$ is continuous of bounded variation on $[0,1]$ as is shown by Miranda in [M], Th. III.

Proof of Theorem 2. Let $u_o^{(n)}(x)$ be absolutely continuous with $(d/dx)\, u_o^{(n)} \in L^2$, $u_o^{(n)}(0) = 0$ if $\mu=1$, such that $\|u_o^{(n)}-u_o\|_\infty \longrightarrow 0$ for $n \longrightarrow \infty$. Let $u(x,t)$ and $u^{(n)}(x,t)$ be the solutions given by i) Th. 1 which correspond to the initial data u_o, v_o and $u_o^{(n)}$, v_o respectively. To prove the theorem it is enough to verify (19) for $u(x,t)$. We know that $\int_0^T dt \int_0^1 |u(x,t) - u^{(n)}(x,t)|^2\, dx$ tends to 0 for $n \longrightarrow \infty$.

Besides, because of lemmas 1 and 2, $u^{(n)}(x,t)$ converges uniformly on W, and necessarily to $u(x,t)$. Then

$$
\sup_W |u(x,t)| = \lim_{n\to\infty} \sup_W |u^{(n)}(x,t)| \leqslant K \lim(\|u_o^{(n)}\|_\infty + |v_o|) =
$$

$$
= K\, (\|u_o\|_\infty + |v_o|) \quad , \quad \text{QED.}
$$

Finally we state a corollary to theorem 2 whose proof we leave to the reader, and which reduces to Th. V of [M] when $\mu=1$.

COROLLARY. *Assume that* $u_o(x)$ *is continuous on* $[0,1]$ *and in case* $\mu=1$, $u_o(0) = 0$. *Then, whenever* $v_o = u_o(1)$, *the solution whose existence is asserted in Theorem 1 is continuous and its defining series* (13) *converges uniformly in* $[0,1] \times [0,\infty)$.

In a forthcoming paper [Z] we discuss another mathematical model from the same point of view as in the present one.

APPENDIX

Here we characterize the expansions to which we referred as Orr expansions, (cf. [BP], Th. 1, (3) and Th. 5).

THEOREM 3. *Assume that* V_k ; $k = 1,2,\dots$ *is a normalized linearly inde-*

pendent set in L^2 *with* g *degrees of freedom such that for any* $v \in L^2$ *there exists an expansion (in* L^2*):* $v = \sum_{k=1}^{\infty} c_k V_k$ *, where each* $c_k = c_k(v)$ *is a continuous linear functional on* v *and has the form*

(27) $$c_j = c_j(v) = (v, V_j)\, \beta_j \quad \textit{whenever} \quad j > j_o ,$$

with β_j *constants (independent of* v*).*
Then the set functionals $\{c_k : k = 1,2,\dots\}$ *is uniquely determined.*

Proof. Let us assume that $\{c_k'\}$ is another set of functionals satisfying the hypothesis. Without loss of generality we can put $j_o = j_o'$. Then for any $v \in L^2$, $\sum_{k=1}^{\infty} (c_k - c_k')(v) V_k = 0$. If $\{\sum_{k=1}^{\infty} N_k^j V_k;\ j=1,2,\dots,g\}$ is a linearly independent set of null series, i.e., the series converge to 0 in L^2 and $\{\{N_k^j ;\ k = 1,2,\dots\};\ j = 1,2,\dots,g\} \subset l^2$ is a linearly independent set, we have:

(28) $$(c_k - c_k')(v) = \sum_{j=1}^{g} a_j(v) N_k^j , \quad \forall\, v \in L^2 , \ \forall\, k .$$

Assume for a moment that there exist $k_1, k_2, \dots, k_g$, $k_i > j_o$, such that

(29) $$\det (N_{k_i}^j) \neq 0 .$$

In this case, for any $v \perp G = [V_{k_1},\dots,V_{k_g}]$, in view of (28),we get

$$\sum_{j=1}^{g} a_j(v) N_{k_i}^j = (\beta_{k_i} - \beta_{k_i}')(v, V_{k_i}) = 0 .$$

Then $a_j(v) = 0 \ \ \forall\, j$ and so

(30) $$c_k'(v) = c_k(v) \quad \forall\ v \in G^\perp = L^2 \ominus G .$$

If $k > J = \max(k_1,\dots,k_g)$ and v is the projection of V_k on $G^\perp$ then $v \neq 0$. Thus, by (30), $\beta_k = \beta_k'$ $\forall\, k > J$. In consequence: $v \in L^2$ implies $0 = \sum_{1}^{J} (c_k - c_k')(v) V_k$. Therefore, $c_k = c_k'$ for $k = 1,2,\dots,J$.

This proves the theorem under hypothesis (29). Let us see that (29) is always satisfied. If not we would have for certain σ and constants γ_s,

(31) $$N_j^\sigma = \sum_{s \neq \sigma} \gamma_s N_j^s \qquad \forall\ j > j_o .$$

Then

$$0 = \sum_{j=1}^{\infty} (N_j^\sigma - \sum_{s\neq\sigma} \gamma_s N_j^s) V_j = \sum_{j=1}^{j_o} (N_j^\sigma - \sum_{s\neq\sigma} \gamma_s N_j^s) V_j \text{ , and in conse-}$$

quence $N_j^\sigma - \sum_{s\neq\sigma} \gamma_s N_j^s = 0$ for $j \leq j_o$. This together with (31) shows that N^σ depends linearly from $\{N^s: s \neq \sigma\}$, a contradiction. QED.

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