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ON THE FRACTIONAL DIFFERENTIATION OF THE COMMUTATOR OF THE HILBERT TRANSFORM *

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT.

In this paper I will derive a result which extends a well known result due to C. Segovia and R. L. Wheeden (see [5]).

Let a(x) be a Bessel Potential of order α, $0 < \alpha < 1$, of a function g(x) belonging to $L^p(R)$, $1 < p \leqslant \infty$ and f(x) a function belonging to $L^q(R)$, $1 < q < \infty$. Consider now the commutator of the Hilbert transform:

1.1) $$aH(f) - H(af) = p.v \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{(x - y)} f(y)\, dy .$$

Call F = aH(f) - H(af) and define the fractional derivative of order α of F in the form:

1.2) $$D^{\alpha}F(x) = p.v \int_{-\infty}^{\infty} \frac{F(x+h) - F(x)}{|h|^{1+\alpha}}\, dh .$$

The above limit is understood pointwise a.e.

Define also the maximal derivative of order α as:

1.3) $$\overset{*}{F}{}^{(\alpha)}(x) = \sup_{\varepsilon>0} \left| \int_{|h|>\varepsilon} \frac{F(x+h) - F(x)}{|h|^{1+\alpha}}\, dh \right| .$$

C. Segovia and R. L. Wheeden proved in [5] that if $\frac{1}{p} + \frac{1}{q} < 1$, then:

1.4) $$\|D^{\alpha}F\|_r < C_{p,q} \|g\|_p \|f\|_q$$

where $r^{-1} = p^{-1} + q^{-1}$; $1 < p \leqslant \infty$, $1 < q < \infty$.

The purpose of this paper is to extend the above result to larger ran-

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ges of p and q, more precisely:

1.5) $$0 < \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha .$$

The main results are contained in the following

THEOREM. *Let* $1 < p < \infty$, $1 < q < \infty$, *and* $\frac{1}{p} + \frac{1}{q} < 1 + \alpha$. *Let* r *be given by* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *then*

i) $$\left(\int_{-\infty}^{\infty} |D^{\alpha}F|^r \, dx\right)^{1/r} < C_1 \ \|a\|_{p,\alpha} \ \|f\|_q$$

ii) $$\left(\int_{-\infty}^{\infty} (\overset{*}{F}{}^{(\alpha)})^r \, dx\right)^{1/r} < C_2 \ \|a\|_{p,\alpha} \ \|f\|_q \ .$$

If $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, *then:*

iii) $$|E\ (|D^{\alpha}F| > \lambda)| < \frac{C_3}{\lambda^r} \ \|a\|^r_{p,\alpha} \ \|f\|^r_q$$

iv) $$|E\ (\overset{*}{F}{}^{(\alpha)} > \lambda)| < \frac{C_4}{\lambda^r} \ \|a\|^r_{p,\alpha} \ \|f\|^r_q \ .$$

The constants C_i *do not depend on* λ , f, *or* a.

2. CONSTRUCTION OF THE SET G(λ) AND RELATED ESTIMATES.

Consider $a(x) = \int_{-\infty}^{\infty} G_\alpha(x-y)\, g(y)\, dy$, where G_α is the Bessel kernel of order α in the real line (see [7] pp. 131-132) and $g \geq 0$, $g \in L^p(R)$. Introduce the maximal function $M_t(\theta)$ as

$$\sup_{\varepsilon>0} \left[\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |\theta|^t \, dy \right]^{1/t} \quad ; \quad t > 0 \ .$$

Call $f^{\urcorner} = |f| + |\tilde{f}|$, where $\tilde{f}$ stands for the Hilbert transform of f. The set $G(\lambda)$ is defined to be:

2.1) $$G(\lambda) = \{x: M_{p_0}(g) > \lambda^{r/p}\} \cup \{x: M_{q_0}(f^{\urcorner}) > \lambda^{r/q}\} \ .$$

Calling $D_1(s)$ the distribution function of $M_{p_0}(g)$ and $D_2(s)$ that of $M_{q_0}(f^{\urcorner})$, we have:

2.2) $$|G(\lambda)| \leq D_1(\lambda^{r/p}) + D_2(\lambda^{r/q}) \ .$$

If $\frac{1}{p} + \frac{1}{q} < 1 + \alpha$, p_0 and q_0 are selected so that:

2.3) $$p_0 < p \quad , \quad q_0 < q \quad , \quad \frac{1}{p_0} + \frac{1}{q_0} = 1 + \alpha \ .$$

If $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$, then, we select $p = p_0$ and $q = q_0$.

Let $\overset{\infty}{\underset{1}{\cup}} I_k$ be a covering in the sense of Whitney for $G(\lambda)$, that is:

2.4) $$|I_k| \leqslant d(I_k, G') \leqslant 4 \ |I_k| \quad ,$$

$$\overset{o}{I}_i \cap \overset{o}{I}_j = \emptyset \quad \text{if} \quad i \neq j$$

d stands for distance and G' for the complement of G. For details see [7] chapters VI and I.

As a consequence of the above properties we have:

2.5) $$\left(\frac{1}{|\ell I_k|} \int_{\ell I_k} g^{p_0} \, dy \right)^{1/p_0} < C_1 \ \lambda^{r/p}$$

2.6) $$\left(\frac{1}{|\ell I_k|} \int_{\ell I_k} (f^{\neg})^{q_0} \, dy \right)^{1/q_0} < C_2 \ \lambda^{r/q}$$

Where ℓI_k stands, as usual, for the dialation of I_k ℓ times about its center, ℓ is taken bounded, that is $1 \leqslant \ell \leqslant 8$.

The constants C_1 and C_2 are universal, they do not depend on λ, f, g, k or ℓ.

Our next step will be to decompose g as $g_1 + g_2$; $g_1 = g$ on R - G and $g_1 = \mu_k^{(1)} + \mu_k^{(2)} (x - x_k)$ on I_k, where:

2.7) $$\mu_k^{(1)} = \frac{1}{|I_k|} \int_{I_k} g(x) \, dx \qquad \mu_k^{(2)} = \frac{3}{4} \frac{1}{|I_k|^3} \int_{I_k} (x - x_k) \ g(x) \, dx$$

in the above formula the x_k are the centers of I_k.

Clearly:

2.8) $$|g_1| \leqslant c \ \lambda^{r/p} \qquad \text{a.e.}$$

Here, c is an universal constant.

Call $\phi_k(x) = [g - \mu_k^{(1)} - \mu_k^{(2)} (x - x_k)] \ \psi_k(x)$, ψ_k is the characteristic function of I_k.

We define g_2 as:

2.9) $$g_2(x) = \sum_1^{\infty} \phi_k(x) \ .$$

Thus:

2.10) $$\int_{I_k} g_2 \, dx = \int_{I_k} \phi_k \, dx = 0$$

2.11) $$\int_{I_k} (x-x_k) \, g_2 dx = \int_{I_k} (x-x_k) \, \phi_k \, dx = 0$$

2.12) $$\int_{I_k} |g_2| \, dx \leqslant c \int_{I_k} g \, dx \leqslant c \, \lambda^{r/p} \, |I_k| \, .$$

Returning to the Bessel kernel $G_\alpha(x)$, we have the estimates:

2.13) $$|G_\alpha(x)| < \frac{c}{|x|^{1-\alpha}} \quad , \quad |G_\alpha^{(\ell)}(x)| < \frac{c}{|x|^{\ell+1-\alpha}} \quad , \ell = 1,2 \, .$$

Call $a_k(x) = G_\alpha * \phi_k$. Using 2.10 to 2.13 we obtain the estimates:

2.14) $$|a_k(x)| \leqslant c \, \lambda^{r/p} \frac{|I_k|^3}{|I_k|^{3-\alpha} + |x-x_k|^{3-\alpha}}$$

for $x; \; |x-x_k| > \frac{3}{2} \, |I_k|$.

ESTIMATES FOR $\sum_1^\infty \int_{|x-y|>\varepsilon} \frac{|a_k(x) - a_k(y)|}{|x-y|^{1+\alpha}} \, f^\urcorner(y) \, dy$ WHERE

$x \in R - \bigcup_1^\infty 6 \, I_k$.

The above sum is dominated by

2.15) $$\sum_1^\infty \int_{3I_k} \frac{|a_k(x) - a_k(y)|}{|x-y|^{1+\alpha}} \, f^\urcorner(y) dy + \sum_1^\infty \int_{(3I_k)'} \frac{|a_k(x)-a_k(y)|}{|x-y|^{1+\alpha}} \, f^\urcorner(y) \, dy.$$

$(3 \, I_k)'$ stands for the complement of $3 \, I_k$.

The first of the series of 2.15) is readily seen to be dominated by:

2.16) $$\sum_1^\infty \frac{|a_k(x)|}{|I_k|^{1+\alpha} + |x-x_k|^{1+\alpha}} \int_{3I_k} f^\urcorner \, dy + \sum_1^\infty \frac{1}{|I_k|^{1+\alpha} + |x-x_k|^{1+\alpha}} \int_{3I_k} |a_k| f^\urcorner \, dy \, .$$

Using 2.14, the fact that $\int_{3I_k} f^\urcorner \, dy < c \, \lambda^{r/q} \, |I_k|$ and the following potential inequality:

2.17) $$\left[\frac{1}{|3I_k|} \int_{3I_k} \left|a_k |I_k|^{-\alpha}\right|^{s_0} dy\right]^{1/s_0} \leqslant c \left(\frac{1}{|I_k|} \int_{I_k} |\phi_k|^{p_0} dy\right)^{1/p_0} \quad \frac{1}{s_0} =$$

$$= \frac{1}{p_0} - \alpha$$

one can see that 2.16 is dominated by:

$$2.18)\qquad \sum_1^\infty c\ .\ \lambda \frac{|I_k|^4}{|I_k|^4 + |x-x_k|^4} +$$

$$+ \ c\ \lambda^{r/p} \sum_1^\infty \frac{|I_k|^{1+\alpha}}{|I_k|^{1+\alpha}+|x-x_k|^{1+\alpha}} \left(\frac{1}{|3I_k|} \int_{I_k} (f^\urcorner)^{q_0}\, dy \right)^{1/q_0} \leqslant$$

$$\leqslant c\ \lambda \sum_1^\infty \left[\frac{|I_k|^4}{|I_k|^4+|x-x_k|^4} + \frac{|I_k|^{1+\alpha}}{|I_k|^{1+\alpha}+|x-x_k|^{1+\alpha}} \right] .$$

Our next step will be to deal with te second term of 2.15. Let us use in this case the fact that $0 \leqslant f^\urcorner \leqslant \lambda^{r/q}$ a.e. in $R - \bigcup_1^\infty 3\ I_k$.

Consequently:

$$2.19)\quad \sum_1^\infty \int_{(3I_k)'} \frac{|a_k(x)-a_k(y)|}{|x-y|^{1+\alpha}}\ f^\urcorner\, dy \leqslant \lambda^{r/q} \sum_1^\infty \int_{(3I_k)} \frac{|a_k(x)-a_k(y)|}{|x-y|^{1+\alpha}}\, dy\ .$$

We have also the estimates:

$$2.20)\qquad \int_{(3I_k)'\cap\{|x-y|>|I_k|\}} \frac{|a_k(x)-a_k(y)|}{|x-y|^{1+\alpha}}\, dy \leqslant c\ \lambda^{r/p} \frac{|I_k|^{3-\alpha}}{|I_k|^{3-\alpha}+|x-x_k|^{3-\alpha}} +$$

$$+ \ c\ \lambda^{r/p} \int_{|x-y|>|I_k|} \frac{1}{|x-y|^{1+\alpha}} \frac{|I_k|^3}{|I_k|^{3-\alpha}+|y-x_k|^{3-\alpha}}\, dy\ .$$

Now, we are going to use the fact that $|x-x_k| > 6\ |I_k|$, the mean value properties of ϕ_k and obtain:

$$2.21)\quad \frac{|a_k(x)-a_k(y)|}{|x-y|} \leqslant c\ \lambda^{r/p} \frac{|I_k|^2}{|I_k|^{3-\alpha}+|x-x_k|^{3-\alpha}} \quad \text{for}\quad |x-y| < |I_k|\ .$$

On account of 2.21), we get the domination:

$$2.22)\qquad \int_{(3I_k)'\cap\{|x-y|<|I_k|\}} \frac{|a_k(x)-a_k(y)|}{|x-y|^{1+\alpha}}\, dy \leqslant c\ \lambda^{r/p} \frac{|I_k|^{3-\alpha}}{|I_k|^{3-\alpha}+|x-x_k|^{3-\alpha}}$$

3. PROOF OF THE MAIN RESULT.

Call $a_1 = G_\alpha * g_1$, $a_2 = G_\alpha * g_2$ and $F_1 = a_1 H(f) - H(a_1 f)$, $F_2 = a_2 H(f) - H(a_2 f)$. From [5], we have:

$$3.1) \qquad \left\| \int_{|h|>\varepsilon} \frac{F_1(x+h) + F_1(x)}{|h|^{1+\alpha}} \, dh \right\|_q \leq c_q \|g_1\|_\infty \|f\|_q \leq c_q \lambda^{r/p} \|f\|_q$$

Applying now results in [6] and [8], we express:

$$3.2) \qquad \int_{|h|>\varepsilon} \frac{F_1(x+h) - F_1(x)}{|h|^{1+\alpha}} \, dh = \int_{-\infty}^{\infty} \varepsilon^{-1} K(\varepsilon^{-1} y) \, \gamma(x-y) \, dy$$

where, $|K| \leq c \, |x|^{\alpha-1}$ if $|x| \leq 1$, $|K| \leq |x|^{\alpha-3}$ if $|x| \geq 1$ and $\|\gamma\|_q \leq c \, \|g_1\|_\infty \|f\|_q \leq c \, \lambda^{r/p} \|f\|_q$.

Call now $D_g[f,\lambda]$ the distribution function of $\overset{*}{F}^{(\alpha)}$ and define:

$$3.3) \qquad \overset{*}{D}(f,\lambda) = \sup_{g: \|g\|_\infty \leq 1} D_g[f,\lambda] .$$

From lemma 1.5 in [2] (p. 146-148) and 3.2 we get:

$$3.4) \qquad \int_0^\infty \overset{*}{D}(f,\lambda) \, \lambda^{q-1} \, d\lambda \leq C_q \|f\|_q^q .$$

From the very definition of $\overset{*}{F}_1^{(\alpha)}$ we have:

$$3.5) \qquad |E(\overset{*}{F}_1^{(\alpha)} > \lambda)| < C \, \overset{*}{D}(f,\lambda^{r/q}) .$$

Let us return to $F_2(x)$ and consider:

$$3.6) \qquad \int_{|h|>\varepsilon} \frac{F_2(x+h) - F_2(x)}{|h|^{1+\alpha}} \, dh .$$

According to [5] see page 348, the above integral is expressible as

$$3.7) \qquad - \int_{|x-y|>\varepsilon} \frac{a_2(x) - a_2(y)}{|x-y|^{1+\alpha}} \, \tilde{f}(y) \, dy \; +$$

$$+ \, b \int_{|x-y|>\varepsilon} \frac{a_2(x) - a_2(y)}{|x-y|^{1+\alpha}} \, sg(x-y) \, f(y) \, dy + T_3 + T_4 .$$

Where T_3 and T_4 are dominated by (see [5] p. 352)

3.8) $$c \; \|g_1\|_\infty \, M_1(f)\,(x) \leq c\, \lambda^{r/p}\, M_1(f)(x) \; .$$

The domination does not depend on $\varepsilon > 0$.

The measure of the set where 3.8) exceeds λ does not exceed:

3.9) $$c\, D_2(\lambda^{r/q}) \; .$$

Call $10\, G(\lambda) = \bigcup_1^\infty 10\, I_k$; where $10\, I_k$ is the dialation of I_k 10 times about its center.

According to the estimates in section 2, if x belongs to the complement of $10\, G(\lambda)$ then

3.10) $$\sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{a_2(x)-a_2(y)}{|x-y|^{1+\alpha}}\, \tilde{f}(y)\, dy \right| +$$

$$\sup_{\varepsilon>0} \left| b \int_{|x-y|>\varepsilon} \frac{a_2(x)-a_2(y)}{|x-y|^{1+\alpha}}\, f(y)\, dy \right|$$

does not exceed:

3.11) $$c\,\lambda \sum_{k=1}^{\infty} \frac{|I_k|^4}{|I_k|^4+|x-x_k|^4} + \frac{|I_k|^{1+\alpha}}{|I_k|^{1+\alpha}+|x-x_k|^{1+\alpha}} + \frac{|I_k|^{3-\alpha}}{|I_k|^{3-\alpha}+|x-x_k|^{3-\alpha}} +$$

$$+ \; c\,\lambda \sum_{k=1}^{\infty} \int_{|x-y|>|I_k|} \frac{1}{|x-y|^{1+\alpha}} \; \frac{|I_k|^3}{|I_k|^{3-\alpha}+|y-x_k|^{3-\alpha}}\, dy \; .$$

Integrating 3.12 over the whole real line we see that its integral does not exceed $c\,\lambda\, |G(\lambda)|$.

Consequently, the measure of the set where 3.11 exceeds λ is less than $C\, |G(\lambda)|$.

If $p = p_0$ and $q = q_0$ the proof stops here.

If $p > p_0$ and $q > q_0$, using the estimates already found we have:

3,12) $$\int_0^\infty |E\, (\overset{*}{F}{}^{(\alpha)} > \lambda)|\, \lambda^{r-1}\, d\lambda \leq$$

$$\leq C\Big(\int_0^\infty \overset{*}{D}(f,\lambda^{r/q})\, \lambda^{r-1}\, d\lambda + \int_0^\infty D_1(\lambda^{r/p})\, \lambda^{r-1}\, d\lambda \; +$$

$$+ \int_0^\infty D_2(\lambda^{r/q})\, \lambda^{r-1}\, d\lambda\Big) \leq C\, (\|g\|_p^p + \|f\|_q^q) \; .$$

This finishes the proof.

BIBLIOGRAPHY

[1] A.P. CALDERON, *Commutators of Singular Integrals*, Proc. Nat. A Sci. U.S.A., 53, 1965, 1092-1099.

[2] C.P. CALDERON, *On Commutators of Singular Integrals*, Studia Ma. 53 (1975), 139-140.

[3] C.P. CALDERON, *On a Singular Integral*, Studia Math. to appear.

[4] R.R. COIFMAN and Y. MEYER, *On Commutators of Singular Integral and Bilinear Singular Integrals*, Trans. Amer. Math. Socie. 212 (1975) 315-332.

[5] C. SEGOVIA and R.L. WHEEDEN, *Fractional Differentiation of the Hilbert Transform*, Journal of Functional Analysis Vol.8, N°3, 1971 (341-358).

[6] E.M. STEIN, *The Characterization of Functions Arising as Potenti als I*. Bull. Amer.Math. Soc. 67 (1961), 102-104; II ibid, 68 (1962) 577-582.

[7] E.M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey 1970.

[8] R.L. WHEEDEN, *On Hypersingular Integrals and Lebesgue Spaces of Differentiable Functions*, Trans. Amer. Math. Soc. 134 (1968), 421-436.

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