Revista de la
Unión Matemática Argentina
Volumen 29, 1980.

THE ISOMETRIES OF H^p AND THEIR LIE ALGEBRA

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The objective of this article is to present some results about the classification of one-parameter groups of isometries of H^p and to calculate their Lie algebra using hermitian operators.

It is known that all conformal self-maps of $|z| < 1$ can be written as

(1) $$\varphi(z) = \frac{az+b}{\overline{b}z+\overline{a}}$$

where a,b are complex numbers and

(2) $$|a|^2 - |b|^2 = 1.$$

The representation (1) is unique up to sign: if a,b are replaced by -a,-b we get the same φ, and only in this case. Thus the group G of conformal self-maps of $|z| < 1$ is isomorphic to the group PQU(2) = = QU(2)/ ± I where QU(2) (see [4]) is the group of 2x2 complex matrices

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$$

with condition (2).

Suppose now that $\{\varphi_t\}$ is a one-parameter subgroup of G represented by

(3) $$\varphi_t(z) = \frac{a(t)z + b(t)}{\overline{b(t)}z + \overline{a(t)}}$$

with (2) holding for each t. The composition rule $\varphi_{t+s} = \varphi_t\varphi_s$ implies

$$a(t+s) = a(t)a(s) + b(t)\overline{b(s)}$$
$$b(t+s) = a(t)b(s) + b(t)\overline{a(s)}$$

Taking derivatives with respect to s yields:

(4) $$a' = \alpha a + \beta b$$
$$b' = \overline{\beta}a + \overline{\alpha}b$$

where $\alpha = a'(0)$, $\bar{\beta} = b'(0)$. In order to solve (4) we need $\exp(tM)$ where

$$M = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

First we differentiate $a(t)\overline{a(t)} - b(t)\overline{b(t)} = 1$ to get $\alpha + \bar{\alpha} = 0$ so that $\bar{\alpha} = -\alpha$ and therefore

$$M = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix}$$

It follows that if $\zeta^2 = |\beta|^2 + \alpha^2 = |\beta|^2 - |\alpha|^2$ we have

$$M^{2n} = \zeta^{2n} I_2$$
$$M^{2n+1} = \zeta^{2n} M$$

where I_2 is the 2x2 identity matrix.

Thus

$$\exp(tM) = I_2 + tM + \frac{t^2}{2!}\zeta^2 I_2 + \frac{t^3}{3!}\zeta^2 M + \ldots =$$
$$= \left(1 + \frac{(t\zeta)^2}{2!} + \frac{(t\zeta)^4}{4!} + \ldots\right) I_2 +$$
$$+ \left(t + \frac{t^3\zeta^2}{3!} + \frac{t^5\zeta^4}{5!} + \ldots\right)M =$$
$$= \operatorname{Cosh}(t\zeta) I_2 + \zeta^{-1} \operatorname{Sinh}(t\zeta) M$$

(and then, quite clearly, the choice of square root of ζ^2 is irrelevant for these formulas).

We conclude that the solution of (4) with initial conditions $a(0) = 1$, $b(0) = 0$ is

$$\text{(5)} \qquad \begin{aligned} a(t) &= \operatorname{Cosh}(t\zeta) + \alpha\zeta^{-1} \operatorname{Sinh}(t\zeta) \\ b(t) &= \bar{\beta}\, \zeta^{-1} \operatorname{Sinh}(t\zeta) \end{aligned}$$

unless $\zeta = 0$ in wich case $\zeta^{-1}\operatorname{Sinh}(t\zeta)$ is to be interpreted as t. The parameters α and β satisfy $(\alpha,\beta) \in i\mathbb{R} \times \mathbb{C}$ and $\zeta^2 = |\beta|^2 - |\alpha|^2$.

Using (3) and (4) we get that $q(z) = \frac{\partial}{\partial t}\varphi_t(z)|_{t=0}$ is the polynomial

$$q(z) = \frac{\partial}{\partial t}\varphi_t(z)|_{t=0} = -\beta z^2 + 2\alpha z + \bar{\beta}$$

whose roots are

$$(6)\qquad \begin{array}{ll} \tau_{1,2} = (\alpha \pm \zeta)/\beta\ , & \text{if}\quad \beta \neq 0, \\ \tau_1,\tau_2 = 0 & \text{, if}\quad \beta = 0. \end{array}$$

One readily verifies that $q(1/\bar{z}) = -\overline{q(z)}/\bar{z}^2$ so that if $|\tau_1| \neq 1$ then necessarily $\tau_2 = 1/\bar{\tau}_1$ (or else τ_1,τ_2 and $1/\bar{\tau}_1$ would all be roots). It is also clear that $|\tau_1|\ |\tau_2| = 1$ when $\beta \neq 0$, since $\tau_1\tau_2 = -\bar{\beta}/\beta$. These considerations simplify the study of $\{\varphi_t\}$ and q done in [1] and [2] (where q was introduced and called the "invariance polynomial" of $\{\varphi_t\}$).

Consider now the representation of pairs $(\alpha,\beta) \in i\mathbb{R} \times \mathbb{C}$ by $\alpha = ix_1$ $\beta = x_2 + ix_3$ with x_1,x_2 and x_3 real. Then the sign of $\zeta^2 = |\beta|^2 - |\alpha|^2 = -x_1^2 + x_2^2 + x_3^2$ changes on a cone $-x_1^2 + x_2^2 + x_3^2 = 0$ and correspondingly we have:

$\zeta^2 < 0$; this means that $|\alpha| > |\beta|$ and α,ζ are both purely imaginary. Thus $|\alpha + \zeta| \neq |\alpha - \zeta|$ and therefore $|\tau_1| \neq |\tau_2|$ which can only occur if $|\tau_1| < 1 < |\tau_2|$ or $|\tau_2| < 1 < |\tau_1|$. In either case q has only one root in $|z| < 1$. $\zeta^2 = 0$; in this case we get $\tau_1 = \tau_2$ from (6) and so also $|\tau_1| = |\tau_2| = 1$. In other words, the roots of q coincide and belong to $|z| = 1$. $\zeta^2 > 0$; in this case ζ is real and α being purely imaginary we get $|\alpha + \beta| = |\alpha - \beta|$ wich implies $|\tau_1| = |\tau_2| = 1$ with $\tau_1 \neq \tau_2$. Thus q has two distinct roots, both in $|z| = 1$.

These three possibilities are referred to as type (i), type (ii), and type (iii), respectively.

In the following we will exclude the case $\alpha = \beta = 0$ (or $\{\varphi_t\} = \{id\}$).

Using the proof of (1.7) in [2] and the discussion above we get the following theorem (that gives additional information or generalizes (1.5), (1.7) and (1.10) in [2] and simplifies (1.6) in [1] (cf. also (1.5) in [1]).

THEOREM 1. *There are three mutually exclusive types of one-parameter groups of conformal self-maps of the disc* $|z| < 1$ *(labeled types (i), (ii) and (iii)), parametrized by three real parameters* x_1,x_2,x_3, *each type corresponding to the set where* $\zeta^2 = -x_1^2 + x_2^2 + x_3^2$ *satisfies* $\zeta^2 < 0$, $\zeta^2 = 0$ *or* $\zeta^2 > 0$, *respectively. For each point* (x_1,x_2,x_3) *the corresponding one-parameter group is given by (5) above, where* $\alpha = ix_1$, *and* $\beta = x_2 + ix_3$, *and the fixed points are given by*

$$\tau_{1,2} = (ix_1 \pm \zeta)/(x_2 + ix_3)$$

$(\tau_1 = \tau_2 = 0$ *if* $x_2 = x_3 = 0)$. *Further* τ_1,τ_2 *are the roots of the invariance polynomial*

$$q(z) = -(x_2 + ix_3)z^3 + 2x_1iz + (x_2 - ix_3)\ .$$

An alternative characterization of $\{\varphi_t\}$ can be obtained from this as follows. First a matrix

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$$

can be represented by $(g_1,g_2,g_3,g_4) \in R^4$ with $a = g_1 + ig_2$, $b = g_3 + ig_4$. Then the group QU(2) can be replaced by the hyperboloid $V = \{g_1^2 + g_2^2 - g_3^2 - g_4^2 = 1\}$, which then becomes a two sheeted cover of G. The one-parameter subgroups of G appear as curves in V, as follows:

type (i): write $\zeta = i\eta$ with η real (η agrees with the "angular velocity" of $\{\varphi_t\}$; see [2] paragraph preceding (1.13)). Then $\{\varphi_t\}$ corresponds to

$$g_1(t) = \cos(\eta t)$$
$$(g_2(t),g_3(t),g_4(t)) = \eta^{-1} \sin(\eta t)\, U$$

where $U \in R^3$,

type (ii): $\{\varphi_t\}$ corresponds to

$$g_1(t) = 1$$
$$(g_2(t),g_3(t),g_4(t)) = tU,$$

type (iii): $\{\varphi_t\}$ corresponds to

$$g_1(t) = \mathrm{Cosh}\ (\zeta t)$$
$$(g_2(t),g_3(t),g_4(t)) = \zeta^{-1}\ \mathrm{Sinh}\ (\zeta t)U.$$

Let now H^p, $1 \leqslant p < \infty$, $p \neq 2$, denote the Hardy space of the disc. It is known (see [1], Th. (2.1)) that each one-parameter group of isometries $\{T_t\}$ of H^p has the form

$$T_t f = \Phi_t f(\varphi_t), \qquad f \in H^p,$$

where $\{\varphi_t\}$ is a one-parameter group in G. Thus the characterization and classification of the theorem above carries over to the $\{T_t\}$. Further, if U^p denotes the group of all isometries of H^p onto H^p , it is proved in [3], Th. 2.1, that U^R is a 4-dimensional Lie group and there is an exact sequence

$$(7) \qquad 1 \longrightarrow \mathbf{T} \xrightarrow{\ \mu\ } U^p \xrightarrow{\ \tau\ } G \longrightarrow 1$$

where $\mathbf{T}$ is the circle group $\mathbf{T} = \mathbf{R/Z}$,$\mu(\theta) = e^{i\theta} I$ and $\tau T = \varphi^{-1}$ if $Tf = \Phi f(\varphi)$. It is also known that the extension (7) does not split, but it suffices to calculate the Lie algebra $L(U^p)$ of U^p as the product

$$L(U^p) = R \oplus L(G)$$

where $L(G)$ is the Lie algebra of G. Since G is isomorphic to $QU(2)/\pm I$, then $L(G)$ is isomorphic to the Lie algebra $qu(2)$ of matrices

$$(8) \qquad M = \begin{pmatrix} \alpha & \beta \\ \bar\beta & -\alpha \end{pmatrix}$$

with $(\alpha,\beta) \in i\mathbf{R} \times \mathbf{C}$ the correspondence being given by assigning to the tangent vector $v = \left.\frac{d\varphi_t}{dt}\right|_{t=0}$, where $\{\varphi_t\}$ is given by (3), the matrix $M(v)$ with $\alpha = a'(0)$, $\beta = \overline{b'(0)}$. (Of course $QU(2) \cong SL(2,\mathbf{R})$ implies also $qu(2) \cong sl(2,\mathbf{R})$).

Our next goal is to interpret $L(U^p)$ in terms of hermitian operators in H^p. Let us recall that an operator A in H^p is *hermitian* if $A = -i \left.\frac{dT_t}{dt}\right|_{t=0}$ for some one-parameter group $\{T_t\}$ in U^p (notice that A will be unbounded, in general). According to (2.4) in [2]:

$$iAf = i\lambda f + gf' + (1/p)q'f$$

where primes denote d/dz, q is the invariance polynomial of $\{\varphi_t\}$ and λ is a real constant. But then the Lie algebra $L(U^p)$ can be identified to the space V^p of operators

$$V(\lambda,q)f = i\lambda f + qf' + (1/p)q'f$$

where $q(z) = -\beta t^2 + 2\alpha z + \bar\beta$, with $(\alpha,\beta) \in i\mathbf{R} \times \mathbf{C}$ and $\lambda \in \mathbf{R}$.

THEOREM 2. *The Lie algebra* $L(U^p)$ *is isomorphic to the algebra* V^p *of all operators* $V(\lambda,q)$ *with the bracket defined by operator composition* $[V_1,V_2] = V_1V_2 - V_2V_1$.

Proof. First we calculate

$$\begin{aligned}
&(V(\lambda_1,q_1) - i\lambda_1 I)(V(\lambda_2,q_2) - i\lambda_2 I)f = \\
&= (V(\lambda_1,q_1) - i\lambda_1 I)(q_2f' + (1/p)q_2'f) = \\
&= q_1(q_2f' + (1/p)q_2'f)' + (1/p)q_1'(q_2f' + (1/p)q_2'f) = \\
&= q_1q_2f'' + \{q_1q_2' + (1/p)(q_1q_2)'\}f' + (1/p^2)q_1'q_2'f,
\end{aligned}$$

so that, using $[\lambda I,T] = 0$ for all λ, T we get

(9) $$[V(\lambda_1,q_1),V(\lambda_2,q_2)] = (q_1q_2' - q_1'q_2)f' + (1/p)(q_1q_2'' - q_1''q_2)f.$$

Observe now that if we put $[\lambda_1,\lambda_2] = 0$ and $[q_1,q_2] = q_1q_2' - q_1'q_2$, then (9) reads

$$[V(\lambda_1,q_1),V(\lambda_2,q_2)] = V([\lambda_1,\lambda_2'],[q_1,q_2]).$$

A routine calculation shows that $[q_1,q_2]$ is again an invariance polynomial (and that $[q_1,q_2]$ is a Lie bracket, although this follows from (9) and the fact that $[T,S]$ is a Lie bracket for operators). Then V^p is isomorphic via $(\lambda,q)\longmapsto V(\lambda,q)$ to $R \oplus Q$ where $Q = \{q\}$ has the bracket $[q_1,q_2] = q_1q_2' - q_1'q_2$. The proof can be finished by obser ving that qu(2) and Q are also isomorphic under the correspondence $q(M) = -\beta z^2 + 2\alpha z + \bar{\beta}$ for M as in (8), i.e., by showing that $q[M_1,M_2] = [q(M_1),q(M_2)]$. This is a routine calculation.

We close with the remark that the identity representation of V^p in H^p is irreducible (see [3]).

This research was partly supported by the Consiglio Nazionale **delle** Ricerche while the author was a Visiting Professor at the **Istituto** Ulisse Dini, in Florence, Italy.

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Recibido en marzo de 1979.