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ON SOME HETERODOX DISTRIBUTIONAL MULTIPLICATIVE PRODUCTS

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Dedicated to Professor Luis A. Santaló

1. Let $g(t)$ $(-\infty < t < \infty)$ be a real function with the following properties:

a) $g(t) \in C^\infty$; b) $\operatorname{supp} g(t) = [-1,1]$; c) $g(t) \geqslant 0$; d) $g(t) = g(-t)$;

e) $g(t)$ is increasing for $-1 \leqslant t \leqslant 0$, and decreasing for $0 \leqslant t \leqslant 1$;

f) $\int_{-\infty}^{\infty} g(t)\, dt = 1$.

Let us consider the sequence $(n = 1,2,\ldots)$

$$g_n(t) = n\, g(nt) \quad . \tag{1.1}$$

It is well known that $g_n(t)$ is a (very smooth) singular kernel; i.e.

$$g_n(t) \xrightarrow[n\to\infty]{} \delta \quad . \tag{1.2}$$

Let S and T be two distributions. We shall define the multiplicative product S.T by the formula

$$S.T = \lim_{n\to\infty} \{S * g_n(t)\}\{T * g_n(t)\} \tag{1.3}$$

if the limit exists. Here the symbol * means, as usual, convolution.

This definition is a slight variant of the definition introduced by Mikusinski in [21].

Our purpose is to evaluate, using definition (1.3), several "heterodox" products that appear in applications, especially in the quantum theory of fields.

Our new results are formulae (2.6), (3.1), (4.1), (4.2), (6.0), (7.1), (7.2), (8.1), (8.2) (9.1), (4,bis.3) and (4,bis.4). The proofs of theorems 5, 6, 7, 10 and 11 are also new.

2. Two distributions will mainly appear in our considerations: $\delta^{[k]}$, which is the k-th derivative of the δ distribution, and x^{-k} $(k = 1,2,\ldots)$ which is defined by the formula

$$x^{-k} = \begin{cases} x_+^{-k} + x_-^{-k} & \text{when } k \text{ is even ,} \\ x_+^{-k} - x_-^{-k} & \text{when } k \text{ is odd .} \end{cases} \tag{2.1}$$

For the definition of x_+^{-k} and x_-^{-k} cf. [2], p. 48. An equivalent definition of x^{-k} is ($\psi \in C_0^\infty$):

$$\langle x^{-2k}, \psi(x) \rangle = \int_0^\infty x^{-2k}\{\psi(x) + \psi(-x) - 2\,[\psi(0) + \frac{x^2}{2!}\psi^{[2]}(0) + \ldots +$$

$$+ \frac{x^{2k-2}}{(2k-2)!}\psi^{[2k-2]}(0)]\}\,dx \quad ; \tag{2.2}$$

$$\langle x^{-2k-1}, \psi(x) \rangle = \int_0^\infty x^{-2k-1}\{\psi(x) - \psi(-x) - 2\,[x\psi^{[1]}(0) + \frac{x^3}{3!}\psi^{[3]}(0) +$$

$$+ \ldots + \frac{x^{2k-1}}{(2k-1)!}\psi^{[2k-1]}(0)]\} \,. \tag{2.3}$$

For future reference we state the known formula

$$\{x^{-1}\}^{[k-1]} = (-1)^{(k-1)}\,(k-1)!\,x^{-k} \,. \tag{2.4}$$

The distribution x^{-k} is usually denoted by fp x^{-k}, where the letters fp denote "finite part", for $k > 1$; and x^{-1} is usually denoted by pv x^{-1}, where the letters pv mean "principal value".
In what follows we shall use repeatedly the formula

$$\delta^{[k]} \,.\, \{x^{-1}\}^{[k]} = \frac{(-1)(k!)^2}{2(2k+1)!}\,\delta^{[2k+1]} \tag{2.5}$$

where k is an arbitrary non-negative integer. This formula was proved by us and Scarfiello in our note [3], pp. 65-67. It was later rediscovered by others: cf. [19], p. 202 and [20], p. 49, formula (3.37).

THEOREM 1. *Let* k *and* ℓ *be non-negative integers. Then*

$$\delta^{[k]}\{x^{-1}\}^{[\ell]} + \delta^{[\ell]}\{x^{-1}\}^{[k]} = (-1)\,\frac{k!\,\ell!}{(k+\ell+1)!}\,\delta^{[k+\ell+1]} \,. \tag{2.6}$$

NOTE. It is important to observe that, in this formula, the left-hand member must be considered *as a single entity*; or, equivalently, each of its two members is meaningless; what has a meaning is *the sum* of both of them, interpreted according to (1.3).

Proof of Theorem 1. Let k be an arbitrary non-negative integer. Then the following formula holds:

$$\delta^{[k]} . \{\frac{1}{x}\}^{[k+1]} + \delta^{[k+1]} . \{\frac{1}{x}\}^{[k]} = (-1) \frac{k!(k+1)!}{(2k+2)!} \delta^{[2k+2]} . \qquad (2.7)$$

The proof of (2.7) follows by differentiating both members of (2.5). If we now differentiate (2.7) we obtain, by using (2.5),

$$\delta^{[k]} . \{\frac{1}{x}\}^{[k+2]} + \delta^{[k+2]} \{\frac{1}{x}\}^{[k]} =$$

$$= (-1)\frac{k!(k+1)!}{(2k+2)!} \delta^{[2k+3]} - 2\left|\delta^{[k+1]} . \{\frac{1}{x}\}^{[k+1]}\right| = (-1)\frac{k!(k+2)!}{(2k+3)!}\delta^{[2k+3]} . \qquad (2.8)$$

This formula is the particular case of (2.6) when $\ell = k+2$. If we proceed with (2.8) as we proceeded with (2.7), i.e., if we differentiate (2.8) and use again (2.5) (where k has been replaced by k+1) we obtain, after easy calculations,

$$\delta^{[k]} \{\frac{1}{x}\}^{[k+3]} + \delta^{[k+3]} \{\frac{1}{x}\}^{[k]} = (-1) \frac{k! \ (k+3)!}{(k+(k+3)+1)!} \delta^{[k+(k+3)+1]} .$$

which is the particular case of (2.6) when $\ell = k+3$.

Iterating the procedure we obtain, after n steps,

$$\delta^{[k]} \{\frac{1}{x}\}^{[k+n]} + \delta^{[k+n]} \{\frac{1}{x}\}^{[k]} = (-1)\frac{k! \ (k+n)!}{(k+(k+n)+1)!} \delta^{[k+(k+n)+1]} . \qquad (2.9)$$

If we put in (2.9) $k+n = \ell$, formula (2.6) is proved.

NOTE. An ingenious proof of (2.6), which does not requiere the use of formula (2.5), and uses instead the theory of Fourier transforms, has been communicated to us by Professor A. P. Calderón.

3. Let S be a distribution with compact support and let us put $T = \frac{1}{x} * S$, $S_n(x) = S * g_n(x)$, $T_n(x) = S_n(x) * \frac{1}{x}$.

THEOREM 2. HYPOTHESIS: *The distribution* S.T *exists.*
THESIS: *The following formula holds:*

$$2 \frac{1}{x} * \{S.T\} = \{T.T. - \pi^2 S.S\} . \qquad (3.1)$$

NOTE. The right-hand side of this formula must be interpreted *as a single entity*; this is the explanation of the brackets which appear in the right-hand member.

Proof of Theorem 2. $S_n(x) \in C_c^\infty$. Therefore, the following formula is valid for $n = 1,2,\ldots$:

$$2\,\frac{1}{x} * \{S_n(x)\; T_n(x)\} = \{T_n(x)\}^2 - \pi^2\,\{S_n(x)\}^2 \quad . \qquad (3.2)$$

When $S_n(x)$ is a step function this relation is due to Cotlar, [4], p. 159. But it can be proved that (3.2) is also valid when $S_n(x) \in C_c^\infty$, $n = 1,2,\ldots$. Since the support of $S_n(x)$ is contained in a fixed compact $K \in R^1$, we can pass to the limit in the left-hand member of (3.2), and the theorem is proved.*

4. We shall see that by using (3.1) it is possible to evaluate some curious heterodox multiplicative products.

THEOREM 3.

$$\{x^{-1}\}^{[S]} \,.\, \{x^{-1}\}^{[S]} - \pi^2\{\delta^{[S]} \,.\, \delta^{[S]}\} = (S!)^2\, x^{-2S-2}. \qquad (4.1)$$

Here S *is a non-negative integer.*

Proof of Theorem 3. Putting in (3.1) $S = \delta^{[S]}$ we obtain, taking into account (2.4) and (2.5)

$$\{x^{-1}\}^{[S]}.\{x^{-1}\}^{[S]} - \pi^2\{\delta^{[S]}.\delta^{[S]}\} = 2x^{-1} * \{\delta^{[S]}.\{x^{-1}\}^{[S]}\} = (S!)^2\, x^{-2S-2}$$

and the theorem is proved. The particular case S=0 has been proved by Mikusinski in [5].

THEOREM 4.

$$\{\{x^{-1}\}^{[k]}.\{x^{-1}\}^{[\ell]} - \pi^2\{\delta^{[k]}.\delta^{[\ell]}\}\} = (-1)^{k+\ell}\, k!\; \ell!\; x^{-k-\ell-2} \quad . \qquad (4.2)$$

Here k *and* ℓ *are non-negative integers.*

Proof of Theorem 4. If we put in (3.1) $S = \delta^{[k]} + \delta^{[\ell]}$ we obtain

$$\{x^{-1}\}^{[k]}.\{x^{-1}\}^{[\ell]} - \pi^2\{\delta^{[k]}.\delta^{[\ell]}\} = x^{-1} * \{\{x^{-1}\}^{[\ell]}.\delta^{[k]} + \{x^{-1}\}^{[k]}.\delta^{[\ell]}\}.$$

The right-hand of this formula reads (if we take into account (2.4) and (2.6)),

* A factor π^2 is missing in the first term of the right-hand side of formula (55), p. 159, of [4].

$$x^{-1} * \{(-1) \frac{k!\ \ell!}{(k+\ell+1)!} \delta^{[k+\ell+1]}\} = (-1)^{k+\ell} k!\ \ell!\ x^{-k-\ell-2}$$

and the theorem is proved.

A natural generalization of Theorem 3 is obtained by putting in (3.1) $S = \sum_{\nu=0}^{r} a_\nu \delta^{[\nu]}$, where the a_ν are complex constants. This will allow (according to a well-known theorem of L. Schwartz ([6], p. 100)), to evaluate explicitly the right-hand side of (3.1) when S is a distribution supported by the origin.

4, bis.* We shall prove here the *existence* of heterodox products similar to - but much more general than - those which appear at the left-hand sides of (2.6) and (3.1); but we shall not evaluate them explicitly, as we have done in (2.6) and (3.1).

THEOREM 4, BIS. HYPOTHESIS. *Let* S *and* T *belong to* $\mathcal{E}'$.** *For their Hilbert transforms we write*

$$\mathcal{H}(S) = S * \frac{1}{\pi}\frac{1}{x} \quad , \qquad (4,\text{bis}.1)$$

$$\mathcal{H}(T) = T * \frac{1}{\pi}\frac{1}{x} \quad . \qquad (4,\text{bis}.2)$$

TESIS. *The products*

$$S \,.\, T - \mathcal{H}(S) \,.\, \mathcal{H}(T) \quad , \qquad (4,\text{bis}.3)$$

and

$$S \,.\, \mathcal{H}(T) + T \,.\, \mathcal{H}(S) \qquad (4,\text{bis}.4)$$

exist always.

NOTE 1. We shall limit ourselves to prove here the existence of the products (4,bis.1) and (4,bis.2) in the sense of (1.3), *using the special mollifier*,

$$g_y(x) = \frac{1}{y}\, g\left(\frac{x}{y}\right) = \frac{1}{\pi}\, \frac{y}{x^2 + y^2} \quad , \qquad (4,\text{bis}.5)$$

* This paragraph has been written following a suggestion of Professor A. P. Calderón.

** More generally, we could assume that S and T belong to the class $\sqrt{1+x^2}\, D'_{L^1}$, introduced by L. Schwartz [23], pp. 6-9. This class can be considered as the most general family of distributions for which the convolution with $\frac{1}{x}$ is possible.

where $g(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$. Here $y > 0$, and y will tend eventually to 0. This generalized mollifier (we say "generalized" because it does not fulfill condition b) is the Poisson singular kernel for the upper half-plane $\Delta = \{z = x+iy \ , \ y > 0\}$. It is very well known that

$$g_y(x) \underset{y\downarrow 0}{\rightarrow} \delta \quad . \qquad (4,bis.6)$$

NOTE 2. In the course of the proof of Theorem 4, bis, the following theorem due to Martineau, [2], p. 203, Proposition 3, will play an essential role.

THEOREM (of Martineau). *For the function* $f(z)$, *holomorphic in* Δ , *to have a limit in* $S'(R)$ *when* $y \downarrow 0$ *it is necessary and sufficient that a positive integer* n *exists such that*

$$\left|\frac{y}{1 + x^2 + y^2}\right|^n . \ |f(z)| \leqslant M \qquad (4,bis.7)$$

for $z \in \Delta$.*

Proof of Theorem 4, bis. We consider the distributions $M = S + i\,\mathcal{H}(S)$, $N = T + i\,\mathcal{H}(T)$, and mollify them by $g_y(x)$:

$$a(x,y) = M * g_y(x) \quad , \qquad (4,bis.8)$$

$$b(x,y) = N * g_y(x) \quad . \qquad (4,bis.9)$$

One verifies without difficulty (by using definitions (4.bis.1) and (4,bis.2)) that the following formulae are valid:

$$- \pi\ i\ a(x,y) = S * \frac{1}{x + iy} \quad ,$$

$$- \pi\ i\ b(x,y) = T * \frac{1}{x + iy} \quad .$$

Therefore $a(x,y)$ and $b(x,y)$, for which we shall write $a(z)$ and $b(z)$, are holomorphic functions in Δ. It is also immediate to verify, taking into account (4,bis.6), that $a(z)$ and $b(z)$ tend, when $y \downarrow 0$, to M and N respectively. Therefore, according to the Theorem of Martineau, $a(z)$ and $b(z)$ satisfy relations of the type

* Condition (4,bis.7) has been communicated to us by Professor A. P. Calderón. The necessary and sufficient condition stated by Martineau is different (and seems to contain an erratum).

$$\left|\frac{y}{1+|z|^2}\right|^{n_1}.|a(z)| \leqslant M_1 \quad ,$$

$$\left|\frac{y}{1+|z|^2}\right|^{n_2}.|b(z)| \leqslant M_2 \quad ,$$

for $z \in \Delta$, where n_1, and n_2 are positive integers. Consequently we have also

$$\left|\frac{y}{1+|z|^2}\right|^{n_1+n_2}.|a(z)b(z)| \leqslant M_3 \quad ,$$

when $z \in \Delta$; and the Theorem of Martineau shows that the function $c(z) = a(z)b(z)$ tends to a limit in $S'(R)$ when $y \downarrow 0$.

Now we have

$$c(z) = a(z)b(z) = \{M * g_y(x)\} . \{N * g_y(x)\} =$$
$$= \{S * g_y(x)\} \{T * g_y(x)\} - \{\mathcal{H}(S) * g_y(x)\} \{\mathcal{H}(T) * g_y(x)\} +$$
$$+ i [\{S * g_y(x)\} \{\mathcal{H}(T) * g_y(x)\} +$$
$$+ (\mathcal{H}(S) * g_y(x))(T * g_y(x))] . \tag{4,bis.10}$$

If we take limits in (4,bis.10) for $y \downarrow 0$ the Theorem is proved.

NOTE. A more elaborated proof shows that Theorem 4,bis continues to hold if we define the products (4,bis.3) and (4,bis.4) according to definition (1,3), the mollifier possessing the five properties a) - f), stated there.

THEOREM 4, TER. HYPOTHESIS. *Let* S_ν $(\nu = 1,2,\dots,n;\ 2 \leqslant n < \infty)$ *belong to* $E'(R_1)$ *or, more generally, to the class* $\sqrt{1+x^2}\, D'_{L_1}$ *of Schwartz. For their Hilbert transforms we shall write* $\{S_\nu\} = S_\nu * \frac{1}{\pi}\frac{1}{x}$, $\nu=1,2,\dots,n$. *Let us put* $M_\nu = S_\nu + i\,\mathcal{H}\{S_\nu\}$, $\nu = 1,2,\dots,n$; *and let us consider the functions* $a_\nu(x,y) = a_\nu(z) = M_\nu * g_y(x)$, $\nu = 1,2,\dots,n$.

THESIS. *The function* $c(z) = \sum_{\nu=1}^{n} a_\nu(z)$, *holomorphic in* Δ , *tends to a limit in* $S'(R)$ *when* $y \downarrow 0$.

Proof of Theorem 4, ter. The proof is identical with that Theorem 4, bis, and is therefore omitted.

NOTE. If we evaluate explicitly the limit of $c(z)$ when $y \downarrow 0$, its real and imaginary parts contain "heterodox" multiplicative products (which reduce to (4,bis.3) and (4,bis.4) in the particular case n=2) which are of interest in the quantum theory of fields.

5. Let us consider the distributions ([2] , p. 94)

$$\{x \pm io\}^{-n} = x^{-n} \mp \pi i \frac{(-1)^{n-1}}{(n-1)!} \delta^{[n-1]} \tag{5.1}$$

where n is a positive integer. An interesting consequence of (2.6) and (4.2) is

THEOREM 5.

$$(x \pm io)^{-k} . (x \pm io)^{-\ell} = (x \pm io)^{-k-\ell} . \tag{5.2}$$

Here k *and* ℓ *are positive integers.*

Proof of Theorem 5. Let us write (5.1) in the equivalent form

$$(x \pm io)^{-n} = \frac{(-1)^{n-1}}{(n-1)!} \{\{x^{-1}\}^{[n-1]} \mp \pi i \delta^{[n-1]}\} \tag{5.3}$$

From (2.6), (4.2) and (5.2) we conclude that the left-hand side of

(5.2) equals $x^{-k-\ell} \mp \pi i \frac{(-1)^{k+\ell-1}}{(k+\ell-1)!} \delta^{[k+\ell-1]}$. This proves the theorem.

Theorem 5 has been proved, using other methods, by Vladimirov ([7], p. 292) * and by Fisher [8]. The particular case k=1, ℓ=1 is due to Mikusinski [5], p. 512.

6. It is possible to generalize to R^n the preceding theorems. One immediate generalization consists in "tensorializing" our one-dimensional formulae.

We shall give only one example. Let $x = (x_1, x_2, \ldots, x_n) \in R^n$ and let us coin the definitions (where the symbol x means "tensorial product")

$$\delta^{[\ell]}(x) = \delta^{[\ell]}(x_1) \times \delta^{[\ell]}(x_2) \times \ldots \times \delta^{[\ell]}(x_n) ,$$

$$\{x^{-1}\}^{[\ell]} = \{x_1^{-1}\}^{[\ell]} \times \{x_2^{-2}\}^{[\ell]} \times \ldots \times \{x_n^{-n}\}^{[\ell]} .$$

Then

* Vladimirov's theorem (and its proof) is the particular case $S_\nu = \frac{1}{x}$ (ν=1,2,...,n) of our Theorem 4,ter, if account is taken of formula (5.1).

$$\delta^{[k]}\{x^{-1}\}^{[k]} = \left\{\frac{(-1)(k!)^2}{2(2k+1)!}\right\}^n \delta^{[2k+1]}(x) \ . \tag{6.0}$$

The particular case k=0 appears in [1], p. 252. Formulae (2.6), (4.1) and (4.2) can be similarly generalized.

More interesting for applications are the multidimensional generalizations obtained by means of a change of variable . Our next theorems will deal with this type of generalization.

Let S_t denote a distribution of one variable t. Let $u \in C^\infty(R^n)$ be such that the (n-1)-dimensional manifold $u(x_1,\ldots,x_n) = 0$ has no critical point. By $S_{u(x)}$ Leray (cf. [9],p. 102) designates the distribution defined on R^n by

$$\langle S_{u(x)}, \phi(x) \rangle = \langle S_t , \psi(t) \rangle \ . \tag{6.1}$$

Here $\phi \in C_0^\infty (R^n)$ and

$$\psi(t) = \int_{u(x)=t} \phi(x) \ w_u(x,dx) \ . \tag{6.2}$$

Here w_u is an (n-1)-dimensional exterior differential form on u defined as follows: $du \wedge dw = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ and the orientation of the manifold $u(x) = t$ is such that $w_u(x,dx) > 0$.

We shall consider the special case

$$u(x) = G(x) - m^2 . \tag{6.3}$$

Here

$$G(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 , \tag{6.4}$$

where p+q = n, and $m^2 > 0$.

We shall consider mainly two distributions, namely $\delta^{[k]}_{G-m^2}$ and $X^{-k}_{G-m^2}$, which, for tipographical convenience, we shall designate by $\delta^{[k]}(G - m^2)$ and $(G - m^2)^{-k}$ respectively.

According to (6.1) they are defined as follows:

$$\langle \delta^{[k]}(G - m^2),\phi(x) \rangle = \langle \delta^{[k]}(t),\psi(t) \rangle = (-1)^k \psi^{[k]}(0) \ ; \tag{6.5}$$

$$\langle (G - m^2)^{-2k},\phi(x) \rangle = \langle t^{-2k},\psi(t) \rangle = \int_0^\infty t^{-2k}\{\psi(t) + \psi(-t) +$$

$$- 2 [\psi(0) + \frac{t^2}{2!} \psi^{[2]}(0) + \ldots + \frac{t^{2k-2}}{(2k-2)!} \psi^{[2k-2]}(0)]\} \ dx \ ; \tag{6.6}$$

$$\langle (G - m^2)^{-2k-1}, \phi(x) \rangle = \langle t^{-2k-1}, \psi(t) \rangle = \int_0^\infty t^{-2k-1}\{\psi(t) - \psi(-t) +$$

$$- 2 [t\, \psi^{[1]}(0) + \frac{t^3}{3!}\psi^{[3]}(0) + \ldots + \frac{t^{2k-1}}{(2k-1)!}\psi^{[2k-1]}(0)]\}\, dx \quad . \qquad (6.7)$$

In these formulae the function $\psi(t)$ is defined, according to (6.2), by the formula

$$\psi(t) = \int_{G(x)-m^2=t} \phi(x) \quad w_{G-m^2}(x,dx) \; . \qquad (6.8)$$

NOTE. The distribution $\{G - m^2\}^{-k}$ can also be defined by the formula (which is a multidimensional analogue of (2.1))

$$\{G - m^2\}^{-k} = \begin{cases} (G - m^2)_+^{-k} + (G - m^2)_-^{-k} & \text{if k is even ,} \\ (G - m^2)_+^{-k} - (G - m^2)_-^{-k} & \text{if k is odd .} \end{cases} \qquad (6.9)$$

In this formulae λ is a complex number, and

$$\langle (G - m^2)_+^\lambda, \phi \rangle = \int_{G-m^2 \geq 0} (G - m^2)^\lambda \phi(x)\, dx \quad , \qquad (6.10)$$

$$\langle (G - m^2)_-^\lambda, \phi \rangle = \int_{G-m^2 < 0} (-G + m^2)^\lambda \phi(x)\, dx \; . \qquad (6.11)$$

The integrals which appear in the right-hand sides of (6.10) and (6.11) converge and are analytical functions of the complex parameter λ, Re $\lambda \geqslant 0$. For Re $\lambda < 0$ they are defined by analytical continuation. The distribution $(G - m^2)^{-k}$, as defined by (6.9) coincides, when $m^2 = 0$, with the distribution which Methée calls σ^{2m}; cf. [18], p.146 and also [6], p. 264.

7. If we change variables in both sides of (2.6) we get

$$[\delta^{[k]}(G - m^2)].[(G - m^2)^{-1}]^{[\ell]} + [\delta^{[\ell]}(G - m^2)].[(G - m^2)^{-1}]^{[k]} =$$

$$= (-1)\frac{k!\,\ell!}{(k+\ell+1)!}\,\delta^{[k+\ell+1]}(G - m^2) \; . \qquad (7.1)$$

Similarly, changing variables in (4.2) we get

$$\left\{\frac{1}{G - m^2}\right\}^{[k]} \cdot \left\{\frac{1}{G - m^2}\right\}^{[\ell]} - \pi^2\{\delta^{[k]}(G - m^2).\delta^{[\ell]}(G - m^2)\} =$$

$$= (-1)^{k+\ell} \quad k!\ \ell! \quad \frac{1}{(G - m^2)^{k+\ell+2}} \quad . \tag{7.2}$$

Let us now consider the distributions (5.1) (or (5.3)). Proceeding as before we obtain

$$(G - m^2 \pm io)^{-k} = \frac{1}{(G - m^2)^k} \mp \frac{\pi i(-1)^{k-1}}{(k-1)!} \cdot \delta^{[k-1]}(G - m^2) \quad , \tag{7.3}$$

or, equivalently,

$$(G - m^2 \pm io)^{-k} = \frac{(-1)^{k-1}}{(k-1)!}\left\{\left(\frac{1}{G - m^2}\right)^{k-1} \mp \pi i \delta^{[k-1]}(G-m^2)\right\} . \tag{7.4}$$

The following proposition is an n-dimensional analogue of Theorem 5.

THEOREM 6.

$$(G - m^2 \pm io)^{-k} . (G - m^2 \pm io)^{-\ell} = (G - m^2 \pm io)^{-k-\ell} \quad . \tag{7.5}$$

Here k *and* ℓ *are positive integers.*

Proof of Theorem 6. It runs exactly along the same lines as those of Theorem 5; one has to use, instead of formulae (2.6) and (4.2), their n-dimensional analogues (7.1) and (7.2).

NOTE. Gelfand and Shilov ([2], p. 289) consider the distribution

$$(m^2 + G \pm io)^{\lambda} = \lim_{\varepsilon \to 0} (m^2 + G \pm i\ \varepsilon(x_1^2 + \ldots + x_n^2))^{\lambda} \quad , \tag{7.6}$$

where $\varepsilon > 0$ and $\lambda \in C$. They have not studied in detail the special case in which λ is a negative integer. This has been done by de Jager [10] and by Bresters [11]. They arrived to results equivalent to our formulae (7.3) and (7.4) by different methods. Trione ([12], p. 24) has proved the formula

$$\{G + m^2 \pm io\}^{\lambda} . \{G + m^2 \pm io\}^{\mu} = \{G + m^2 \pm io\}^{\lambda+\mu}$$

for λ and $\mu \in C$.

Let us state the following equivalent version of Theorem 6.

THEOREM 7. *Let* k *be any integer* $\geqslant 1$. *Then*

$$(G - m^2 \pm io)^{-1} \underset{\text{k-times}}{\dots} (G - m^2 \pm io)^{-1} = (G - m^2 \pm io)^{-k} . \quad (7.7)$$

8. It is important to observe that for (7.1) and (7.2) to hold, m *must be different from zero*. Indeed, formula (6.1) may not hold if we put in it $u(x) = G(x)$. This is due to the fact that the cone $G(x) = 0$ has a critical point (namely, the origin). A consequence of this is that the distribution $\delta^{[k]}(G)$ exists only if $k < \frac{n-2}{2}$ (cf. [2], p. 250).

This entails that formulae (7.1), (7.2), (7.3), (7.4), Theorem 6 and Theorem 7 do not hold for every k when m = 0; and one verifies that the analogues of these formulae and theorems read as follows.

THEOREM 8. *Let the non-negative integers* k *and* ℓ *be such that* $k+\ell+1 < \frac{n-2}{2}$. *Then*

$$\delta^{[k]}(G) . \left[\frac{1}{G}\right]^{[\ell]} + \delta^{[\ell]}(G) . \left[\frac{1}{G}\right]^{[k]} = (-1)\frac{k!\ell!}{(k+\ell+1)!} \delta^{[k+\ell+1]}(G). \quad (8.1)$$

THEOREM 9. *Let the non-negative integers* k *and* ℓ *be such that* $k+\ell+2 < \frac{n-2}{2}$. *Then*

$$\left[\frac{1}{G}\right]^{[k]} . \left[\frac{1}{G}\right]^{[\ell]} - \pi^2\{\delta^{[k]}(G).\delta^{[\ell]}(G)\} = (-1)^{k+\ell}k!\ell! \frac{1}{G^{k+\ell+2}} . \quad (8.2)$$

THEOREM 10. *Let* k *and* ℓ *be positive integers such that* $k+\ell-1 < \frac{n-2}{2}$. *Then*

$$\{G \pm io\}^{-k} . \{G \pm io\}^{-\ell} = \{G \pm io\}^{-k-\ell} . \quad (8.3)$$

THEOREM 11. *If* k *is a positive integer such that* $k < \frac{n}{2}$ *the following formula holds:*

$$(G \pm io)^{-1} \underset{\text{k-times}}{\dots} (G \pm io)^{-1} = (G \pm io)^{-k} . \quad (8.4)$$

NOTE. Gelfand and Shilov ([2], p. 275) consider the distributions

$$(G \pm io)^{\lambda} = \lim_{\varepsilon \to 0} \{G \pm i \varepsilon(x_1^2 + \dots + x_n^2)\}^{\lambda} , \quad (8.5)$$

where $\varepsilon > 0$ and $\lambda \in \mathbb{C}$. They show that this distribution is an holomorphic (distribution-valued) function of λ, everywhere except at $\lambda = -\frac{n}{2} - k$, where k is a non-negative integer. At these points $(G \pm io)^{\lambda}$ has first-order poles, and one has ([2], p. 276)

$$\operatorname*{res}_{\lambda=-\frac{n}{2}-k} [G \pm io]^{\lambda} = \frac{e^{\mp \pi q i} \pi^{\frac{n}{2}}}{4^k k! \, \Gamma(\frac{n}{2}+k)} \cdot \{L^k\} \delta . \tag{8.6}$$

Here

$$L = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} . \tag{8.7}$$

They did not study in detail the special case in which λ is a negative integer which is not a pole of (8.6). This has been done by de Jager [10] and by Bresters [11]. We remark, finally, that an equivalent version of Theorem 10 has been proved by Trione in her thesis ([12], p. 28).

9. As an application of Theorem 10 we shall evaluate an heterodox product which appears in the quantum theory of fields.

THEOREM 12. *Let* m *be a positive integer such that* $m < \frac{n}{2}$ *; and let* k *be a non-negative integer. Then*

$$\{G \pm io\}^{-m} . \{[L^k]\delta\} = C\,[L^{k+m}]\delta . \tag{9.1}$$

Here

$$C = \{4^m (k+1) \dots (k+m)(\tfrac{n}{2}+k) \dots (\tfrac{n}{2}+k+m-1)\}^{-1} . \tag{9.2}$$

Proof of Theorem 12. It follows from (8.7) that

$$\lim_{\alpha\to 0} \frac{4^k k! \, \Gamma(\frac{n}{2}+k)\alpha}{e^{\mp i \frac{\pi}{2} q} \pi^{\frac{n}{2}}} \{G \pm io\}^{\alpha-\frac{n}{2}-k} = \{L^k\}\delta . \tag{9.3}$$

Let us call P the left-hand side of (9.1). We have, according to (9.3) and Theorem 10,

$$P = \lim_{\alpha\to 0} \frac{4^k k! \, \Gamma(\frac{n}{2}+k)\alpha}{e^{\mp i \frac{\pi}{2} q} \pi^{\frac{n}{2}}} \{G \pm io\}^{\alpha-\frac{n}{2}-k-m} =$$

$$= \lim_{\alpha\to 0} \frac{4^k k! \, \Gamma(\frac{n}{2}+k)}{e^{\mp i \frac{\pi}{2} q} \pi^{\frac{n}{2}}} \frac{C_\pm}{C_\pm} \alpha \{G \pm io\}^{\alpha-\frac{n}{2}-k-m} =$$

$$= \frac{4^k k! \, \Gamma(\frac{n}{2}+k)}{e^{\mp \frac{\pi}{2} qi} \pi^{\frac{n}{2}} C_\pm} \{L^{k+m}\}\delta . \tag{9.4}$$

Here we have put

$$C_{\pm} = \frac{4^{k+m}(k+m)!\ \Gamma(\frac{n}{2}+k+m)}{e^{\mp\frac{\pi}{2}qi}\ \pi^{\frac{n}{2}}}$$

One verifies immediately that

$$\frac{4^k k!\ \Gamma(\frac{n}{2}+k)}{e^{\mp\frac{\pi}{2}qi}\ \pi^{\frac{n}{2}}\ C_{\pm}} = \{4^m(k+1)\ldots(k+m)(\tfrac{n}{2}+k)\ldots(\tfrac{n}{2}+k+m-1)\}^{-1}. \tag{9.5}$$

The theorem follows from (9.4) and (9.5). The particular case $n=4$, $q=1$, $m=1$, $k=0$ reads

$$\{x_1^2 + x_2^2 + x_3^2 - x_4^2 \pm io\}^{-1}.\delta = \frac{1}{8}\Box\delta \quad , \tag{9.6}$$

where

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} . \tag{9.7}$$

Formula (9.6), which is used in the quantum theory of fields, belongs to Guerra, cf. [13], p. 530. A theorem similar to Theorem 12 appears in our joint note [14].

10. We shall finish with a remark about a connection of our heterodox products with quantum electrodynamics.

We have proved that formula (8.3) is valid only when $k-1 < \frac{n-2}{2}$.

Now, we may define $(G \pm io)^{-k}$, when $k = \frac{n}{2}, \frac{n}{2}+1, \ldots$, *by taking its finite part.*

This finite part has been explicitly evaluated by Trione, cf. [7], p. 252, formula (1.7). Her result reads as follows ($n \geqslant 4$, $m = 0,1,2,\ldots$):

$$\mathrm{Pf}\{G \pm io\}^{-\frac{n}{2}-m} = C_1[n,m,q]\int_{R^n} e^{i\langle x,y\rangle} Q^m \lg[Q \mp io]\,dy + C_2(n,m,q)\ L^m\delta . \tag{10.1}$$

In this formula we have put

$$C_1(n,m,q) = \frac{\pi^{-\frac{n}{2}}\ 2^{-n-2m}\ e^{\mp\frac{\pi}{2}qi}\ (-1)^{m+1}}{m!\ \Gamma(\frac{n}{2}+m)} , \tag{10.2}$$

$$C_2(n,m,q) = \left\{2\lg 2 + [1 + \frac{1}{2} + \ldots + \frac{1}{m}] - \nu + \frac{\Gamma'(\frac{n}{2}+m)}{\Gamma(\frac{n}{2}+m)}\right\} .$$

$$\cdot \left\{ \frac{\pi^{\frac{n}{2}} \, 2^{-n-2m} \, e^{\mp \frac{\pi}{2} qi}}{m! \; \Gamma(\frac{n}{2} - m)} \right\} \; . \tag{10.3}$$

Here ν denotes the Euler constant.

When m=0 the expression $1 + \frac{1}{2} + \ldots + \frac{1}{m}$ which appears in C_2 must be replaced by 0.

NOTE. It is interesting to observe, with Trione ([7], p. 252), that putting in (10.1) q=0 and taking Fourier transforms in both sides, one obtains an important formula due to Schwartz (stated by him without proof); cf. [6], p. 258, formula (VII, 7;4).

11. The particular case n=4, q=1 of (10.1) is especially important. If we put in it $\frac{n}{2} + m = \ell$, it reads as follows:

$$\mathrm{Pf}\, \{x_1^2 + x_2^2 + x_3^2 - x_4^2 \pm io\}^{-\ell} =$$

$$= C_1(4,\ell-2,1) \int_{R^4} e^{i\langle x,y \rangle} (y_1^2+y_2^2+y_3^2-y_4^2)^{\ell-2} \; \lg[\, y_1^2+y_2^2+y_3^2-y_4^2 \mp io]\, dy \; +$$

$$+ \; C_2(4,\ell-2,1) \; \Box^{\ell-2} \, \delta \quad . \tag{11.1}$$

In this formula the constants C_1 and C_2 are defined by (10.2) and (10.3) respectively, and $\Box$ is the wave operator (9.7). The expression $1 + \frac{1}{2} + \ldots + \frac{1}{\ell-2}$ which appears in C $(4,\ell-2,1)$ must be replaced by 0 when $\ell=2$.

Let us consider the distribution

$$D_F^c(x) = \frac{i}{4\pi^2} \{x_1^2 + x_2^2 + x_3^2 - x_4^2 - io\}^{-1} \quad . \tag{11.2}$$

This distribution is the famous "photonic causal delta", of Feynman. In quantum electrodynamics appear the multiplicative products $\{D_F^c\}^\ell$. We know (cf. Theorem 11) that these products are infinite for $\ell \geqslant 2$. In order to give a sense to these products (or to "regularize" them, as physicists say) it is a very natural prescription (as it seems to us) *to define them by their finite parts*. If we accept this prescription we get, taking into account (11.1),

$$\{D_F^c\}^\ell = \left\{\frac{i}{4\pi^2}\right\}^\ell C_1(4,\ell-2,1) \int_{R^4} e^{i\langle x,y \rangle} (y_1^2 + y_2^2 + y_3^2 - y_4^2)^{\ell-2} \; .$$

$$\cdot \ \lg(y_1^2 + y_2^2 + y_3^2 - y_4^2 - io)\ dy + \left\{\frac{i}{4\pi^2}\right\}^{\ell} C_2(4,\ell-2,1)\Box^{\ell-2}\,\delta. \tag{11.3}$$

This relation is an example of "analytical regularization", a prescription introduced in [15] for eliminating the infinities from the series solution of the Schrödinger equation of quantum electrodynamics. The method has been much developed by others, notably by Speer ; cf. [16]

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